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GENERALIZATION OF POPOVICIU TYPE INEQUALITIES VIA GREEN'S FUNCTION AND FINK'S IDENTITY

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Abstract. We obtain some useful identities via Green's function and Fink's identity, and apply them to generalize the known Popoviciu's inequality for convex functions to higher order convex functions. Then we investigate the bounds for the identities related to the generalization of the Popoviciu inequality by using inequalities for the Čebyšev functional. Some results relating to the Grüss and Ostrowski type inequalities are also obtained. Finally, we construct new families of exponentially convex functions and Cauchy-type means by exploring at linear functionals associated with the obtained inequalities.

1. Introduction

Many areas in modern analysis directly or indirectly involve the applications of convex functions; further, convex functions are closely related to the theory of inequalities and many important inequalities are consequences of the applications of convex functions (see [10]). Divided differences are found to be very helpful when we are dealing with functions having different degrees of smoothness. The following definition of divided difference is given in [10, p. 14].

DEFINITION 1. The nth-order divided difference of a function $f:[a,b]\to\mathbb{R}$ at mutually distinct points $x_0,\ldots,x_n\in[a,b]$ is defined recursively by

$$[x_i;f]=f(x_i),\quad i=0,\ldots,n,$$
$$[x_0,\ldots,x_n;f]=\frac{[x_1,\ldots,x_n;f]-[x_0,\ldots,x_{n-1};f]}{x_n-x_0}. \tag{1}$$

It is easy to see that (1) is equivalent to

$$[x_0,\ldots,x_n;f]=\sum_{i=0}^{n}\frac{f(x_i)}{q'(x_i)},\quad \text{where } q(x)=\prod_{j=0}^{n}(x-x_j).$$

The following definition of a real valued convex function is characterized by nth-order divided difference (see [10, p. 15]).

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DEFINITION 2. A function $f:[a,b]\to\mathbb{R}$ is said to be n-convex $(n\geqslant 0)$ if and only if for all choices of $(n+1)$ distinct points $x_0,\dots,x_n\in[a,b]$, $[x_0,\dots,x_n;f]\geqslant 0$ holds.

If this inequality is reversed, then f is said to be n-concave. If the inequality is strict, then f is said to be a strictly n-convex (n-concave) function.

REMARK 1. Note that 0-convex functions are non-negative functions, 1-convex functions are increasing functions, and 2-convex functions are simply the convex functions.

The following theorem gives an important criteria to examine the n-convexity of a function f (see [10, p. 16]).

THEOREM 1. *If $f^{(n)}$ exists, then f is n-convex if and only if $f^{(n)}\geqslant 0$.*

In 1965, T. Popoviciu introduced a characterization of convex functions [11]. The inequality of Popoviciu as given by Vasić and Stanković in [12] can be written in the following form (see [10, p. 173]):

THEOREM 2. *Let $m,k\in\mathbb{N}$, $m\geqslant 3$, $2\leqslant k\leqslant m-1$, $[\alpha,\beta]\subset\mathbb{R}$, $\mathbf{x}=(x_1,\dots,x_m)\in[\alpha,\beta]^m$, $\mathbf{p}=(p_1,\dots,p_m)$ be a positive m-tuple such that $\sum_{i=1}^m p_i=1$. Also let $f:[\alpha,\beta]\to\mathbb{R}$ be a convex function. Then*

$$p_{k,m}(\mathbf{x},\mathbf{p};f)\leqslant\frac{m-k}{m-1}p_{1,m}(\mathbf{x},\mathbf{p};f)+\frac{k-1}{m-1}p_{m,m}(\mathbf{x},\mathbf{p};f),\tag{2}$$

where

$$p_{k,m}(\mathbf{x},\mathbf{p};f)=p_{k,m}(\mathbf{x},\mathbf{p};f(x)):=\frac{1}{C_{k-1}^{m-1}}\sum_{1\leqslant i_1<\dots<i_k\leqslant m}\left(\sum_{j=1}^k p_{i_j}\right)f\left(\frac{\sum_{j=1}^k p_{i_j}x_{i_j}}{\sum_{j=1}^k p_{i_j}}\right)$$

is the linear functional with respect to f.

In what follows in inequality (2), we will write

$$\Upsilon(\mathbf{x},\mathbf{p};f):=\frac{m-k}{m-1}p_{1,m}(\mathbf{x},\mathbf{p};f)+\frac{k-1}{m-1}p_{m,m}(\mathbf{x},\mathbf{p};f)-p_{k,m}(\mathbf{x},\mathbf{p};f).\tag{3}$$

REMARK 2. It is important to note that under the assumptions of Theorem 2, if the function f is convex then $\Upsilon(\mathbf{x},\mathbf{p};f)\geqslant 0$, and $\Upsilon(\mathbf{x},\mathbf{p};f)=0$ for $f(x)=x$ or when f is a constant function.

Consider the Green function $G:[\alpha,\beta]\times[\alpha,\beta]\to\mathbb{R}$ defined as

$$G(t,s)=\begin{cases}\frac{(t-\beta)(s-\alpha)}{\beta-\alpha}, & \alpha\leqslant s\leqslant t;\\ \frac{(s-\beta)(t-\alpha)}{\beta-\alpha}, & t\leqslant s\leqslant\beta.\end{cases}\tag{4}$$

The function G is convex and continuous w.r.t s and due to symmetry also w.r.t t.

For any function $\lambda : [\alpha,\beta] \to \mathbb{R}$, $\lambda \in C^2([\alpha,\beta])$, we have

$$\lambda(x) = \frac{\beta - x}{\beta - \alpha}\lambda(\alpha) + \frac{x-\alpha}{\beta-\alpha}\lambda(\beta) + \int_{\alpha}^{\beta} G(x,s)\lambda''(s)ds, \tag{5}$$

where the function G is defined in (4) (see [13]).

In the present paper, we use A. M. Fink's identity and prove many interesting results. The following theorem is proved by A. M. Fink in [6].

THEOREM 3. *Let $a,b \in \mathbb{R}$, $\lambda : [a,b] \to \mathbb{R}$, $n \geqslant 1$ and $\lambda^{(n-1)}$ is absolutely continuous on $[a,b]$. Then*

$$\begin{aligned}\lambda(x) = {} & \frac{n}{b-a}\int_a^b \lambda(t)\,dt \\ & - \sum_{w=1}^{n-1}\left(\frac{n-w}{w!}\right)\left(\frac{\lambda^{(w-1)}(a)(x-a)^w - \lambda^{(w-1)}(b)(x-b)^w}{b-a}\right) \\ & + \frac{1}{(n-1)!(b-a)}\int_a^b (x-t)^{n-1} w^{[a,b]}(t,x)\lambda^{(n)}(t)\,dt,\end{aligned} \tag{6}$$

where

$$w^{[a,b]}(t,x) = \begin{cases} t-a, & a \leqslant t \leqslant x \leqslant b, \\ t-b, & a \leqslant x < t \leqslant b. \end{cases} \tag{7}$$

The organization of the paper is as follows: In Section 2, we use Fink's identity and the n-convexity of the function λ to establish a generalization of Popoviciu's inequality. In Section 3, we present some interesting results by employing Čebyšev functional and Grüss-type inequalities, also results relating to the Ostrowski-type inequality. At the end we study the functional defined as the difference between the R.H.S. and the L.H.S. of the generalized inequality. Here our objective is to investigate the properties of the functional, such as n-exponential and logarithmic convexity.

2. Generalization of Popoviciu's inequality for n-convex functions via Green function and A. M. Fink's identity

Motivated by the identity (3), we use (5) and Fink's identity to prove the following generalized identity.

THEOREM 4. *Let $\lambda : [\alpha,\beta] \to \mathbb{R}$ be such that for $n \geqslant 3$, $\lambda^{(n-1)}$ is absolutely continuous and let $m,k \in \mathbb{N}$, $m \geqslant 3$, $2 \leqslant k \leqslant m-1$, $[\alpha,\beta] \subset \mathbb{R}$, $\mathbf{x} = (x_1,\dots,x_m) \in [\alpha,\beta]^m$, $\mathbf{p} = (p_1,\dots,p_m)$ be a real m-tuple such that $\sum_{j=1}^{k} p_{i_j} \neq 0$ for any $1 \leqslant i_1 < \dots < i_k \leqslant m$ and $\sum_{i=1}^{m} p_i = 1$. Also let $\frac{\sum_{j=1}^{k} p_{i_j}x_{i_j}}{\sum_{j=1}^{k} p_{i_j}} \in [\alpha,\beta]$ for any $1 \leqslant i_1 < \dots < i_k \leqslant m$ with G and*

$w^{[\alpha,\beta]}(t,x)$ be the same as defined in (4) and (7) respectively. Then we have the following identity:

$$\begin{aligned}
&\Upsilon(\mathbf{x},\mathbf{p};\lambda(x)) \\
&= (n-2)\left(\frac{\lambda^{(1)}(\beta)-\lambda^{(1)}(\alpha)}{\beta-\alpha}\right)\int_\alpha^\beta \Upsilon(\mathbf{x},\mathbf{p};G(x,s))ds + \frac{1}{(\beta-\alpha)}\int_\alpha^\beta \Upsilon(\mathbf{x},\mathbf{p};G(x,s)) \\
&\times\left(\sum_{w=1}^{n-3}\left(\frac{n-2-w}{w!}\right)\left(\lambda^{(w+1)}(\beta)(s-\beta)^w-\lambda^{(w+1)}(\alpha)(s-\alpha)^w\right)\right)ds \\
&+\frac{1}{(n-3)!(\beta-\alpha)}\int_\alpha^\beta \lambda^{(n)}(t)\left(\int_\alpha^\beta \Upsilon(\mathbf{x},\mathbf{p};G(x,s))(s-t)^{n-3}w^{[\alpha,\beta]}(t,s)ds\right)dt. \quad (8)
\end{aligned}$$

Proof. Using (5) in (3) and following the linearity of $\Upsilon(\mathbf{x},\mathbf{p};\lambda(x))$, we have

$$\Upsilon(\mathbf{x},\mathbf{p};\lambda(s)) = \int_\alpha^\beta \Upsilon(\mathbf{x},\mathbf{p};G(x,s))\lambda''(s)ds. \quad (9)$$

Differentiating (6), twice with respect variable s, we get

$$\begin{aligned}
\lambda''(s) = &\sum_{w=0}^{n-3}\left(\frac{n-2-w}{w!}\right)\left(\frac{\lambda^{(w+1)}(\beta)(s-\beta)^w-\lambda^{(w+1)}(\alpha)(s-\alpha)^w}{\beta-\alpha}\right) \\
&+\frac{1}{(n-3)!(\beta-\alpha)}\int_\alpha^\beta (s-t)^{n-3}w^{[\alpha,\beta]}(t,s)\lambda^{(n)}(t)dt \\
= &\sum_{w=1}^{n-2}\left(\frac{n-1-w}{(w-1)!}\right)\left(\frac{\lambda^{(w)}(\beta)(s-\beta)^{w-1}-\lambda^{(w)}(\alpha)(s-\alpha)^{w-1}}{\beta-\alpha}\right) \\
&+\frac{1}{(n-3)!(\beta-\alpha)}\int_\alpha^\beta (s-t)^{n-3}w^{[\alpha,\beta]}(t,s)\lambda^{(n)}(t)dt \\
&=(n-2)\left(\frac{\lambda^{(1)}(\beta)-\lambda^{(1)}(\alpha)}{\beta-\alpha}\right) \\
&+\sum_{w=2}^{n-2}\left(\frac{n-1-w}{(w-1)!}\right)\left(\frac{\lambda^{(w)}(\beta)(s-\beta)^{w-1}-\lambda^{(w)}(\alpha)(s-\alpha)^{w-1}}{\beta-\alpha}\right) \\
&+\frac{1}{(n-3)!(\beta-\alpha)}\int_\alpha^\beta (s-t)^{n-3}w^{[\alpha,\beta]}(t,s)\lambda^{(n)}(t)dt. \quad (10)
\end{aligned}$$

Using (10) in (9) and applying Fubini's Theorem in the last term we get (8).

Alternatively, we use formula (6) for the function λ'' and replace n by $n-2$ $(n \geqslant 3)$, to get

$$\begin{aligned}\lambda''(s) = &\ (n-2)\left(\frac{\lambda^{(1)}(\beta)-\lambda^{(1)}(\alpha)}{\beta-\alpha}\right)\\ &+\sum_{w=1}^{n-3}\left(\frac{n-2-w}{w!}\right)\left(\frac{\lambda^{(w+1)}(\beta)(s-\beta)^{w}-\lambda^{(w+1)}(\alpha)(s-\alpha)^{w}}{\beta-\alpha}\right)\\ &+\frac{1}{(n-3)!(\beta-\alpha)}\int_{\alpha}^{\beta}(s-t)^{n-3}w^{[\alpha,\beta]}(t,s)\lambda^{(n)}(t)\,dt \qquad (11)\\ = &\ (n-2)\left(\frac{\lambda^{(1)}(\beta)-\lambda^{(1)}(\alpha)}{\beta-\alpha}\right)\\ &+\sum_{w=2}^{n-2}\left(\frac{n-1-w}{(w-1)!}\right)\left(\frac{\lambda^{(w)}(\beta)(s-\beta)^{w-1}-\lambda^{(w)}(\alpha)(s-\alpha)^{w-1}}{\beta-\alpha}\right)\\ &+\frac{1}{(n-3)!(\beta-\alpha)}\int_{\alpha}^{\beta}(s-t)^{n-3}w^{[\alpha,\beta]}(t,s)\lambda^{(n)}(t)\,dt.\end{aligned}$$

Now using (11) in (9) and applying Fubini's Theorem in the last term we get (8). □

The following theorem gives a generalization of Popoviciu's inequality for n-convex functions.

THEOREM 5. *Let all the assumptions of Theorem 4 be satisfied and let for* $n \geqslant 3$

$$\int_{\alpha}^{\beta}\Upsilon(\mathbf{x},\mathbf{p};G(x,s))(s-t)^{n-3}w^{[\alpha,\beta]}(t,s)ds \geqslant 0,\ t\in[\alpha,\beta]. \qquad (12)$$

If λ *is* n*-convex function such that* $\lambda^{(n-1)}$ *is absolutely continuous, then we have*

$$\begin{aligned}&\Upsilon(\mathbf{x},\mathbf{p};\lambda(x))\\ &\geqslant (n-2)\left(\frac{\lambda^{(1)}(\beta)-\lambda^{(1)}(\alpha)}{\beta-\alpha}\right)\int_{\alpha}^{\beta}\Upsilon(\mathbf{x},\mathbf{p};G(x,s))ds+\frac{1}{(\beta-\alpha)}\int_{\alpha}^{\beta}\Upsilon(\mathbf{x},\mathbf{p};G(x,s))\\ &\times\left(\sum_{w=1}^{n-3}\left(\frac{n-2-w}{w!}\right)\left(\lambda^{(w+1)}(\beta)(s-\beta)^{w}-\lambda^{(w+1)}(\alpha)(s-\alpha)^{w}\right)\right)ds. \qquad (13)\end{aligned}$$

Proof. Since $\lambda^{(n-1)}$ is absolutely continuous on $[\alpha,\beta]$, $\lambda^{(n)}$ exists almost everywhere. As λ is n-convex, applying Theorem 1, we have, $\lambda^{(n)} \geqslant 0$ for all $x\in[\alpha,\beta]$. Hence we can apply Theorem 4 to obtain (13). □

Now we obtain a generalization of Popoviciu's inequality for m-tuples.

THEOREM 6. *Let in addition to the assumptions of Theorem 4,* $\mathbf{p}=(p_1,\ldots,p_m)$ *be a positive* m*-tuple such that* $\sum_{i=1}^{m} p_i = 1$*, and* $\lambda:[\alpha,\beta]\to\mathbb{R}$ *be an* n*-convex function.*

(i) If n is even and $n > 3$, then (13) holds.

(ii) Let the inequality (13) be satisfied and

$$\sum_{w=0}^{n-3}\left(\frac{n-2-w}{w!}\right)\left(\lambda^{(w+1)}(\beta)(s-\beta)^{w}-\lambda^{(w+1)}(\alpha)(s-\alpha)^{w}\right)\geqslant 0. \tag{14}$$

Then we have

$$\Upsilon(\mathbf{x},\mathbf{p};\lambda(x))\geqslant 0. \tag{15}$$

Proof.

(i) Since Green's function $G(x,s)$ is convex and the weights are positive. So $\Upsilon(\mathbf{x},\mathbf{p};G(x,s))\geqslant 0$ by virtue of Remark 2. Also, since

$$\vartheta(s):=(s-t)^{n-3}w^{[\alpha,\beta]}(t,s)=\begin{cases}(s-t)^{n-3}(t-\alpha), & \alpha\leqslant t\leqslant s\leqslant\beta,\\ (s-t)^{n-3}(t-\beta), & \alpha\leqslant s<t\leqslant\beta,\end{cases}$$

ϑ is positive for even n, where $n>3$. So, (12) holds for even n. Now following Theorem 5, we can obtain (13).

(ii) Using (14) in (13), we get (15). □

3. Bounds for identities related to generalization of Popoviciu's inequality

In this section we present some interesting results by using Čebyšev functional and Grüss type inequalities. For two Lebesgue integrable functions $f,h:[\alpha,\beta]\to\mathbb{R}$, we consider the Čebyšev functional

$$\Delta(f,h)=\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}f(t)h(t)dt-\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}f(t)dt.\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}h(t)dt.$$

The following Grüss type inequalities are given in [5].

THEOREM 7. *Let $f:[\alpha,\beta]\to\mathbb{R}$ be a Lebesgue integrable function and $h:[\alpha,\beta]\to\mathbb{R}$ be an absolutely continuous function with $(.-\alpha)(\beta-.)[h']^2\in L[\alpha,\beta]$. Then we have the inequality*

$$|\Delta(f,h)|\leqslant\frac{1}{\sqrt{2}}[\Delta(f,f)]^{\frac{1}{2}}\frac{1}{\sqrt{\beta-\alpha}}\left(\int_{\alpha}^{\beta}(x-\alpha)(\beta-x)[h'(x)]^2dx\right)^{\frac{1}{2}}. \tag{16}$$

The constant $\frac{1}{\sqrt{2}}$ in (16) is the best possible.

THEOREM 8. *Assume that* $h : [\alpha,\beta] \to \mathbb{R}$ *is monotonic nondecreasing on* $[\alpha,\beta]$ *and* $f : [\alpha,\beta] \to \mathbb{R}$ *be an absolutely continuous with* $f' \in L_\infty[\alpha,\beta]$. *Then we have the inequality*

$$|\Delta(f,h)| \leqslant \frac{1}{2(\beta-\alpha)}||f'||_\infty \int_\alpha^\beta (x-\alpha)(\beta-x)dh(x). \tag{17}$$

The constant $\frac{1}{2}$ *in* (17) *is the best possible.*

In the sequel, we consider above theorems to derive generalizations of the results proved in the previous section. In what follows we let

$$\mathfrak{O}(t) = \int_\alpha^\beta \Upsilon(\mathbf{x},\mathbf{p};G(x,s))(s-t)^{n-3}w^{[\alpha,\beta]}(t,s)ds \geqslant 0,\ t \in [\alpha,\beta]. \tag{18}$$

THEOREM 9. *Let* $\lambda : [\alpha,\beta] \to \mathbb{R}$ *be such that for* $n \geqslant 3$, $\lambda^{(n)}$ *is absolutely continuous with* $(.-\alpha)(\beta-.)[\lambda^{(n+1)}]^2 \in L[\alpha,\beta]$. *Let* $m,k \in \mathbb{N}$, $m \geqslant 3$, $2 \leqslant k \leqslant m-1$, $[\alpha,\beta] \subset \mathbb{R}$, $\mathbf{x} = (x_1,\ldots,x_m) \in [\alpha,\beta]^m$, $\mathbf{p} = (p_1,\ldots,p_m)$ *be a real* m*-tuple such that* $\sum_{j=1}^k p_{i_j} \neq 0$ *for any* $1 \leqslant i_1 < \ldots < i_k \leqslant m$ *and* $\sum_{i=1}^m p_i = 1$. *Also let* $\frac{\sum_{j=1}^k p_{i_j}x_{i_j}}{\sum_{j=1}^k p_{i_j}} \in [\alpha,\beta]$ *for any* $1 \leqslant i_1 < \ldots < i_k \leqslant m$ *with* $\mathfrak{O}$ *defined in* (18). *Then the remainder* $\mathfrak{K}_n(\alpha,\beta;\lambda)$ *given in the following identity*

$$\begin{aligned}\Upsilon(\mathbf{x},\mathbf{p};\lambda(x)) = &\int_\alpha^\beta \Upsilon(\mathbf{x},\mathbf{p};G(x,s)) \\ &\times \sum_{w=0}^{n-3}\left(\frac{n-2-w}{w!}\right)\left(\frac{\lambda^{(w+1)}(\beta)(s-\beta)^w - \lambda^{(w+1)}(\alpha)(s-\alpha)^w}{\beta-\alpha}\right)ds \\ &+ \frac{\lambda^{(n-1)}(\beta)-\lambda^{(n-1)}(\alpha)}{(\beta-\alpha)^2(n-3)!}\int_\alpha^\beta \mathfrak{O}(t)dt + \mathfrak{K}_n(\alpha,\beta;\lambda),\end{aligned} \tag{19}$$

satisfies the bound

$$|\mathfrak{K}_n(\alpha,\beta;\lambda)| \leqslant \frac{1}{\sqrt{2}(n-3)!}[\Delta(\mathfrak{O},\mathfrak{O})]^{\frac{1}{2}}\frac{1}{\sqrt{\beta-\alpha}}\left|\int_\alpha^\beta (t-\alpha)(\beta-t)[\lambda^{(n+1)}(t)]^2dt\right|^{\frac{1}{2}}.$$

Proof. The proof is the direct application of Theorem 7 by making substitutions $f \to \mathfrak{O}$ and $h \to \lambda^{(n)}$. □

The following Grüss type inequalities can be obtained by using Theorem 8.

THEOREM 10. *Let* $\lambda : [\alpha,\beta] \to \mathbb{R}$ *be such that for* $n \geqslant 3$, $\lambda^{(n)}$ *is absolutely continuous and let* $\lambda^{(n+1)} \geqslant 0$ *on* $[\alpha,\beta]$ *with* $\mathfrak{O}$ *defined in* (18). *Then in the representation* (19) *the remainder* $\mathfrak{K}_n(\alpha,\beta;\lambda)$ *satisfies the estimate*

$$|\mathfrak{K}_n(\alpha,\beta;\lambda)| \leqslant \frac{||\mathfrak{O}'||_\infty}{(n-3)!}\left[\frac{\lambda^{(n-1)}(\beta)+\lambda^{(n-1)}(\alpha)}{2} - \frac{\lambda^{(n-2)}(\beta)-\lambda^{(n-2)}(\alpha)}{\beta-\alpha}\right].$$

Proof. Applying Theorem 8 for $f \to \mathfrak{O}$ and $h \to \lambda^{(n)}$ and following the steps of Theorem 3.4 in [3] (see also [4]), we get above result. $\square$

Next we present an Ostrowski type inequality related to generalizations of Popoviciu's inequality.

THEOREM 11. *Suppose all the assumptions of Theorem 4 be satisfied. Moreover, assume that (p,q) is a pair of conjugate exponents, that is $p,q \in [1,\infty]$ such that $1/p+1/q=1$. Let $|\lambda^{(n)}|^p : [\alpha,\beta] \to \mathbb{R}$ be a R-integrable function for some $n \geqslant 3$. Then, we have*

$$\Bigg|\Upsilon(\mathbf{x},\mathbf{p};\lambda(x)) - \int_\alpha^\beta \Upsilon(\mathbf{x},\mathbf{p};G(x,s)) \times \sum_{w=0}^{n-3}\left(\frac{n-2-w}{w!}\right)\left(\frac{\lambda^{(w+1)}(\beta)(s-\beta)^w - \lambda^{(w+1)}(\alpha)(s-\alpha)^w}{\beta-\alpha}\right)ds\Bigg|$$
$$\leqslant \frac{1}{(n-3)!\beta-\alpha}||\lambda^{(n)}||_p\left(\int_\alpha^\beta\left|\int_\alpha^\beta \Upsilon(\mathbf{x},\mathbf{p};G(x,s))(s-t)^{n-3}w^{[\alpha,\beta]}(t,s)ds\right|^q dt\right)^{1/q}. \quad (20)$$

The constant on the R.H.S. of (20) is sharp for $1 < p \leqslant \infty$ and the best possible for $p=1$.

Proof. The proof is similar to the Theorem 3.5 in [3] (see also [4]). $\square$

4. Mean value theorems and n-exponential convexity

In the present section, we construct a positive linear functional and then give mean value theorems of Lagrange and Cauchy type.

REMARK 3. In virtue of Theorem 5, we can define the positive linear functional with respect to n-convex function λ as follows

$$\Gamma(\lambda) := \Upsilon(\mathbf{x},\mathbf{p};\lambda(x)) - \int_\alpha^\beta \Upsilon(\mathbf{x},\mathbf{p};G(x,s)) \times \sum_{w=0}^{n-3}\left(\frac{n-2-w}{w!}\right)\left(\frac{\lambda^{(w+1)}(\beta)(s-\beta)^w - \lambda^{(w+1)}(\alpha)(s-\alpha)^w}{\beta-\alpha}\right)ds \geqslant 0. \quad (21)$$

Lagrange and Cauchy type mean value theorems related to above functional are given in the following theorems.

THEOREM 12. *Let $\lambda : [\alpha,\beta] \to \mathbb{R}$ be such that $\lambda \in C^n[\alpha,\beta]$. If the inequality in (12) holds, then there exist $\xi \in [\alpha,\beta]$ such that*

$$\Gamma(\lambda) = \lambda^{(n)}(\xi)\Gamma(\varphi),$$

where $\varphi(x) = \frac{x^n}{n!}$ and $\Gamma(\cdot)$ is defined by (21).

Proof. Similar to the proof of Theorem 4.1 in [8] (see also [1]). □

THEOREM 13. *Let* $\lambda,\psi:[\alpha,\beta]\to\mathbb{R}$ *be such that* $\lambda,\psi\in C^n[\alpha,\beta]$. *If the inequality in (12) holds, then there exist* $\xi\in[\alpha,\beta]$ *such that*

$$\frac{\Gamma(\lambda)}{\Gamma(\psi)}=\frac{\lambda^{(n)}(\xi)}{\psi^{(n)}(\xi)},$$

provided that the denominators are non-zero, where $\Gamma(\cdot)$ *is defined by (21).*

Proof. Similar to the proof of Corollary 4.2 in [8] (see also [1]). □

Theorem 13 enables us to define Cauchy means, in fact

$$\xi=\left(\frac{\lambda^{(n)}}{\psi^{(n)}}\right)^{-1}\left(\frac{\Gamma(\lambda)}{\Gamma(\psi)}\right),$$

means that ξ is the mean of α, β for given functions λ and ψ.

We conclude our paper with the following remark.

REMARK 4. One can construct the non trivial examples of $n-$exponentially and exponentially convex functions from positive linear functional $\Gamma(\cdot)$ by following the $n-$exponentially method introduced by Pečarić et.al. in [7] and [9] (see also [2], [3] and [4]). As an application to Cauchy means, it enables us to construct a large families of functions which are exponentially convex.

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