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ON A PROBLEM CONNECTED WITH STRONGLY CONVEX FUNCTIONS

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Abstract. In this paper we show that the result obtained by Nikodem and Páles in [3] can by extended to a more general case. In particular, for a non-negative function F defined on a real vector space we define F-strongly convex functions and show that such functions are in the form $g+F^*$, where g is a convex function and F^* is a function associated with function F, iff F^* is a quadratic function. Using this result, we get a characterization of quadratic functions.

1. Introduction

Let $(X,\|\cdot\|)$ be a real normed space, D stand for a convex subset of X and c be a positive constant. A function $f : D \to \mathbb{R}$ is called *strongly convex with modulus* c if

$$f(tx+(1-t)y) \leqslant tf(x)+(1-t)f(y)-ct(1-t)\|x-y\|^2, \tag{1}$$

for all $x,y \in D$ and $t \in (0,1)$.

Such functions have been introduced by Polyak in [4] and as in turns out they play an important role in optimization theory. Strongly convex functions have also been studied by many authors, among others, see [1], [5], [6]. A function $f : D \to \mathbb{R}$ is called *strongly Jensen convex with modulus* c if

$$f\left(\frac{x+y}{2}\right) \leqslant \frac{f(x)+f(y)}{2} - \frac{c}{4}\|x-y\|^2, \tag{2}$$

for all $x,y \in D$.

In [3] the authors present relations between strongly convex (strongly Jensen convex) and convex (Jensen convex) functions. In particular, they show that each strongly convex function (strongly Jensen convex function) is in the form $g+\|\cdot\|^2$, where g is a convex function (Jensen convex function) iff the space $(X,\|\cdot\|)$ is an inner product space.

Now, if in (1) and (2) we replace $c\|\cdot\|^2$, with a non-negative function F defined on X we get the following inequalities, respectively.

$$f(tx+(1-t)y) \leqslant tf(x)+(1-t)f(y)-t(1-t)F(x-y),$$

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for all $x,y \in D$ and $t \in (0,1)$.

$$f\Big(\frac{x+y}{2}\Big) \leqslant \frac{f(x)+f(y)}{2} - \frac{1}{4}F(x-y),$$

for all $x,y \in D$.

The main goal of this paper is to resolve a problem of whether for such functions a similar result as Nikodem and Páles got in [3] is possible to obtain.

2. Main result

At the beginning we formally introduce two definitions of functions aforementioned in the introduction.

DEFINITION 1. Let X be a real vector space, D be a nonempty convex subset of X and $F : X \to [0,\infty)$ be a given function. A function $f : D \to \mathbb{R}$ will be called F-*strongly convex* if

$$f\big(tx+(1-t)y\big) \leqslant tf(x)+(1-t)f(y)-t(1-t)F(x-y)$$

for all $x,y \in D$ and $t \in (0,1)$.

DEFINITION 2. Let X be a real vector space, D be a nonempty convex subset of X and $F : X \to [0,\infty)$ be a given function. A function $f : D \to \mathbb{R}$ we will call F-*strongly J-convex* if

$$f\Big(\frac{x+y}{2}\Big) \leqslant \frac{f(x)+f(y)}{2} - \frac{1}{4}F(x-y)$$

for all $x,y \in D$.

Notice that in Definition 1 parameter t is arbitrary from the segment $(0,1)$. Thus, function f is F-strongly convex if and only if

$$f\big(tx+(1-t)y\big) \leqslant tf(x)+(1-t)f(y)-t(1-t)F(y-x)$$

for all $x,y \in D$ and $t \in (0,1)$. So, defining the function F^* by setting

$$F^*(x) := max\{F(-x),F(x)\}, \quad x \in X,$$

we have the following observation.

OBSERVATION 1. Let X be a real vector space, D be a nonempty convex subset of X and $F : X \to [0,\infty)$ be a given function. A function $f : D \to \mathbb{R}$ is F-strongly convex (F-strongly J-convex) if and only if a function $f : D \to \mathbb{R}$ is F^*-strongly convex (F^*-strongly J-convex).

2.1. F-strongly J-convexity

In this section we will consider F-strongly J-convex functions and we shall start with three useful lemmas.

LEMMA 1. *Let X be a real vector space, D be a nonempty convex subset of X and $F : X \to [0,\infty)$ be a given quadratic function (i.e. $F(x+y)+F(x-y) = 2F(x)+2F(y)$). A function $f : D \to \mathbb{R}$ is F-strongly J-convex if and only if the function $g = f-F$ is J-convex.*

Proof. F is a quadratic function thus

$$\frac{1}{4}F(x-y) = -F\left(\frac{x+y}{2}\right)+\frac{1}{2}F(x)+\frac{1}{2}F(y).$$

Now, the inequality

$$f\left(\frac{x+y}{2}\right) \leqslant \frac{f(x)+f(y)}{2} - \frac{1}{4}F(x-y)$$

can by written in an equivalent form

$$f\left(\frac{x+y}{2}\right) - F\left(\frac{x+y}{2}\right) \leqslant \frac{f(x)-F(x)+f(y)-F(y)}{2}.$$

Taking $g := f-F$ we get

$$g\left(\frac{x+y}{2}\right) \leqslant \frac{g(x)+g(y)}{2}. \quad \square$$

LEMMA 2. *Let X be a real vector space and $F : X \to [0,\infty)$ be a function that is even. If the function F is F-strongly J-convex, then $F(\frac{1}{2}x) = \frac{1}{4}F(x)$ far all $x \in X$.*

Proof. We are assuming the inequality

$$F\left(\frac{x+y}{2}\right) \leqslant \frac{F(x)+F(y)}{2} - \frac{1}{4}F(x-y), \quad x,y \in X. \tag{3}$$

From the above inequality with $x = y = 0$ and non-negativity of F we get $F(0) = 0$. Now, taking $y = 0$ we have

$$F\left(\frac{x}{2}\right) \leqslant \frac{1}{4}F(x), \quad x \in X.$$

Moreover, putting $y = -x$, substituting x with $\frac{x}{2}$ in (3) and using evenness of F we obtain

$$\frac{1}{4}F(x) \leqslant F\left(\frac{x}{2}\right), \quad x \in X.$$

Thus

$$F\left(\frac{x}{2}\right) = \frac{1}{4}F(x), \quad x \in X. \tag{4}$$

$\square$

LEMMA 3. *Let X be a real vector space and $F : X \to [0,\infty)$ be a function that is even. The function F is F-strongly J-convex if and only if F is a quadratic function.*

Proof. Suppose that F is F-strongly J-convex. From Definition 2 and Lemma 2 we get

$$F(x+y)+F(x-y) \leqslant 2F(x)+2F(y), \quad x,y \in X.$$

Now, putting $x+y=u$ and $x-y=v$ in the above inequality a using once more Lemma 2 we obtain

$$F(u+v)+F(u-v) \geqslant 2F(u)+2F(v), \quad u,v \in X.$$

Thus

$$F(x+y)+F(x-y) = 2F(x)+2F(y), \quad x,y \in X.$$

The reverse implication is obviously true. □

The next result gives the solution of the problem aforementioned in case of F-strongly J-convexity. Moreover, we obtain a characterization of quadratic functions.

THEOREM 1. *Let X be a real vector space, D be a nonempty convex subset of X and $F : X \to [0,\infty)$ be a given function. The following conditions are equivalent:*

1. *For all function $f : D \to \mathbb{R}$, f is F-strongly J-convex if and only if the function $g = f - F^*$ is J-convex;*
2. *The function F^* is F^*-strongly J-convex;*
3. *The function F^* is a quadratic function.*

Proof. Assuming *(1)* and taking $g=0$ we obtain that $F^* = f$. Thus F^* is F-strongly J-convex and from Observation 1, F^* is F^*-strongly J-convex. So, we have *(2)*. Lemma 3 follows the implication *(2)⇒(3)* and from Lemma 1 and Observation 1 we deduce the implication *(3)⇒(1)*. □

It is well known, that each quadratic and continuous function $F : \mathbb{R}^n \to \mathbb{R}$ can be written in the following form $F(x) = xAx^T$, where A is a symmetric matrix of degree n. Therefore, from Theorem 1 we have the following corollary.

COROLLARY 1. For ℓ_p^n spaces where $n \geqslant 2$ we have

$$2^{\frac{2-s}{2}} \|x+y\|_p^s + 2^{\frac{s-2}{2}} \|x-y\|_p^s \leqslant 2^{\frac{s}{2}} \|x\|_p^s + 2^{\frac{s}{2}} \|y\|_p^s, \quad x,y \in \mathbb{R}^n,$$

if and only if $p = s = 2$.

In order to substantiate this corollary let us observe that, multiplying the above inequality by $2^{-\frac{s+2}{2}}$, the function $F(x) := \|x\|_p^s$ must be F-strongly convex and, of course, $F = F^*$. Therefore, from Theorem 1 the function F must be quadratic and consequently we have that

$$F(x) = ax_1^2 + bx_1x_2 + cx_2^2, \tag{5}$$

for $x = (x_1, x_2, 0, \ldots, 0) \in \mathbb{R}^n$. From the definition of F and (5), it follows that $a = c$, because $F(x_1, x_2, 0, \ldots, 0) = F(x_2, x_1, 0, \ldots, 0)$. Now taking $x_1 = 1$, $x_2 = 0$ we conclude that $a = 1$. Hence, for $x_2 = 0$ with arbitrary x_1 we get $s = 2$. Thus

$$\sqrt[p]{|x_1|^p + |x_2|^p} = \sqrt{x_1^2 + bx_1x_2 + x_2^2}.$$

Dividing this equality by $|x_1|$ and tending with $|x_1|$ to infinity we obtain $b = 0$. Thus

$$\sqrt[p]{|x_1|^p + |x_2|^p} = \sqrt{x_1^2 + x_2^2}.$$

Finally, taking in the above equality $x_1 = x_2$ we obtain $p = 2$.

Observe that if we take $n = 1$ in the previous corollary we also get $s = 2$ and, of course, the value of p is unimportant.

At the end of this section, notice that if we additionally assume the continuity of F in Theorem 1, then the function F^* must also be homogeneous of degree 2 (i.e. $F^*(tx) = t^2F^*(x)$ for all $x \in X$ and $t \in \mathbb{R}$) and consequently, using the well known Jordan-von Neumann theorem presented in [2], defines a symmetric bilinear form, thus X is an inner product space.

2.2. Strongly F-convexity

In this section F-strongly convex functions will be considered. We start with three lemmas which are analogous to the lemmas presented in the previous section, respectively.

LEMMA 4. *Let X be a real vector space, D be a nonempty convex subset of X and $F : X \to [0, \infty)$ be a given F-strongly affine function (i.e. we have "$=$" instead of "$\leqslant$" in Definition 1). A function $f : D \to \mathbb{R}$ is F-strongly convex if and only if the function $g = f - F$ is convex.*

Proof. F is a F-strongly affine function thus

$$t(1-t)F(x-y) = -F(tx + (1-t)y) + tF(x) + (1-t)F(y).$$

Now, the inequality

$$f(tx + (1-t)y) \leqslant tf(x) + (1-t)f(y) - t(1-t)F(x-y)$$

can be written in an equivalent form

$$f(tx + (1-t)y) - F(tx + (1-t)y) \leqslant t(f(x) - F(x)) + (1-t)(f(y) - F(y)).$$

Taking $g := f - F$ we get

$$g(tx + (1-t)y) \leqslant tg(x) + (1-t)g(y). \quad \square$$

LEMMA 5. *Let X be a real vector space and $F : X \to [0, \infty)$ be a function that is even. If the function F is F-strongly convex, then F is homogeneous of degree 2.*

Proof. From the assumption the following inequality holds true

$$F(tx+(1-t)y) \leqslant tF(x)+(1-t)F(y)-t(1-t)F(x-y), \tag{6}$$

for all $t \in (0,1)$ and $x,y \in X$. Putting in this inequality $y=0$ and using the fact that $F(0)=0$ we get

$$F(tx) \leqslant tF(x)+t(1-t)F(x)=t^2F(x),$$

thus

$$F(tx) \leqslant t^2F(x), \tag{7}$$

for all $t \in (0,1)$ and $x \in X$. In order to show the reverse inequality, we put $x=(1-t)u$, $y=-tu$ in (6) and using the fact that $F(0)=0$ we have

$$0=F(0) \leqslant tF((1-t)u)+(1-t)F(-tu)-t(1-t)F(u).$$

Now, using the above inequality, (7) and evenness of F we obtain

$$t(1-t)F(u) \leqslant tF((1-t)u)+(1-t)F(tu) \leqslant t(1-t)^2F(u)+(1-t)F(tu).$$

Dividing this inequality by $1-t$ we get

$$tF(u) \leqslant t(1-t)F(u)+F(tu),$$

thus

$$F(tu) \geqslant t^2F(u), \tag{8}$$

for all $t \in (0,1)$ and $u \in X$. From (7) and (8) we conclude that

$$F(tx)=t^2F(x), \tag{9}$$

for all $t \in (0,1)$ and $x \in X$. Moreover, if we substitute in the above equality x with $\frac{x}{t}$ we conclude that (9) holds also for $t>1$. Which together with the evenness of F gives the equality

$$F(tx)=t^2F(x), \tag{10}$$

for all $t \in \mathbb{R}$ and $x \in X$. $\square$

LEMMA 6. *Let X be a real vector space and $F: X \to [0,\infty)$ be a function that is even. The function F is F-strongly convex if and only if F is a F-strongly affine function.*

Proof. Assume that F is F-strongly convex, i.e.

$$F(tx+(1-t)y) \leqslant tF(x)+(1-t)F(y)-t(1-t)F(x-y), \tag{11}$$

for all $t \in (0,1)$ and $x,y \in X$. Now, putting $tx+(1-t)y=u\sqrt{t}$ and $x-y=\frac{v}{\sqrt{t}}$ in (11) and using Lemma 5 we get

$$tF(u) \leqslant F(tu+(1-t)v)+t(1-t)F(u-v)-(1-t)F(v)$$

or equivalently

$$F(tu+(1-t)v) \geqslant tF(u)+(1-t)F(v)-t(1-t)F(u-v),$$

for all $t \in (0,1)$ and $u,v \in X$. This together with (11) implies that

$$F(tu+(1-t)v) = tF(u)+(1-t)F(v)-t(1-t)F(u-v),$$

for all $t \in (0,1)$ and $x,y \in X$, i.e. the function F is F-strongly affine.

The reverse implication is obvious. □

The next result gives the solution of the problem aforementioned in case of F-strongly convexity.

THEOREM 2. *Let X be a real vector space, D be a nonempty convex subset of X and $F : X \to [0,\infty)$ be a given function. The following conditions are equivalent:*

1. *For all function $f : D \to \mathbb{R}$, f is F-strongly convex if and only if the function $g = f - F^*$ is convex;*
2. *The function F^* is F^*-strongly convex;*
3. *The function F^* is F^*-strongly affine (and of course quadratic and homogeneous of degree 2, and X is an inner product space).*

Proof. The implication *(1)*⇒*(2)* we argue as in the proof of Theorem 1. By virtue of Lemma 6 we have the implication *(2)*⇒*(3)*. Finally, using Lemma 4 and Observation 1 we obtain the implication *(3)*⇒*(1)*. □

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