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ON WEIGHTED INTEGRAL AND DISCRETE OPIAL–TYPE INEQUALITIES

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Abstract. In this paper some multidimensional integral and discrete Opial-type inequalities due to Agarwal, Pang and Sheng are considered. Theirs generalizations and extensions using submultiplicative convex functions, appropriate integral representations of functions, appropriate summation representations of discrete functions and inequalities involving means are presented.

1. Introduction

In 1960, Z. Opial [10] proved next integral inequality:

Let $x(t) \in C^1[0,h]$ be such that $x(0) = x(h) = 0$ and $x(t) > 0$ for $t \in (0,h)$. Then

$$\int_0^h \left|x(t)x'(t)\right| dt \leqslant \frac{h}{4}\int_0^h \left(x'(t)\right)^2 dt, \tag{1}$$

where constant $\frac{h}{4}$ is the best possible.

Over the last five decades, an enormous amount of work has been done on this integral inequality, dealing with new proofs, various generalizations, extensions and discrete analogues. Opial's inequality is recognized as fundamental result in the analysis of qualitative properties of solution of differential equations (see [3, 9] and the references cited therein).

The aim of this paper is to generalize and extend some integral and discrete Opial-type inequalities due to Agarwal, Pang and Sheng ([1, 2, 6]). To establish these inequalities, we will use some elementary techniques such as appropriate integral representations of functions, appropriate summation representations of the discrete functions and inequalities involving means. We start each section with inequality involving a submultiplicative convex function. Recall that function $f : [0,\infty) \to [0,\infty)$ is called *submultiplicative* function if it satisfies the inequality

$$f(xy) \leqslant f(x)f(y), \quad \text{for all } x, y \in [0,\infty)$$

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(see for example [8]).

One of such submultiplicative functions, which is also convex and increasing, is $f(x) = x^p \log(e+x)$, where $p \geqslant \frac{1+\sqrt{5}}{2}$. The obtained results will give in a special case improvements of corresponding inequalities in [1, 2, 6], and, at the same time, they will simplify proofs of the corresponding theorems in [4, 5, 7].

For the following inequalities we present obtained generalizations, extensions and improvements: first is a result by Agarwal and Pang from [1], observed in Section 2. Recall, $AC[0,h]$ is the space of all absolutely continuous functions on $[0,h]$. Also, let B denotes the beta function.

THEOREM 1. [1] *Let $\lambda \geqslant 1$ be a given real number and let p be a nonnegative and continuous function on $[0,h]$. Further, let $x \in AC[0,h]$ be such that $x(0) = x(h) = 0$. Then the following inequality holds*

$$\int_0^h p(t)|x(t)|^\lambda dt \leqslant \frac{1}{2}\left(\int_0^h (t(h-t))^{\frac{\lambda-1}{2}} p(t)\,dt\right)\int_0^h |x'(t)|^\lambda dt. \tag{2}$$

For a constant function p, the inequality (2) reduces to

$$\int_0^h |x(t)|^\lambda dt \leqslant \frac{h^\lambda}{2} B\left(\frac{\lambda+1}{2}, \frac{\lambda+1}{2}\right)\int_0^h |x'(t)|^\lambda dt. \tag{3}$$

Next is a multidimensional Poincaré-type inequality by Agarwal and Sheng from [6], observed in Section 3. This inequality involves a special class of continuous functions, a class $G(\Omega)$, whose definition and properties are given at the beginning of Section 3.

THEOREM 2. [6] *Let $\lambda, \mu \geqslant 1$ and let $u \in G(\Omega)$. Then the following inequality holds*

$$\int_\Omega |u(x)|^\lambda dx \leqslant K(\lambda,\mu)\int_\Omega \|\operatorname{grad} u(x)\|_\mu^\lambda dx,$$

where

$$K(\lambda,\mu) = \frac{1}{2m} B\left(\frac{1+\lambda}{2}, \frac{1+\lambda}{2}\right) C\left(\frac{\lambda}{\mu}\right) G_m\left((b-a)^\lambda\right), \tag{4}$$

$$C(\alpha) = \begin{cases} 1, & \alpha \geqslant 1, \\ m^{1-\alpha}, & 0 \leqslant \alpha \leqslant 1. \end{cases} \tag{5}$$

Finally, a discrete inequality, observed in Section 4, is a result by Agarwal and Pang from [2]. A definition of a class $G(\Omega)$ for the discrete case and a definition of forward difference operator Δ_i are given at the beginning of Section 4.

THEOREM 3. [2] *Let $\lambda \geqslant 1$ and let $u \in G(\Omega)$. Then the following inequality holds*

$$\sum_{x=1}^{X-1} |u(x)|^\lambda \leqslant K(\lambda) \sum_{x=0}^{X-1}\left(\sum_{i=1}^m |\Delta_i u(x)|^2\right)^{\frac{\lambda}{2}},$$

where

$$K(\lambda)=\frac{1}{m}C\left(\frac{\lambda}{2}\right)\prod_{i=1}^{m}\left(\sum_{x_i=1}^{X_i-1}\frac{1}{2}\left(x_i(X_i-x_i)\right)^{\frac{\lambda-1}{2}}\right)^{\frac{1}{m}} \tag{6}$$

and C is defined by (5).

2. Integral inequalities in one variable

First we give a generalization of Theorem 1 involving submultiplicative convex functions. In a special case (Corollary 4) this theorem will improve result from Theorem 1.

THEOREM 4. *Let $n\in\mathbb{N}$ and let f_i be increasing, submultiplicative convex functions on $[0,\infty)$, $i=1,\dots,n$. Let p be a nonnegative and integrable function on $[0,h]$. Further, let $x_i\in AC[0,h]$ be such that $x_i(0)=x_i(h)=0$ for $i=1,\dots,n$. Then the following inequality holds*

$$\begin{aligned}&\int_0^h p(t)\prod_{i=1}^{n}f_i\left(|x_i(t)|\right)dt\\ &\leqslant\left(\int_0^h p(t)\prod_{i=1}^{n}\left(\frac{t}{f_i(t)}+\frac{h-t}{f_i(h-t)}\right)^{-1}dt\right)\prod_{i=1}^{n}\int_0^h f_i\left(|x_i'(t)|\right)dt\,.\end{aligned} \tag{7}$$

Proof. As in [1], for each fixed i, $i=1,\dots,n$, from the hypotheses of the theorem we have

$$x_i(t)=\int_0^t x_i'(s)ds\,,$$

$$x_i(t)=-\int_t^h x_i'(s)ds\,.$$

Since f_i is an increasing and convex function, we use Jensen's inequality to obtain

$$f_i\left(|x_i(t)|\right)\leqslant f_i\left(\frac{1}{t}\int_0^t t\,|x_i'(s)|ds\right)\leqslant\frac{1}{t}\int_0^t f_i\left(t\,|x_i'(s)|\right)ds$$

and by submultiplicativity of f_i follows

$$f_i\left(|x_i(t)|\right)\leqslant\frac{1}{t}\int_0^t f_i(t)\,f_i(|x_i'(s)|)\,ds=\frac{f_i(t)}{t}\int_0^t f_i(|x_i'(s)|)\,ds\,. \tag{8}$$

Analogously we obtain

$$\begin{aligned} f_i(|x_i(t)|) &\leqslant f_i\left(\frac{1}{h-t}\int_t^h (h-t)\,|x_i'(s)|ds\right) \\ &\leqslant \frac{1}{h-t}\int_t^h f_i\left((h-t)\,|x_i'(s)|\right)ds \\ &\leqslant \frac{1}{h-t}\int_t^h f_i(h-t)\,f_i(|x_i'(s)|)\,ds \\ &= \frac{f_i(h-t)}{h-t}\int_t^h f_i(|x_i'(s)|)\,ds. \end{aligned} \tag{9}$$

Multiplying (8) by $\frac{t}{f_i(t)}$ and (9) by $\frac{h-t}{f_i(h-t)}$ and adding these inequalities, we find

$$\left(\frac{t}{f_i(t)}+\frac{h-t}{f_i(h-t)}\right)f_i(|x_i(t)|) \leqslant \int_0^h f_i\left(|x_i'(s)|\right)ds,$$

i.e.

$$f_i(|x_i(t)|) \leqslant \left(\frac{t}{f_i(t)}+\frac{h-t}{f_i(h-t)}\right)^{-1}\int_0^h f_i\left(|x_i'(s)|\right)ds. \tag{10}$$

This gives us

$$\prod_{i=1}^n f_i(|x_i(t)|) \leqslant \prod_{i=1}^n\left[\left(\frac{t}{f_i(t)}+\frac{h-t}{f_i(h-t)}\right)^{-1}\int_0^h f_i\left(|x_i'(s)|\right)ds\right]. \tag{11}$$

Now multiplying (11) by p and integrating on $[0,h]$ we obtain

$$\begin{aligned} &\int_0^h p(t)\prod_{i=1}^n f_i(|x_i(t)|)\,dt \\ &\leqslant \int_0^h p(t)\prod_{i=1}^n\left[\left(\frac{t}{f_i(t)}+\frac{h-t}{f_i(h-t)}\right)^{-1}\int_0^h f_i\left(|x_i'(s)|\right)ds\right]dt, \end{aligned}$$

which is the inequality (7). □

REMARK 1. For a special class of a submultiplicative convex functions f_i on $[0,\infty)$ with $f_i(0)=0$ $(i=1,\ldots,n)$, Theorem 4 also holds. Namely, submultiplicativity of a function implies its positivity, and if f_i is a convex, nonnegative function on $[0,\infty)$ with $f_i(0)=0$, then f_i is obviously an increasing function.

COROLLARY 1. *Let $n\in\mathbb{N}$ and let f_i be increasing, submultiplicative convex functions on $[0,\infty)$, $i=1,\ldots,n$. Let p be a nonnegative and integrable function on $[0,h]$. Further, let $x_i\in AC[0,h]$ be such that $x_i(0)=x_i(h)=0$ for $i=1,\ldots,n$. Then the following inequality holds*

$$\begin{aligned} &\int_0^h p(t)\prod_{i=1}^n f_i(|x_i(t)|)\,dt \\ &\leqslant \frac{1}{2^n}\left(\int_0^h p(t)\prod_{i=1}^n\left(\frac{f_i(t)\,f_i(h-t)}{t\,(h-t)}\right)^{\frac{1}{2}}dt\right)\prod_{i=1}^n\int_0^h f_i\left(|x_i'(t)|\right)dt. \end{aligned} \tag{12}$$

Proof. The inequality (12) follows by the harmonic-geometric inequality

$$2\left(\frac{t}{f_i(t)}+\frac{h-t}{f_i(h-t)}\right)^{-1} \leqslant \left(\frac{f_i(t)\,f_i(h-t)}{t\,(h-t)}\right)^{\frac{1}{2}}. \quad \square$$

For $n=1$ we have two following results.

COROLLARY 2. *Let f be an increasing, submultiplicative convex function on $[0,\infty)$ and let p be a nonnegative and integrable function on $[0,h]$. Further, let $x \in AC[0,h]$ be such that $x(0)=x(h)=0$. Then the following inequality holds*

$$\int_0^h p(t)f(|x(t)|)\,dt \leqslant \left(\int_0^h p(t)\left(\frac{t}{f(t)}+\frac{h-t}{f(h-t)}\right)^{-1} dt\right)\int_0^h f\left(|x'(t)|\right)dt. \quad (13)$$

COROLLARY 3. *Let f be an increasing, submultiplicative convex function on $[0,\infty)$ and let p be a nonnegative and integrable function on $[0,h]$. Further, let $x \in AC[0,h]$ be such that $x(0)=x(h)=0$. Then the following inequality holds*

$$\int_0^h p(t)f(|x(t)|)\,dt \leqslant \frac{1}{2}\left(\int_0^h p(t)\left(\frac{f(t)\,f(h-t)}{t\,(h-t)}\right)^{\frac{1}{2}} dt\right)\int_0^h f\left(|x'(t)|\right)dt. \quad (14)$$

Next result was proven by Brnetić and Pečarić in [7]. Here it is merely a consequence, a special case of Corollary 2 (as we can see from its proof). By the harmonic-geometric inequality, it is clear that (15) improves (2).

COROLLARY 4. *Let $\lambda \geqslant 1$ be a given real number and let p be a nonnegative and continuous function on $[0,h]$. Further, let $x \in AC[0,h]$ be such that $x(0)=x(h)=0$. Then the following inequality holds*

$$\int_0^h p(t)|x(t)|^{\lambda}dt \leqslant \left(\int_0^h \left(t^{1-\lambda}+(h-t)^{1-\lambda}\right)^{-1}p(t)\,dt\right)\int_0^h |x'(t)|^{\lambda}dt. \quad (15)$$

Proof. The inequality (15) will follow if we use the function $f(t)=t^{\lambda}$ and apply Corollary 2. $\square$

3. Multidimensional integral inequalities

Let Ω be a bounded domain in $\mathbb{R}^m$ defined by $\Omega=\prod_{j=1}^m[a_j,b_j]$. Let $x=(x_1,\ldots,x_m)$ be a general point in Ω and $dx=dx_1\ldots dx_m$. For any continuous real-valued function u defined on Ω we denote $\int_\Omega u(x)\,dx$ the m-fold integral $\int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m}u(x_1,\ldots,x_m)\,dx_1\ldots dx_m$. Let $D_k u(x_1,\ldots,x_m)=\frac{\partial}{\partial x_k}u(x_1,\ldots,x_m)$ and $D^k u(x_1,\ldots,x_m)=D_1\cdots D_k u(x_1,\ldots,x_m)$, $1\leqslant k\leqslant m$.

We denote by $G(\Omega)$ the class of continuous functions $u:\Omega\to\mathbb{R}$ for which $D^m u(x)$ exists with $u(x)|_{x_j=a_j}=u(x)|_{x_j=b_j}=0$, $1\leqslant j\leqslant m$.

Further, let $u(x;s_j)=u(x_1,\dots,x_{j-1},s_j,x_{j+1},\dots,x_m)$, and

$$\|\operatorname{grad} u(x)\|_\mu=\left(\sum_{j=1}^m\left|\frac{\partial}{\partial x_j}u(x)\right|^\mu\right)^{\frac{1}{\mu}}.$$

Also let $\alpha=(\alpha_1,\dots,\alpha_m)$ and $\alpha^\lambda=(\alpha_1^\lambda,\dots,\alpha_m^\lambda)$, $\lambda\in\mathbb{R}$. In particular, $(b-a)=(b_1-a_1,\dots,b_m-a_m)$ and $(b-a)^\lambda=((b_1-a_1)^\lambda,\dots,(b_m-a_m)^\lambda)$. For the geometric and the harmonic means of $\alpha_1,\dots,\alpha_m$ we will use $G_m(\alpha)$ and $H_m(\alpha)$, respectively. Let $M^{[k]}(\alpha)$ denote the mean of order k of $\alpha_1,\dots,\alpha_m$.

We start with a weighted extension of Theorem 2 involving submultiplicative convex function. Again, in a special case (Corollary 6) this theorem will improve result from Theorem 2.

THEOREM 5. *Let f be an increasing, submultiplicative convex function on $[0,\infty)$. Let p be a nonnegative and integrable function on Ω and $u\in G(\Omega)$. Then the following inequality holds*

$$\int_\Omega p(x)f(|u(x)|)\,dx\leqslant\frac{1}{m}H_m(\alpha)\int_\Omega\left(\sum_{i=1}^m f\left(\left|\frac{\partial}{\partial x_i}u(x)\right|\right)\right)dx, \tag{16}$$

where $\alpha=(\alpha_1,\dots,\alpha_m)$ and

$$\alpha_i=\int_{a_i}^{b_i}\left(\frac{x_i-a_i}{f(x_i-a_i)}+\frac{b_i-x_i}{f(b_i-x_i)}\right)^{-1}p(x)\,dx_i,\quad i=1,\dots,m.$$

Proof. For each fixed i, $i=1,\dots,m$, we have

$$u(x)=\int_{a_i}^{x_i}\frac{\partial}{\partial s_i}u(x;s_i)\,ds_i$$

and

$$u(x)=-\int_{x_i}^{b_i}\frac{\partial}{\partial s_i}u(x;s_i)\,ds_i.$$

First we use Jensen's inequality (since f is an increasing convex function) and then submultiplicativity of f, to obtain

$$\begin{aligned}f(|u(x)|)&\leqslant f\left(\frac{1}{x_i-a_i}\int_{a_i}^{x_i}(x_i-a_i)\left|\frac{\partial}{\partial s_i}u(x;s_i)\right|ds_i\right)\\&\leqslant\frac{1}{x_i-a_i}\int_{a_i}^{x_i}f\left((x_i-a_i)\left|\frac{\partial}{\partial s_i}u(x;s_i)\right|\right)ds_i\\&\leqslant\frac{1}{x_i-a_i}\int_{a_i}^{x_i}f(x_i-a_i)f\left(\left|\frac{\partial}{\partial s_i}u(x;s_i)\right|\right)ds_i\\&=\frac{f(x_i-a_i)}{x_i-a_i}\int_{a_i}^{x_i}f\left(\left|\frac{\partial}{\partial s_i}u(x;s_i)\right|\right)ds_i\end{aligned}\tag{17}$$

and analogously

$$f(|u(x)|) \leqslant \frac{f(b_i - x_i)}{b_i - x_i} \int_{x_i}^{b_i} f\left(\left|\frac{\partial}{\partial s_i} u(x;s_i)\right|\right) ds_i\,, \tag{18}$$

for $i = 1,\dots,m$. Multiplying (17) by $\frac{x_i - a_i}{f(x_i - a_i)}$ and (18) by $\frac{b_i - x_i}{f(b_i - x_i)}$ and adding these inequalities, we find

$$\left(\frac{x_i - a_i}{f(x_i - a_i)} + \frac{b_i - x_i}{f(b_i - x_i)}\right) f(|u(x)|) \;\leqslant\; \int_{a_i}^{b_i} f\left(\left|\frac{\partial}{\partial s_i} u(x;s_i)\right|\right) ds_i\,,$$

i.e.

$$f(|u(x)|) \;\leqslant\; \left(\frac{x_i - a_i}{f(x_i - a_i)} + \frac{b_i - x_i}{f(b_i - x_i)}\right)^{-1} \int_{a_i}^{b_i} f\left(\left|\frac{\partial}{\partial s_i} u(x;s_i)\right|\right) ds_i\,, \tag{19}$$

for $i = 1,\dots,m$. Now multiplying (19) by p and integrating on Ω we obtain

$$\begin{aligned}\int_\Omega p(x)\, f(|u(x)|)\, dx \leqslant \int_{a_i}^{b_i} &\left(\frac{x_i - a_i}{f(x_i - a_i)} + \frac{b_i - x_i}{f(b_i - x_i)}\right)^{-1} p(x)\, dx_i \\ &\times \int_\Omega f\left(\left|\frac{\partial}{\partial x_i} u(x)\right|\right) dx\,, \end{aligned} \tag{20}$$

i.e.

$$\begin{aligned}\left(\int_{a_i}^{b_i} \left(\frac{x_i - a_i}{f(x_i - a_i)} + \frac{b_i - x_i}{f(b_i - x_i)}\right)^{-1} p(x)\, dx_i\right)^{-1} &\int_\Omega p(x)\, f(|u(x)|)\, dx \\ &\leqslant \int_\Omega f\left(\left|\frac{\partial}{\partial x_i} u(x)\right|\right) dx\,, \end{aligned} \tag{21}$$

for $i = 1,\dots,m$. Notice that

$$\alpha_i^{-1} = \left(\int_{a_i}^{b_i} \left(\frac{x_i - a_i}{f(x_i - a_i)} + \frac{b_i - x_i}{f(b_i - x_i)}\right)^{-1} p(x)\, dx_i\right)^{-1}, \quad i = 1,\dots,m.$$

Now, by summing these m inequalities (21), we find

$$\sum_{i=1}^m \alpha_i^{-1} \int_\Omega p(x)\, f(|u(x)|)\, dx \leqslant \sum_{i=1}^m \int_\Omega f\left(\left|\frac{\partial}{\partial x_i} u(x)\right|\right) dx\,,$$

which is the same as the inequality (16). □

COROLLARY 5. *Let f be an increasing, submultiplicative convex function on $[0,\infty)$. Let p be a nonnegative and integrable function on Ω and let $u \in G(\Omega)$. Then the following inequality holds*

$$\int_\Omega p(x)\, f(|u(x)|)\, dx \leqslant \frac{1}{2m} H_m(\beta) \int_\Omega \left(\sum_{i=1}^m f\left(\left|\frac{\partial}{\partial x_i} u(x)\right|\right)\right) dx\,, \tag{22}$$

where $\beta = (\beta_1, \dots, \beta_m)$ *and*

$$\beta_i = \int_{a_i}^{b_i} \left(\frac{f(x_i - a_i)\, f(b_i - x_i)}{(x_i - a_i)\,(b_i - x_i)} \right)^{\frac{1}{2}} p(x)\, dx_i\,, \quad i = 1, \dots, m.$$

Proof. By harmonic-geometric inequality we have

$$2\left(\frac{x_i - a_i}{f(x_i - a_i)} + \frac{b_i - x_i}{f(b_i - x_i)} \right)^{-1} \leqslant \left(\frac{f(x_i - a_i)\, f(b_i - x_i)}{(x_i - a_i)\,(b_i - x_i)} \right)^{\frac{1}{2}}.$$

Applying this and using $H_m(\frac{1}{2}\gamma) = \frac{1}{2} H_m(\gamma)$, the inequality (22) follows. □

Next result was proven by Agarwal, Brnetić and Pečarić in [4]. Here we use Theorem 5 applied on a constant function p and the function $f(t) = t^{\lambda}$ to prove the inequality (23). By the harmonic-geometric inequality, it follows that Corollary 6 improves Theorem 2.

COROLLARY 6. *Let* $\lambda, \mu \geqslant 1$ *and let* $u \in G(\Omega)$. *Then the following inequality holds*

$$\int_{\Omega} |u(x)|^{\lambda}\, dx \leqslant K_1(\lambda, \mu) \int_{\Omega} \|\mathrm{grad}\, u(x)\|_{\mu}^{\lambda}\, dx, \tag{23}$$

where

$$K_1(\lambda, \mu) = \frac{1}{m} I(\lambda) C\left(\frac{\lambda}{\mu}\right) H_m\left((b - a)^{\lambda}\right), \tag{24}$$

$$I(\lambda) = \int_0^1 \left(t^{1-\lambda} + (1 - t)^{1-\lambda} \right)^{-1} dt \tag{25}$$

and C *is defined by* (5).

Proof. We follow steps from the proof of Theorem 5, using the function $f(t) = t^{\lambda}$, up to the inequality (20), which is now equal to

$$\int_{\Omega} |u(x)|^{\lambda}\, dx \leqslant \int_{a_i}^{b_i} \left((x_i - a_i)^{1-\lambda} + (b_i - x_i)^{1-\lambda} \right)^{-1} dx_i \int_{\Omega} \left| \frac{\partial}{\partial x_i} u(x) \right|^{\lambda} dx \tag{26}$$

for $i = 1, \dots, m$. However, since

$$\begin{aligned} \int_{a_i}^{b_i} \left((x_i - a_i)^{1-\lambda} + (b_i - x_i)^{1-\lambda} \right)^{-1} dx_i &= (b_i - a_i)^{\lambda} \int_0^1 \left(t^{1-\lambda} + (1 - t)^{1-\lambda} \right)^{-1} dt \\ &= (b_i - a_i)^{\lambda}\, I(\lambda), \end{aligned}$$

the inequality (26) can be written as

$$\int_{\Omega} |u(x)|^{\lambda}\, dx \leqslant (b_i - a_i)^{\lambda}\, I(\lambda) \int_{\Omega} \left| \frac{\partial}{\partial x_i} u(x) \right|^{\lambda} dx. \tag{27}$$

Multiplying both sides of the inequality (27) by $(b_i - a_i)^{-\lambda}$, $i = 1,\dots,m$, and then summing these inequalities, we obtain

$$\sum_{i=1}^{m}(b_i - a_i)^{-\lambda}\int_{\Omega}|u(x)|^{\lambda}\,dx \leqslant I(\lambda)\int_{\Omega}\left(\sum_{i=1}^{m}\left|\frac{\partial}{\partial x_i}u(x)\right|^{\lambda}\right)dx,$$

i.e.

$$\int_{\Omega}|u(x)|^{\lambda}\,dx \leqslant \frac{1}{m}I(\lambda)H_m\left((b-a)^{\lambda}\right)\int_{\Omega}\left(\sum_{i=1}^{m}\left|\frac{\partial}{\partial x_i}u(x)\right|^{\lambda}\right)dx. \tag{28}$$

Our result now follows from (28) and the elementary inequality

$$\sum_{i=1}^{m}a_i^{\alpha} \leqslant C(\alpha)\left(\sum_{i=1}^{m}a_i\right)^{\alpha}, \quad a_i \geqslant 0. \tag{29}$$

□

4. Multidimensional discrete inequalities

Let $x, X \in \mathbb{N}_0^m$ be such that $x \leqslant X$, i.e., $x_i \leqslant X_i$, $i = 1,\dots,m$. Let $\Omega = [0,X]$, where $[0,X] \subset \mathbb{N}_0^m$. We denote by $G(\Omega)$ the class of functions $u : \Omega \to \mathbb{R}$, which satisfies conditions $u(x)|_{x_i=0} = u(x)|_{x_i=X_i} = 0$, $i = 1,\dots,m$. For u we define forward difference operators Δ_i, $i = 1,\dots,m$, as

$$\Delta_i u(x) \;=\; u(x_1,\dots,x_{i-1},x_i+1,x_{i+1},\dots,x_m) - u(x).$$

As in a previous section, let $u(x;s_i)$ stand for $u(x_1,\dots,x_{i-1},s_i,x_{i+1},\dots,x_m)$, $\alpha = (\alpha_1,\dots,\alpha_m)$ and let $H_m(\alpha)$ denote the harmonic mean of $\alpha_1,\dots,\alpha_m$. Also, let $\sum\limits_{x=1}^{X-1}$ denote $\sum\limits_{j=1}^{m}\sum\limits_{x_j=1}^{X_j-1}$.

First we present a weighted extension of Theorem 3 involving submultiplicative convex functions.

THEOREM 6. *Let $n \in \mathbb{N}$ and let f_j be submultiplicative convex functions on $[0,\infty)$ with $f_j(0) = 0$, $j = 1,\dots,n$. Let p be a nonnegative function on Ω and let $u_j \in G(\Omega)$ for $j = 1,\dots,n$. Then the following inequality holds*

$$\sum_{x=1}^{X-1}p(x)\prod_{j=1}^{n}f_j\left(|u_j(x)|\right) \leqslant \frac{1}{m}H_m(\alpha)\sum_{i=1}^{m}\prod_{j=1}^{n}\sum_{x=0}^{X-1}f_j\left(|\Delta_i u_j(x)|\right), \tag{30}$$

where $\alpha = (\alpha_1,\dots,\alpha_m)$ and

$$\alpha_i = \sum_{x_i=1}^{X_i-1}p(x)\prod_{j=1}^{n}\left(\frac{x_i}{f_j(x_i)} + \frac{X_i - x_i}{f_j(X_i - x_i)}\right)^{-1}, \quad i = 1,\dots,m. \tag{31}$$

Proof. For each fixed i $(i=1,\dots,m)$ and each fixed j $(j=1,\dots,n)$ we have

$$u_j(x)=\sum_{s_i=0}^{x_i-1}\Delta_i u_j(x;s_i), \qquad u_j(x)=-\sum_{s_i=x_i}^{X_i-1}\Delta_i u_j(x;s_i).$$

From the discrete case of Jensen's inequality (since f_j is an increasing convex function) and the submultiplicativity of f_j, we have

$$\begin{aligned} f_j(|u_j(x)|) &\leqslant f_j\left(\frac{1}{x_i}\sum_{s_i=0}^{x_i-1} x_i\left|\Delta_i u_j(x;s_i)\right|\right) \\ &\leqslant \frac{1}{x_i}\sum_{s_i=0}^{x_i-1} f_j\left(x_i\left|\Delta_i u_j(x;s_i)\right|\right) \\ &\leqslant \frac{1}{x_i}\sum_{s_i=0}^{x_i-1} f_j(x_i)\, f_j\left(\left|\Delta_i u_j(x;s_i)\right|\right) \\ &= \frac{f_j(x_i)}{x_i}\sum_{s_i=0}^{x_i-1} f_j\left(\left|\Delta_i u_j(x;s_i)\right|\right) \end{aligned} \tag{32}$$

and analogously

$$f_j(|u_j(x)|) \leqslant \frac{f_j(X_i-x_i)}{X_i-x_i}\sum_{s_i=x_i}^{X_i-1} f_j\left(\left|\Delta_i u_j(x;s_i)\right|\right) \tag{33}$$

for $i=1,\dots,m$. We multiply (32) by $\frac{x_i}{f_j(x_i)}$ and (33) by $\frac{X_i-x_i}{f_j(X_i-x_i)}$. Then we add these resulting inequalities, to obtain

$$\left(\frac{x_i}{f_j(x_i)}+\frac{X_i-x_i}{f_j(X_i-x_i)}\right) f_j(|u_j(x)|) \leqslant \sum_{s_i=0}^{X_i-1} f_j\left(\left|\Delta_i u_j(x;s_i)\right|\right),$$

i.e.

$$f_j(|u_j(x)|) \leqslant \left(\frac{x_i}{f_j(x_i)}+\frac{X_i-x_i}{f_j(X_i-x_i)}\right)^{-1} \sum_{s_i=0}^{X_i-1} f_j\left(\left|\Delta_i u_j(x;s_i)\right|\right) \tag{34}$$

for $i=1,\dots,m$. This gives us

$$\prod_{j=1}^{n} f_j(|u_j(x)|) \leqslant \prod_{j=1}^{n}\left[\left(\frac{x_i}{f_j(x_i)}+\frac{X_i-x_i}{f_j(X_i-x_i)}\right)^{-1} \sum_{s_i=0}^{X_i-1} f_j\left(\left|\Delta_i u_j(x;s_i)\right|\right)\right] \tag{35}$$

for $i=1,\dots,m$. Now multiplying (35) by p we get

$$\begin{aligned} &p(x)\prod_{j=1}^{n} f_j(|u_j(x)|) \\ &\leqslant p(x)\left[\prod_{j=1}^{n}\left(\frac{x_i}{f_j(x_i)}+\frac{X_i-x_i}{f_j(X_i-x_i)}\right)^{-1}\right]\left[\prod_{j=1}^{n}\sum_{s_i=0}^{X_i-1} f_j\left(\left|\Delta_i u_j(x;s_i)\right|\right)\right] \end{aligned}$$

for $i = 1,\dots,m$. Summing from $x = 1$ to $x = X-1$, we get

$$\sum_{x=1}^{X-1} p(x) \prod_{j=1}^{n} f_j(|u_j(x)|) \leqslant \left(\sum_{x_i=1}^{X_i-1} p(x) \prod_{j=1}^{n} \left(\frac{x_i}{f_j(x_i)} + \frac{X_i - x_i}{f_j(X_i - x_i)} \right)^{-1} \right) \prod_{j=1}^{n} \sum_{x=0}^{X-1} f_j\left(\left|\Delta_i u_j(x)\right|\right) \tag{36}$$

for $i = 1,\dots,m$. Multiplying both sides of the inequality (36) by α_i^{-1} and then adding these m inequalities , we obtain

$$\sum_{i=1}^{m} \alpha_i^{-1} \sum_{x=1}^{X-1} p(x) \prod_{j=1}^{n} f_j\left(\left|u_j(x)\right|\right) \leqslant \sum_{i=1}^{m} \prod_{j=1}^{n} \sum_{x=0}^{X-1} f_j\left(\left|\Delta_i u_j(x)\right|\right),$$

which is the same as the inequality (30). □

COROLLARY 7. *Let $n \in \mathbb{N}$ and let f_j be submultiplicative convex functions on $[0,\infty)$ with $f_j(0) = 0$, $j = 1,\dots,n$. Let p be a nonnegative function on Ω and let $u_j \in G(\Omega)$ for $j = 1,\dots,n$. Then the following inequality holds*

$$\sum_{x=1}^{X-1} p(x) \prod_{j=1}^{n} f_j\left(\left|u_j(x)\right|\right) \leqslant \frac{1}{2^n m} H_m(\beta) \sum_{i=1}^{m} \prod_{j=1}^{n} \sum_{x=0}^{X-1} f_j\left(\left|\Delta_i u_j(x)\right|\right), \tag{37}$$

where $\beta = (\beta_1,\dots,\beta_m)$ and

$$\beta_i = \sum_{x_i=1}^{X_i-1} p(x) \prod_{j=1}^{n} \left(\frac{f_j(x_i)\, f_j(X_i - x_i)}{x_i\,(X_i - x_i)} \right)^{\frac{1}{2}}, \quad i = 1,\dots,m. \tag{38}$$

Proof. By harmonic-geometric inequality we have

$$2\left(\frac{x_i}{f_j(x_i)} + \frac{X_i - x_i}{f_j(X_i - x_i)} \right)^{-1} \leqslant \left(\frac{f_j(x_i)\, f_j(X_i - x_i)}{x_i\,(X_i - x_i)} \right)^{\frac{1}{2}}.$$

Applying this and using $H_m(\frac{1}{2^n}\beta) = \frac{1}{2^n} H_m(\beta)$, the inequality (37) follows. □

Next are special results for $n = 1$.

COROLLARY 8. *Let f be a submultiplicative convex function on $[0,\infty)$ with $f(0) = 0$. Let p be a nonnegative function on Ω and $u \in G(\Omega)$. Then the following inequality holds*

$$\sum_{x=1}^{X-1} p(x) f(|u(x)|) \leqslant \frac{1}{m} H_m(\alpha) \sum_{x=0}^{X-1} \sum_{i=1}^{m} f(|\Delta_i u(x)|), \tag{39}$$

where $\alpha = (\alpha_1,\dots,\alpha_m)$ is defined by (31).

COROLLARY 9. *Let f be a submultiplicative convex function on $[0,\infty)$ with $f(0)=0$. Let p be a nonnegative function on Ω and $u\in G(\Omega)$. Then the following inequality holds*

$$\sum_{x=1}^{X-1} p(x)\, f(|u(x)|) \leqslant \frac{1}{2m} H_m(\beta) \sum_{x=0}^{X-1}\sum_{i=1}^{m} f(|\Delta_i u(x)|), \tag{40}$$

where $\beta=(\beta_1,\dots,\beta_m)$ is defined by (38).

An improvement of Theorem 3 is the following result, given also Agarwal, Brnetić and Pečarić in [5]. Here we use Corollary 8 applied on a constant function p and the function $f(t)=t^{\lambda}$ to prove the inequality (41). Thus again, by the harmonic-geometric inequality, it follows that Corollary 10 for $\mu=2$ improves Theorem 3.

COROLLARY 10. *Let $\lambda,\mu\geqslant 1$ and let $u\in G(\Omega)$. Then the following inequality holds*

$$\sum_{x=1}^{X-1} |u(x)|^{\lambda} \leqslant K_1(\lambda,\mu) \sum_{x=0}^{X-1}\left(\sum_{i=1}^{m} |\Delta_i u(x)|^{\mu}\right)^{\frac{\lambda}{\mu}}, \tag{41}$$

where

$$K_1(\lambda,\mu) = \frac{1}{m} C\left(\frac{\lambda}{\mu}\right) H_m(h(x,X,\lambda)), \tag{42}$$

$$h(x,X,\lambda) = (h_1(x,X,\lambda),\dots,h_m(x,X,\lambda)), \tag{43}$$

$$h_i(x,X,\lambda) = \sum_{x_i=1}^{X_i-1}\left(x_i^{1-\lambda}+(X_i-x_i)^{1-\lambda}\right)^{-1}, \quad i=1,\dots,m$$

and C is defined by (5).

Proof. From Corollary 8 using the function $f(t)=t^{\lambda}$ (and a constant function p) we have

$$\sum_{x=1}^{X-1} |u(x)|^{\lambda} \leqslant \frac{1}{m} H_m(h(x,X,\lambda)) \sum_{x=0}^{X-1}\sum_{i=1}^{m} |\Delta_i u(x)|^{\lambda}. \tag{44}$$

The inequality (41) now follows from (44) and the elementary inequality (29). □

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