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POPOVICIU TYPE INEQUALITIES VIA HERMITE'S POLYNOMIAL

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Abstract. We obtain useful identities via Hermite interpolation polynomial, by which the inequality of Popoviciu for convex functions is generalized for higher order convex functions. We investigate the bounds for the identities extracted by the generalization of the Popoviciu inequality using inequalities for the Čebyšev functional. Some results relating to the Grüss and Ostrowski type inequalities are constructed.

1. Introduction

A characterization of convex function established by T. Popoviciu [11] is studied by many people (see [12, 10] and references with in). For recent work, we refer [5, 7, 8, 9]. The following form of Popoviciu's inequality is established by Vasić and Stanković in [12] (see also page 173 [10]):

THEOREM 1. *Let* $z, w \in \mathbb{N}$, $z \geqslant 3$, $2 \leqslant w \leqslant z-1$, $[\alpha, \beta] \subset \mathbb{R}$, $\mathbf{x} = (x_1, \ldots, x_z) \in [\alpha, \beta]^z$, $\mathbf{p} = (p_1, \ldots, p_z)$ *be a positive* z*-tuple such that* $\sum_{u=1}^{z} p_u = 1$. *Also let* $f : [\alpha, \beta] \to \mathbb{R}$ *be a convex function. Then*

$$p_{w,z}(\mathbf{x}, \mathbf{p}; f) \leqslant \frac{z-w}{z-1} p_{1,z}(\mathbf{x}, \mathbf{p}; f) + \frac{w-1}{z-1} p_{z,z}(\mathbf{x}, \mathbf{p}; f), \tag{1}$$

where

$$p_{w,z}(\mathbf{x}, \mathbf{p}; f) = p_{w,z}(\mathbf{x}, \mathbf{p}; f(x)) := \frac{1}{\binom{z-1}{w-1}} \sum_{1 \leqslant u_1 < \ldots < u_l \leqslant z} \left(\sum_{v=1}^{w} p_{u_v} \right) f\left(\frac{\sum_{v=1}^{w} p_{u_v} x_{u_v}}{\sum_{v=1}^{w} p_{u_v}} \right)$$

is the linear functional with respect to f.

By inequality (1), we write

$$\mathbf{P}(\mathbf{x}, \mathbf{p}; f) := \frac{z-w}{z-1} p_{1,z}(\mathbf{x}, \mathbf{p}; f) + \frac{w-1}{z-1} p_{z,z}(\mathbf{x}, \mathbf{p}; f) - p_{w,z}(\mathbf{x}, \mathbf{p}; f); \quad 2 \leqslant w \leqslant z-1. \tag{2}$$

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REMARK 1. It is important to note that under the assumptions of Theorem 1, if the function f is convex then $\mathbf{P}(\mathbf{x},\mathbf{p};f) \geqslant 0$ and $\mathbf{P}(\mathbf{x},\mathbf{p};f) = 0$ if f is an identity or constant function.

The mean value theorems and exponential convexity of the linear functional $\mathbf{P}(\mathbf{x},\mathbf{p};f)$ are given in [7] for positive tuples $\mathbf{p}$. Some special classes of convex functions are considered to construct the exponential convexity of $\mathbf{P}(\mathbf{x},\mathbf{p};f)$ in [7].

Let $-\infty < \alpha < \beta < \infty$ and $\alpha = a_1 < a_2 \ldots < a_r = \beta$, $(r \geqslant 2)$ be the given points. For $\psi \in C^n[\alpha,\beta]$ a unique polynomial $\rho_H(s)$ of degree $(n-1)$ exists satisfying any of the following conditions:
Hermite conditions:

$$\rho_H^{(i)}(a_j) = \psi^{(i)}(a_j); \quad 0 \leqslant i \leqslant k_j, \quad 1 \leqslant j \leqslant r, \quad \sum_{j=1}^{r} k_j + r = n. \tag{H}$$

It is of great interest to note that Hermite conditions include the following particular cases:
Lagrange conditions: ($r = n$, $k_j = 0$ for all j)

$$\rho_L(a_j) = \psi(a_j), \quad 1 \leqslant j \leqslant n,$$

Type $(m, n-m)$ conditions: ($r = 2$, $1 \leqslant m \leqslant n-1$, $k_1 = m-1$, $k_2 = n-m-1$)

$$\rho_{(m,n)}^{(i)}(\alpha) = \psi^{(i)}(\alpha), \quad 0 \leqslant i \leqslant m-1,$$

$$\rho_{(m,n)}^{(i)}(\beta) = \psi^{(i)}(\beta), \quad 0 \leqslant i \leqslant n-m-1,$$

Two-point Taylor conditions: ($n = 2m$, $r = 2$, $k_1 = k_2 = m-1$)

$$\rho_{2T}^{(i)}(\alpha) = \psi^{(i)}(\alpha), \quad \rho_{2T}^{(i)}(\beta) = \psi^{(i)}(\beta), \quad 0 \leqslant i \leqslant m-1.$$

We have the following result from [1].

THEOREM 2. *Let $-\infty < \alpha < \beta < \infty$ and $\alpha \leqslant a_1 < a_2 \ldots < a_r \leqslant \beta$, $(r \geqslant 2)$ be the given points, and $\psi \in C^n([\alpha,\beta])$. Then we have*

$$\psi(t) = \rho_H(t) + R_H(\psi,t) \tag{3}$$

where $\rho_H(t)$ is the Hermite interpolating polynomial, i.e.

$$\rho_H(t) = \sum_{j=1}^{r} \sum_{i=0}^{k_j} H_{ij}(t)\psi^{(i)}(a_j);$$

the H_{ij} are fundamental polynomials of the Hermite basis defined by

$$H_{ij}(t) = \frac{1}{i!} \frac{\omega(t)}{(t-a_j)^{k_j+1-i}} \sum_{k=0}^{k_j-i} \frac{1}{k!} \frac{d^k}{dt^k} \left(\frac{(t-a_j)^{k_j+1}}{\omega(t)} \right) \Bigg|_{t=a_j} (t-a_j)^k, \tag{4}$$

with

$$\omega(t) = \prod_{j=1}^{r} (t - a_j)^{k_j+1},$$

and the remainder is given by

$$R_H(\psi,t) = \int_{\alpha}^{\beta} G_{H,n}(t,s)\psi^{(n)}(s)ds$$

where $G_{H,n}(t,s)$ *is defined by*

$$G_{H,n}(t,s) = \begin{cases} \sum_{j=1}^{l}\sum_{i=0}^{k_j} \frac{(a_j-s)^{n-i-1}}{(n-i-1)!} H_{ij}(t);\ s \leqslant t, \\ -\sum_{j=l+1}^{r}\sum_{i=0}^{k_j} \frac{(a_j-s)^{n-i-1}}{(n-i-1)!} H_{ij}(t);\ s \geqslant t, \end{cases} \tag{5}$$

for all $a_l \leqslant s \leqslant a_{l+1}$*;* $l = 0,\dots,r$ *with* $a_0 = \alpha$ *and* $a_{r+1} = \beta$.

REMARK 2. In particular cases, for Lagrange conditions, from Theorem 2 we have

$$\psi(t) = \rho_L(t) + R_L(\psi,t)$$

where $\rho_L(t)$ is the Lagrange interpolating polynomial i.e,

$$\rho_L(t) = \sum_{j=1}^{n} \prod_{\substack{k=1 \\ k\neq j}}^{n} \left(\frac{t-a_k}{a_j-a_k}\right)\psi(a_j)$$

and the remainder $R_L(\psi,t)$ is given by

$$R_L(\psi,t) = \int_{\alpha}^{\beta} G_L(t,s)\psi^{(n)}(s)ds$$

with

$$G_L(t,s) = \frac{1}{(n-1)!}\begin{cases} \sum_{j=1}^{l} (a_j-s)^{n-1} \prod_{\substack{k=1 \\ k\neq j}}^{n} \left(\frac{t-a_k}{a_j-a_k}\right), & s \leqslant t \\ -\sum_{j=l+1}^{n} (a_j-s)^{n-1} \prod_{\substack{k=1 \\ k\neq j}}^{n} \left(\frac{t-a_k}{a_j-a_k}\right), & s \geqslant t \end{cases} \tag{6}$$

$a_l \leqslant s \leqslant a_{l+1}$, $l = 1,2,\dots,n-1$ with $a_1 = \alpha$ and $a_n = \beta$.

For type $(m,n-m)$ conditions, from Theorem 2 we have

$$\psi(t) = \rho_{(m,n)}(t) + R_{(m,n)}(\psi,t)$$

where $\rho_{(m,n)}(t)$ is $(m,n-m)$ interpolating polynomial, i.e

$$\rho_{(m,n)}(t) = \sum_{i=0}^{m-1} \tau_i(t)\psi^{(i)}(\alpha) + \sum_{i=0}^{n-m-1} \eta_i(t)\psi^{(i)}(\beta),$$

with

$$\tau_i(t) = \frac{1}{i!}(t-\alpha)^i\left(\frac{t-\beta}{\alpha-\beta}\right)^{n-m}\sum_{k=0}^{m-1-i}\binom{n-m+k-1}{k}\left(\frac{t-\alpha}{\beta-\alpha}\right)^k \tag{7}$$

and

$$\eta_i(t) = \frac{1}{i!}(t-\beta)^i\left(\frac{t-\alpha}{\beta-\alpha}\right)^{m}\sum_{k=0}^{n-m-1-i}\binom{m+k-1}{k}\left(\frac{t-\beta}{\alpha-\beta}\right)^k. \tag{8}$$

and the remainder $R_{(m,n)}(\psi,t)$ is given by

$$R_{(m,n)}(\psi,t) = \int_{\alpha}^{\beta} G_{(m,n)}(t,s)\psi^{(n)}(s)ds$$

with

$$G_{(m,n)}(t,s) = \begin{cases} \sum_{j=0}^{m-1}\left[\sum_{p=0}^{m-1-j}\binom{n-m+p-1}{p}\left(\frac{t-\alpha}{\beta-\alpha}\right)^p\right] & \\ \times\frac{(t-\alpha)^j(\alpha-s)^{n-j-1}}{j!(n-j-1)!}\left(\frac{\beta-t}{\beta-\alpha}\right)^{n-m}, & \alpha\leqslant s\leqslant t\leqslant\beta \\ -\sum_{i=0}^{n-m-1}\left[\sum_{q=0}^{n-m-i-1}\binom{m+q-1}{q}\left(\frac{\beta-t}{\beta-\alpha}\right)^q\right] & \\ \times\frac{(t-\beta)^i(\beta-s)^{n-i-1}}{i!(n-i-1)!}\left(\frac{t-\alpha}{\beta-\alpha}\right)^{m}, & \alpha\leqslant t\leqslant s\leqslant\beta. \end{cases} \tag{9}$$

For Type Two-point Taylor conditions, from Theorem 2 we have

$$\psi(t) = \rho_{2T}(t) + R_{2T}(\psi,t)$$

where $\rho_{2T}(t)$ is the two-point Taylor interpolating polynomial i.e,

$$\rho_{2T}(t) = \sum_{i=0}^{m-1}\sum_{k=0}^{m-1-i}\binom{m+k-1}{k}\left[\frac{(t-\alpha)^i}{i!}\left(\frac{t-\beta}{\alpha-\beta}\right)^m\left(\frac{t-\alpha}{\beta-\alpha}\right)^k\psi^{(i)}(\alpha) + \frac{(t-\beta)^i}{i!}\left(\frac{t-\alpha}{\beta-\alpha}\right)^m\left(\frac{t-\beta}{\alpha-\beta}\right)^k\psi^{(i)}(\beta)\right]$$

and the remainder $R_{2T}(\psi,t)$ is given by

$$R_{2T}(\psi,t) = \int_{\alpha}^{\beta} G_{2T}(t,s)\psi^{(n)}(s)ds$$

with

$$G_{2T}(t,s) = \begin{cases} \frac{(-1)^m}{(2m-1)!}p^m(t,s)\sum_{j=0}^{m-1}\binom{m-1+j}{j}(t-s)^{m-1-j}q^j(t,s), & s\leqslant t; \\ \frac{(-1)^m}{(2m-1)!}q^m(t,s)\sum_{j=0}^{m-1}\binom{m-1+j}{j}(s-t)^{m-1-j}p^j(t,s), & s\geqslant t; \end{cases} \tag{10}$$

where $p(t,s) = \frac{(s-\alpha)(\beta-t)}{\beta-\alpha}$, $q(t,s) = p(t,s), \forall t,s\in[\alpha,\beta]$.

The following Lemma describes the positivity of Green's function (5) see (Beesack [2] and Levin [6]).

LEMMA 1. *The Green's function $G_{H,n}(t,s)$ has the following properties:*

(i) $\frac{G_{H,n}(t,s)}{w(t)} > 0,\ a_1 \leqslant t \leqslant a_r,\ a_1 \leqslant s \leqslant a_r;$

(ii) $G_{H,n}(t,s) \leqslant \frac{1}{(n-1)!(\beta-\alpha)}|w(t)|;$

(iii) $\int_\alpha^\beta G_{H,n}(t,s)ds = \frac{w(t)}{n!}.$

The organization of the paper is as follows: In Section 2, we use Hermite interpolating polynomial and the n-convexity of the function ψ (defined in Theorem 2) to establish a generalization of Theorem 1 for real weights. We discuss the results for particular cases namely, Lagrange interpolating polynomial, $(m,n-m)$ interpolating polynomial, two-point Taylor interpolating polynomial. In Section 3, we present some interesting results about upper bounds by employing Čebyšev functional and Grüss-type inequalities, also results relating to the Ostrowski-type inequality.

2. Popoviciu's inequality by Hermite interpolating polynomial

We begin this section with the proof of our main identity related to generalizations of Popoviciu's inequality.

THEOREM 3. (Main) *Let $z,w \in \mathbb{N}$, $z \geqslant 3$, $2 \leqslant w \leqslant z-1$, $[\alpha,\beta] \subset \mathbb{R}$, $\mathbf{x} = (x_1,\dots,x_z) \in [\alpha,\beta]^z$, $\mathbf{p} = (p_1,\dots,p_z)$ be a real z-tuple such that $\sum_{v=1}^{w} p_{u_v} \neq 0$ for any $1 \leqslant u_1 < \dots < u_w \leqslant z$ $\sum_{u=1}^{z} p_u = 1$ and $\frac{\sum\limits_{v=1}^{w} p_{u_v}x_{u_v}}{\sum\limits_{v=1}^{w} p_{u_v}} \in [\alpha,\beta]$ for any $1 \leqslant u_1 < \dots < u_w \leqslant z$. Also let $\alpha = a_1 < a_2 \dots < a_r = \beta$, $(r \geqslant 2)$ be the given points, and $\psi \in C^n([\alpha,\beta])$. Moreover, H_{ij} be the fundamental polynomials of the Hermite basis and $G_{H,n}$ be the green functions as defined by (4) and (5) respectively. Then we have the following identity:*

$$\mathbf{P}(\mathbf{x},\mathbf{p};\psi(x)) = \sum_{j=1}^{r}\sum_{i=0}^{k_j} \psi^{(i)}(a_j)\mathbf{P}(\mathbf{x},\mathbf{p};H_{ij}(x)) + \int_\alpha^\beta \mathbf{P}(\mathbf{x},\mathbf{p};G_{H,n}(x,s))\psi^{(n)}(s)ds. \tag{11}$$

Proof. Using (3) in (2) and following the linearity of $\mathbf{P}(\mathbf{x},\mathbf{p};\cdot)$, we get (11). □

For n-convex functions, we can give the following form of new identity (11).

THEOREM 4. *Let all the assumptions of Theorem 3 be satisfied and ψ be an n-convex function. Then we have the following result:*

If

$$\mathbf{P}(\mathbf{x},\mathbf{p};G_{H,n}(x,s)) \geqslant 0, \quad s \in [\alpha,\beta] \tag{12}$$

then

$$\mathbf{P}(\mathbf{x},\mathbf{p};\psi(x)) \geqslant \sum_{j=1}^{r}\sum_{i=0}^{k_j} \psi^{(i)}(a_j)\mathbf{P}(\mathbf{x},\mathbf{p};H_{ij}(x)). \tag{13}$$

Proof. Since the function ψ is n-convex, therefore without loss of generality we can assume that ψ is n-times differentiable and $\psi^{(n)}(x) \geqslant 0$ for all $x \in [\alpha,\beta]$ (see [10], p. 16). Hence we can apply Theorem 3 to obtain (13). □

REMARK 3. The inequality (13) hold in reverse directions if the inequality in (12) is reversed.

By using Lagrange conditions we can give the following result.

COROLLARY 1. *Let all the assumptions of Theorem 3 be satisfied and G_L be the green function as defined in (6). Also let ψ be n-convex function, then we have the following result:*

If

$$\mathbf{P}(\mathbf{x},\mathbf{p};G_L(x,s)) \geqslant 0, \quad s \in [\alpha,\beta]$$

then

$$\mathbf{P}(\mathbf{x},\mathbf{p};\psi(x)) \geqslant \sum_{j=1}^{n} \psi(a_j)\mathbf{P}\Big(\mathbf{x},\mathbf{p};\prod_{\substack{k=1\\k\neq j}}^{n}\Big(\frac{x-a_k}{a_j-a_k}\Big)\Big). \tag{14}$$

By using type $(m,n-m)$ conditions we can give the following result.

COROLLARY 2. *Let all the assumptions of Theorem 3 be satisfied and $G_{(m,n)}$ be the Green function as defined by (9) and τ_i,η_i be as defined in (7) and (8) respectively. Also let ψ be n-convex function, then we have the following result:*

If

$$\mathbf{P}(\mathbf{x},\mathbf{p};G_{(m,n)}(x,s)) \geqslant 0, \quad s \in [\alpha,\beta]$$

then

$$\mathbf{P}(\mathbf{x},\mathbf{p};\psi(x)) \geqslant \sum_{i=0}^{m-1} \psi^{(i)}(\alpha)\mathbf{P}(\mathbf{x},\mathbf{p};\tau_i(x)) + \sum_{i=0}^{n-m-1} \psi^{(i)}(\beta)\mathbf{P}(\mathbf{x},\mathbf{p};\eta_i(x)).$$

By using Two-point Taylor conditions we can give the following result.

COROLLARY 3. *Let all the assumptions of Theorem 3 be satisfied and G_{2T} be the green function as defined by (10). Also let ψ be n-convex function, then we have the following result:*

If

$$\mathbf{P}(\mathbf{x},\mathbf{p};G_{2T}(x,s)) \geqslant 0, \quad s \in [\alpha,\beta]$$

then

$$\mathbf{P}(\mathbf{x},\mathbf{p};\psi(x)) \geqslant \sum_{i=0}^{m-1}\sum_{k=0}^{m-1-i}\binom{m+k-1}{k}\Big[\psi^{(i)}(\alpha)\mathbf{P}\Big(\mathbf{x},\mathbf{p};\frac{(x-\alpha)^i}{i!}\Big(\frac{x-\beta}{\alpha-\beta}\Big)^m\Big(\frac{x-\alpha}{\beta-\alpha}\Big)^k\Big)$$
$$+\psi^{(i)}(\beta)\mathbf{P}\Big(\mathbf{x},\mathbf{p};\frac{(x-\beta)^i}{i!}\Big(\frac{x-\alpha}{\beta-\alpha}\Big)^m\Big(\frac{x-\beta}{\alpha-\beta}\Big)^k\Big)\Big].$$

Now we obtain a generalization of Popoviciu's type inequality for z-tuples.

THEOREM 5. *Let in addition to the assumptions of Theorem 3, $\mathbf{p}=(p_1,\ldots,p_z)$ be a positive z-tuple such that $\sum_{u=1}^{z}p_u=1$, and $\psi:[\alpha,\beta]\to\mathbb{R}$ be an n-convex function. Assume further that the inequality (13) be satisfied and the function*

$$F(x)=\sum_{j=1}^{r}\sum_{i=0}^{k_j}\psi^{(i)}(a_j)H_{ij}(x) \tag{15}$$

is convex. Then we have

$$\mathbf{P}(\mathbf{x},\mathbf{p};\psi(x))\geqslant 0. \tag{16}$$

Proof. $\mathbf{P}$ is a linear functional, so we can rewrite the R.H.S. of (13) in the form $\mathbf{P}(x,p;F(x))$ where F is defined in (15) and will be obtained after reorganization of this side. Since F is assumed to be convex, therefore using the given conditions and by following Remark 1, the non negativity of the R.H.S. of (13) is immediate and we have (16) for positive z-tuples. □

3. Bounds for identities related to generalization of Popoviciu's inequality

In this section we present some interesting results by using Čebyšev functional and Grüss type inequalities. For two Lebesgue integrable functions $f,h:[\alpha,\beta]\to\mathbb{R}$, we consider the Čebyšev functional

$$\Delta(f,h)=\frac{1}{\beta-\alpha}\int_\alpha^\beta f(t)h(t)dt-\frac{1}{\beta-\alpha}\int_\alpha^\beta f(t)dt.\frac{1}{\beta-\alpha}\int_\alpha^\beta h(t)dt.$$

The following Grüss type inequalities are given in [4].

THEOREM 6. *Let $f:[\alpha,\beta]\to\mathbb{R}$ be a Lebesgue integrable function and $h:[\alpha,\beta]\to\mathbb{R}$ be an absolutely continuous function with $(.-\alpha)(\beta-.)[h']^2\in L[\alpha,\beta]$. Then we have the inequality*

$$|\Delta(f,h)|\leqslant\frac{1}{\sqrt{2}}[\Delta(f,f)]^{\frac{1}{2}}\frac{1}{\sqrt{\beta-\alpha}}\left(\int_\alpha^\beta(x-\alpha)(\beta-x)[h'(x)]^2dx\right)^{\frac{1}{2}}. \tag{17}$$

The constant $\frac{1}{\sqrt{2}}$ in (17) is the best possible.

THEOREM 7. *Assume that* $h:[\alpha,\beta]\to\mathbb{R}$ *is monotonic nondecreasing on* $[\alpha,\beta]$ *and* $f:[\alpha,\beta]\to\mathbb{R}$ *be an absolutely continuous with* $f'\in L_\infty[\alpha,\beta]$. *Then we have the inequality*

$$|\Delta(f,h)| \leqslant \frac{1}{2(\beta-\alpha)}||f'||_\infty \int_\alpha^\beta (x-\alpha)(\beta-x)dh(x). \tag{18}$$

The constant $\frac{1}{2}$ *in* (18) *is the best possible.*

In the sequel, we consider above theorems to derive generalizations of the results proved in the previous section. In order to avoid many notions let us denote

$$\tilde{\mathfrak{H}}(s) = \mathbf{P}(\mathbf{x},\mathbf{p};G_H(x,s)), \quad s\in[\alpha,\beta]. \tag{19}$$

THEOREM 8. *Let all the assumptions of Theorem 3 be valid with* $-\infty<\alpha<\beta<\infty$ *and* $\alpha=a_1<a_2\ldots<a_r=\beta$, $(r\geqslant 2)$ *be the given points. Moreover,* $\psi\in C^n([\alpha,\beta])$ *such that* $\psi^{(n)}$ *is absolutely continuous with* $(.-\alpha)(\beta-.)[\psi^{(n+1)}]^2\in L[\alpha,\beta]$. *Also let* H_{ij} *be the fundamental polynomials of the Hermite basis and the functions* G_H *and* $\tilde{\mathfrak{H}}$ *be defined by* (5) *and* (19) *respectively. Then*

$$\begin{aligned}\mathbf{P}(\mathbf{x},\mathbf{p};\psi(x)) &= \sum_{j=1}^{r}\sum_{i=0}^{k_j}\psi^{(i)}(a_j)\mathbf{P}(\mathbf{x},\mathbf{p};H_{ij}(x))\\ &\quad+\frac{\psi^{(n-1)}(\beta)-\psi^{(n-1)}(\alpha)}{(\beta-\alpha)}\int_\alpha^\beta \tilde{\mathfrak{H}}(s)ds+\tilde{\mathfrak{K}}_n(\alpha,\beta;\psi).\end{aligned} \tag{20}$$

where the remainder $\tilde{\mathfrak{K}}_n(\alpha,\beta;\psi)$ *satisfy the bound*

$$|\tilde{\mathfrak{K}}_n(\alpha,\beta;\psi)| \leqslant [\Delta(\tilde{\mathfrak{H}},\tilde{\mathfrak{H}})]^{\frac{1}{2}}\sqrt{\frac{\beta-\alpha}{2}}\left|\int_\alpha^\beta (s-\alpha)(\beta-s)[\psi^{(n+1)}(s)]^2ds\right|^{\frac{1}{2}}.$$

Proof. The proof is similar to the Theorem 9 in [3]. □

The following Grüss type inequalities can be obtained by using Theorem 7.

THEOREM 9. *Let all the assumptions of Theorem 3 be valid with* $-\infty<\alpha<\beta<\infty$ *and* $\alpha=a_1<a_2\ldots<a_r=\beta$, $(r\geqslant 2)$ *be the given points. Moreover,* $\psi\in C^n([\alpha,\beta])$ *such that* $\psi^{(n)}$ *is absolutely continuous and let* $\psi^{(n+1)}\geqslant 0$ *on* $[\alpha,\beta]$ *with* $\tilde{\mathfrak{H}}$ *defined in* (19). *Then the representation* (20) *and the remainder* $\tilde{\mathfrak{K}}_n(\alpha,\beta;\psi)$ *satisfies the estimation*

$$|\tilde{\mathfrak{K}}_n(\alpha,\beta;\psi)| \leqslant (\beta-\alpha)||\tilde{\mathfrak{H}}'||_\infty\left[\frac{\psi^{(n-1)}(\beta)+\psi^{(n-1)}(\alpha)}{2}-\frac{\psi^{(n-2)}(\beta)-\psi^{(n-2)}(\alpha)}{(\beta-\alpha)}\right].$$

Proof. The proof is similar to the Theorem [10] in [3]. □

Now we intend to give the Ostrowski type inequalities related to generalizations of Popoviciu's inequality.

THEOREM 10. *Suppose all the assumptions of Theorem 3 be satisfied. Moreover, assume* (p,q) *is a pair of conjugate exponents, that is* $p,q \in [1,\infty]$ *such that* $1/p+1/q=1$. *Let* $|\psi^{(n)}|^p : [\alpha,\beta] \to \mathbb{R}$ *be a R-integrable function for some* $n \geqslant 2$. *Then, we have*

$$\left| \mathbf{P}(\mathbf{x},\mathbf{p};\psi(x)) - \sum_{j=1}^{r}\sum_{i=0}^{k_j} \psi^{(i)}(a_j)\mathbf{P}(\mathbf{x},\mathbf{p};H_{ij}(x)) \right| \leqslant ||\psi^{(n)}||_p \left(\int_{\alpha}^{\beta} \left| \mathbf{P}(\mathbf{x},\mathbf{p};G_H(x,s)) \right|^q ds \right)^{1/q}. \tag{21}$$

The constant on the R.H.S. of (21) *is sharp for* $1 < p \leqslant \infty$ *and the best possible for* $p=1$, *respectively.*

Proof. The proof is similar to the Theorem 11 in [3]. □

REMARK 4. We can give all the above results of this sections for the Lagrange conditions, Type $(m,n-m)$ conditions, Two-point Taylor conditions.

REMARK 5. Analogous to Section 4 and Section 5 of [3], the n-exponential convexity, mean value theorems and related monotonic Cauchy means (along with examples) can be constructed for the functional defined as the difference between the R.H.S and the L.H.S of the generalized inequality (13).

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