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ON CONVERGENCE PROPERTIES OF GAMMA–STANCU OPERATORS BASED ON q–INTEGERS

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Abstract. In this paper we introduce Stancu type generalization of Gamma operators based on the concept of q-integers. We first establish local approximation theorems for these operators. Next, we investigate the weighted approximation properties and give an estimate for the rate of convergence using classical modulus of continuity. Lastly, we obtain a Voronovskaya type theorem.

1. Introduction

Let f be a function defined on $[0,\infty)$ and satisfies the growth condition:

$$f(t) \leqslant Me^{\beta t} \qquad (M \geqslant 0;\ \beta \geqslant 0;\ t \to \infty).$$

In 2005, Zeng [20] defined the following Gamma operators

$$G_n(f;x) = \frac{1}{x^n\Gamma(n)} \int_0^\infty f\left(\frac{t}{n}\right) t^{n-1} e^{-\frac{t}{x}} dt, \quad x > 0, \tag{1}$$

for functions satisfying exponential growth condition. He studied the approximation properties of these operators to the locally bounded functions and the absolutely continuous functions. One of the important generalizations of the Gamma operators is due to Mazhar [13], namely

$$F_n(f;x) := \int_0^\infty \int_0^\infty g_n(x,u) g_{n-1}(u,t) f(t) du dt = \frac{(2n)!x^{n+1}}{n!(n-1)!} \int_0^\infty \frac{t^{n-1}}{(x+t)^{2n+1}} f(t) dt,$$

where $g_n(x,u) = \frac{x^{n+1}}{n!} e^{-xu} u^n$, $n > 1$, $x > 0$. In [8] Karslı considered the following Gamma type linear and positive operators

$$L_n(f;x) := \int_0^\infty \int_0^\infty g_{n+2}(x,u) g_n(u,t) f(t) du dt = \frac{(2n+3)!x^{n+3}}{n!(n+2)!} \int_0^\infty \frac{t^n}{(x+t)^{2n+4}} f(t) dt, \tag{2}$$

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for $x > 0$, and obtained some approximation results. Karslı, Gupta and İzgi [9] gave an estimate for the rate of convergence of these operators on a Lebesgue point of a function f of bounded variation defined on the interval $(0,\infty)$. In [10], Karslı and Özarslan also gave some local and global approximation results for the same operators.

In the last two decades q-calculus are intensively used in the area of approximation theory. Pioneer work is due to Lupaş [12] and Phillips [14], as they introduce the q-analogue of well-known Bernstein polynomials. After q-Bernstein operators, several operators' q-generalizations are defined and approximation properties are investigated. We now recall some concepts from q-calculus. Details can be found in [7].

For any real number $q > 0$, the q-integer and the q-factorial of a nonnegative integer k are defined as

$$[k]_q := [k] = \begin{cases} \dfrac{1-q^k}{1-q}, & q \neq 1 \\ k, & q = 1. \end{cases}$$

$$[k]_q! := [k]! = \begin{cases} [k]\,[k-1]\ldots[1], & k = 1,2,\ldots \\ 1, & k = 0 \end{cases}$$

respectively. For the integers n and k, the q-binomial coefficients are also defined as

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]!\,[n-k]!} \qquad (n \geqslant k \geqslant 0).$$

The q-integral and the q-improper integral are defined as

$$\int_0^a f(x)d_qx = (1-q)\sum_{j=0}^{\infty} aq^j f(aq^j).$$

$$\int_0^{\infty/A} f(x)\,d_qx = (1-q)\sum_{n=-\infty}^{\infty} f\left(\frac{q^n}{A}\right)\frac{q^n}{A}, \quad A > 0, \tag{3}$$

respectively, provided the sums converge absolutely. The classical exponential function e^x has the following two q-analogues:

$$e_q(x) = \sum_{k=0}^{\infty} \frac{x^k}{[k]!} = \frac{1}{(1-(1-q)x)_q^{\infty}}$$

$$E_q(x) = \sum_{k=0}^{\infty} q^{k(k-1)/2}\frac{x^k}{[k]!} = (1+(1-q)x)_q^{\infty}, \tag{4}$$

where $(1+x)_q^{\infty} = \prod_{j=0}^{\infty}(1+q^jx)$. The q-Gamma function was introduced by Thomae [18] and later by Jackson [6] as the infinite product

$$\Gamma_q(t) = (1-q)^{1-t}\prod_{n=0}^{\infty}\frac{1-q^{n+1}}{1-q^{n+t}}.$$

The integral representation of q-Gamma function [15] is given by

$$\Gamma_q(t) = \int_0^{\frac{1}{1-q}} x^{t-1} E_q(-qx) d_q x \tag{5}$$

which satisfies the functional equations

$$\begin{aligned} \Gamma_q(t+1) &= [t]\Gamma_q(t), \quad t > 0, \quad \Gamma_q(1) = 1, \\ \Gamma_q(t) &= [t-1]!, \quad t > 0. \end{aligned} \tag{6}$$

Note that (5) can also be rewritten via an improper integral as

$$\Gamma_q(t) = \int_0^{\infty/1-q} x^{t-1} E_q(-qx) d_q x \tag{7}$$

since $E_q\left(-\frac{q^n}{1-q}\right) = 0$ for $n \leqslant 0$. For more detailed information, see [1] and [16].

As the q-generalizations of linear positive operators have been introduced by several authors, similar studies are also valid for the Gamma type operators. In [4], Cai introduced a q-analogue of the Gamma operators defined by (1) as

$$G_{n,q}(f;x) = \frac{1}{x^n \Gamma_q(n)} \int_0^{\infty/A} f\left(\frac{t}{[n]}\right) t^{n-1} E_q\left(-\frac{qt}{x}\right) d_q t \tag{8}$$

and investigated the approximation properties of these operators. In [3], Cai and Zeng introduced a q-generalization of gamma type operators given by (2) using the concept of q-integral. They estimated the rate of convergence and examined the weighted approximation properties. Later then Zhao et. al. [21] proposed the Stancu type generalization of the same q-Gamma operators and studied similar concepts. Stancu [17] was the first to modify Bernstein operators as

$$P_n^{\alpha,\beta}(f;x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k+\alpha}{n+\beta}\right)$$

for $x \in [0,1]$ and α, β are any numbers satisfying $0 \leqslant \alpha \leqslant \beta$. For more studies on q-Gamma type operators see also [11] and references therein. Another operator that is related to our study is the Post-Widder operators. Ünal et. al. [19] studied the statistical approximation properties of real and complex Post-Widder operators based on the q-integers. Recently Aydın et. al. [2] also introduced a generalization of q-Post-Widder operators and studied approximation properties.

In the present paper, we introduce the Stancu type modification of the q-Gamma operators. We study the approximation theorems for these operators. Local approximation theorems, weighted approximation and rate of convergence results are investigated. Voronovskaya type theorem is also obtained in the last section.

2. Construction of the operators

By $C_B(0,\infty)$, we denote the space of real valued functions defined on $(0,\infty)$ which are bounded and continuous with the norm $\|f\| = \sup_{x>0} |f(x)|$.

We define the Stancu type generalization of the q-Gamma operators for $n \in \mathbb{N}$, $0 < q < 1$ as

$$G_{n,q}^{\alpha,\beta}(f;x) = \frac{1}{x^n \Gamma_q(n)} \int_0^{\infty/\frac{1-q}{x}} f\left(\frac{t+\alpha}{[n]+\beta}\right) t^{n-1} E_q\left(-\frac{qt}{x}\right) d_q t, \qquad x > 0. \tag{9}$$

where α and β are two numbers satisfying $0 \leqslant \alpha \leqslant \beta$. It is easy to check that (9) is linear and positive. By putting $\alpha = \beta = 0$, the operators reduces to the Gamma operators defined by (8) with $A = \frac{1-q}{x}$.

We first give the following Lemma in order to investigate the approximation theorems in the proceeding sections.

REMARK 1. Note that we take $\frac{1-q}{x}$ instead of A in the definition of the improper integral, so that we are able to get the function $\Gamma_q(n)$ after making the change of variable when finding the test functions. We have to be careful when writing the upper limit of the integral after making the change of variable as it differs from the ordinary calculus.

LEMMA 1. *For $q \in (0,1)$, $x \in (0,\infty)$, we have the following test functions for the operator defined in (9).*

$$\begin{aligned}
G_{n,q}^{\alpha,\beta}(1;x) &= 1 \\
G_{n,q}^{\alpha,\beta}(t;x) &= \frac{[n]}{[n]+\beta}x + \frac{\alpha}{[n]+\beta} \\
G_{n,q}^{\alpha,\beta}(t^2;x) &= \frac{[n]^2}{([n]+\beta)^2}\left(1+\frac{q^n}{[n]}\right)x^2 + \frac{2[n]\alpha}{([n]+\beta)^2}x + \frac{\alpha^2}{([n]+\beta)^2} \\
G_{n,q}^{\alpha,\beta}(t^3;x) &= \frac{[n]^3}{([n]+\beta)^3}\left(1+\frac{q^n(2+q)}{[n]}+\frac{[2]q^{2n}}{[n]^2}\right)x^3 \\
&\quad + \frac{3[n]^2\alpha}{([n]+\beta)^3}\left(1+\frac{q^n}{[n]}\right)x^2 + \frac{3[n]\alpha^2}{([n]+\beta)^3}x + \frac{\alpha^3}{([n]+\beta)^3} \\
G_{n,q}^{\alpha,\beta}(t^4;x) &= \frac{[n]^4}{([n]+\beta)^4}\left(1+\frac{(1+[2]+[3])q^n}{[n]}+\frac{([2]+[3]+[2][3])q^{2n}}{[n]^2}+\frac{[2][3]q^{3n}}{[n]^3}\right)x^4 \\
&\quad + \frac{4[n]^3\alpha}{([n]+\beta)^4}\left(1+\frac{q^n(2+q)}{[n]}+\frac{[2]q^{2n}}{[n]^2}\right)x^3 \\
&\quad + \frac{6[n]^2\alpha^2}{([n]+\beta)^4}\left(1+\frac{q^n}{[n]}\right)x^2 + \frac{4[n]\alpha^3}{([n]+\beta)^4}x + \frac{\alpha^4}{([n]+\beta)^4}.
\end{aligned}$$

Proof. We prove the first equality by making the change of variable $t = ux$.

$$\begin{aligned} G_{n,q}^{\alpha,\beta}(1;x) &= \frac{1}{x^n\Gamma_q(n)}\int_0^{\infty/\frac{1-q}{x}} t^{n-1}E_q\left(-\frac{qt}{x}\right)d_qt \\ &= \frac{1}{x^n\Gamma_q(n)}\int_0^{\infty/1-q} u^{n-1}x^{n-1}E_q(-qu)\,xd_qu \\ &= 1. \end{aligned}$$

For $\alpha = 0,\ \beta \neq 0,$ and $k = 0,1,2,\ldots,$ using the identity (6), we can write

$$\begin{aligned} G_{n,q}^{0,\beta}\left(t^k;x\right) &= \frac{1}{x^n\Gamma_q(n)}\int_0^{\infty/\frac{1-q}{x}}\left(\frac{t}{[n]+\beta}\right)^k t^{n-1}E_q\left(-\frac{qt}{x}\right)d_qt \\ &= \frac{1}{x^n\Gamma_q(n)}\int_0^{\infty/1-q}\frac{u^kx^k}{([n]+\beta)^k}u^{n-1}x^{n-1}E_q(-qu)\,xd_qu \\ &= \frac{x^k\Gamma_q(n+k)}{([n]+\beta)^k\Gamma_q(n)} = \frac{[n+k-1]!}{[n-1]!}\frac{x^k}{([n]+\beta)^k}. \end{aligned}$$

Now in the light of the above equality, we can write $G_{n,q}^{\alpha,\beta}\left(t^k;x\right)$ in terms of $G_{n,q}^{0,\beta}\left(t^k;x\right)$ as,

$$\begin{aligned} G_{n,q}^{\alpha,\beta}\left(t^k;x\right) &= \frac{1}{x^n\Gamma_q(n)}\int_0^{\infty/\frac{1-q}{x}}\left(\frac{t+\alpha}{[n]+\beta}\right)^k t^{n-1}E_q\left(-\frac{qt}{x}\right)d_qt \\ &= \sum_{j=0}^{k}\binom{k}{j}\left(\frac{\alpha}{[n]+\beta}\right)^j\frac{1}{x^n\Gamma_q(n)}\int_0^{\infty/\frac{1-q}{x}}\frac{t^{k-j}}{([n]+\beta)^{k-j}}t^{n-1}E_q\left(\frac{-qt}{x}\right)d_qt \\ &= \sum_{j=0}^{k}\binom{k}{j}\left(\frac{\alpha}{[n]+\beta}\right)^j G_{n,q}^{0,\beta}\left(t^{k-j};x\right). \end{aligned}$$

Using the identity $[n+k] = [n] + q^n[k]$, for $k \geqslant 0$, one can obtain the desired equalities after simple calculations. $\square$

COROLLARY 1. *For every* $q \in (0,1)$, $x \in (0,\infty)$, *we have*

$$G_{n,q}^{\alpha,\beta}(t-x;x) = \left(\frac{[n]}{[n]+\beta}-1\right)x + \frac{\alpha}{[n]+\beta}$$

$$\begin{aligned} G_{n,q}^{\alpha,\beta}\left((t-x)^2;x\right) &= \left(\frac{[n]^2}{([n]+\beta)^2}\left(1+\frac{q^n}{[n]}\right) - \frac{2[n]}{[n]+\beta}+1\right)x^2 \\ &\quad +2\alpha\left(\frac{[n]}{([n]+\beta)^2}-\frac{1}{[n]+\beta}\right)x + \frac{\alpha^2}{([n]+\beta)^2} \end{aligned} \tag{10}$$

$$
\begin{aligned}
G_{n,q}^{\alpha,\beta}\left((t-x)^4;x\right) = & \left\{ \frac{[n]^4}{([n]+\beta)^4} + \frac{[n]^3}{([n]+\beta)^4}(1+[2]+[3])q^n \right. \\
& + \frac{[n]^2([2]+[3]+[2][3])q^{2n}}{([n]+\beta)^4} \\
& + \frac{[n][2][3]q^{3n}}{([n]+\beta)^4} - \frac{4[n]^3}{([n]+\beta)^3}\left(1+\frac{q^n(2+q)}{[n]}+\frac{[2]q^{2n}}{[n]^2}\right) \\
& \left. +6\frac{[n]^2}{([n]+\beta)^2}\left(1+\frac{q^n}{[n]}\right) - 4\frac{[n]}{[n]+\beta}+1 \right\} x^4 \\
& + \left\{ \frac{4[n]^3\alpha}{([n]+\beta)^4}\left(1+\frac{q^n(2+q)}{[n]}+\frac{[2]q^{2n}}{[n]^2}\right) \right. \\
& \left. -\frac{12[n]^2\alpha}{([n]+\beta)^3}\left(1+\frac{q^n}{[n]}\right)+\frac{12[n]\alpha}{([n]+\beta)^2}-\frac{4\alpha}{[n]+\beta} \right\} x^3 \\
& + \left\{ \frac{6[n]^2\alpha^2}{([n]+\beta)^4}\left(1+\frac{q^n}{[n]}\right) - \frac{12[n]\alpha^2}{([n]+\beta)^3} + \frac{6\alpha^2}{([n]+\beta)^2} \right\} x^2 \\
& + \left\{ \frac{4[n]\alpha^3}{([n]+\beta)^4} - \frac{4\alpha^3}{([n]+\beta)^3} \right\} x + \frac{\alpha^4}{([n]+\beta)^4}
\end{aligned} \tag{11}
$$

THEOREM 1. *Let $q=(q_n)$ be a sequence satisfing*

$$
(q_n)\in(0,1), \quad \lim_{n\to\infty} q_n = 1 \quad \text{and} \quad \lim_{n\to\infty} q_n^n = a, \quad a\neq 0. \tag{12}
$$

For each $f\in C_B(0,\infty)$, the sequence $\left(G_{n,q_n}^{\alpha,\beta}(f;.)\right)$ converges uniformly to the function f on every compact subset of $(0,\infty)$.

Proof. Let $\lim_{n\to\infty} q_n = 1$. Since $\lim_{n\to\infty} q_n^n = a$, $a\neq 0$, we see that

$$
\begin{aligned}
G_{n,q_n}^{\alpha,\beta}(1;.) &\rightrightarrows 1, \\
G_{n,q_n}^{\alpha,\beta}(t;.) &\rightrightarrows x, \\
G_{n,q_n}^{\alpha,\beta}(t^2;.) &\rightrightarrows x^2
\end{aligned}
$$

from Lemma 1. Therefore from the well known Korovkin's Theorem we get the desired result. □

3. Local approximation

Recall that the first-order and second-order modulus of continuities of the function $f \in C_B(0,\infty)$ is defined by for $\delta > 0$

$$\begin{aligned} w(f;\delta) &= \sup\{|f(x+h)-f(x)| : x>0,\ 0 \leqslant h \leqslant \delta\}, \\ w_2(f;\delta) &= \sup\{|f(x+2h)-2f(x+h)+f(x)| : x>0,\ 0 \leqslant h \leqslant \delta\}. \end{aligned}$$

The Peetre's K-functional of the function $f \in C_B(0,\infty)$ is defined by

$$K_2(f;\delta) = \inf_{g \in C_B^2(0,\infty)} \left\{ \|f-g\| + \delta \left\|g''\right\| \right\}.$$

Here $C_B^2(0,\infty)$ is the space of functions f such that $f,\ f',\ f'' \in C_B(0,\infty)$. The norm on C_B^2 is defined as

$$\|g\|_{C_B^2} = \|g\|_{C_B} + \left\|g'\right\|_{C_B} + \left\|g''\right\|_{C_B}.$$

It is known that there exists a positive constant $C > 0$ such that

$$K_2(f;\delta) \leqslant C w_2(f;\sqrt{\delta}). \tag{13}$$

LEMMA 2. *For $f \in C_B(0,\infty)$ one has*

$$\left| G_{n,q}^{\alpha,\beta}(f;x) \right| \leqslant \|f\|.$$

Proof. The proof follows from the linearity of the operator $G_{n,q}^{\alpha,\beta}$ and from the first identity of Lemma 1. □

Here is our direct local approximation theorem for the operators $G_{n,q}^{\alpha,\beta}$.

THEOREM 2. *Let $f \in C_B(0,\infty)$ and $0 < q < 1$. For each $x \in (0,\infty)$*

$$\left| G_{n,q}^{\alpha,\beta}(f;x) - f(x) \right| \leqslant C w_2\left(f;\sqrt{\delta_{n,q}(x)}\right) + w(f;\mu_{n,q}(x))$$

for some positive constant C, where

$$\delta_{n,q}(x) = G_{n,q}^{\alpha,\beta}\left((t-x)^2;x\right) + \left(\frac{[n]}{[n]+\beta}x + \frac{\alpha}{[n]+\beta} - x\right)^2$$

and

$$\mu_{n,q}(x) = \left| \frac{[n]}{[n]+\beta}x + \frac{\alpha}{[n]+\beta} - x \right|$$

Proof. For $x \in (0,\infty)$ consider the following auxiliary operator $\widetilde{G}_{n,q}^{\alpha,\beta}(f;x)$ defined by

$$\widetilde{G}_{n,q}^{\alpha,\beta}(f;x) = G_{n,q}^{\alpha,\beta}(f;x) + f(x) - f\left(\frac{[n]}{[n]+\beta}x + \frac{\alpha}{[n]+\beta}\right). \tag{14}$$

From Corollary 1, $\widetilde{G}_{n,q}^{\alpha,\beta}$ reproduce linear functions, i.e.

$$\widetilde{G}_{n,q}^{\alpha,\beta}(t-x;x)=0.$$

Let $x\in(0,\infty)$ and $g\in C_B^2(0,\infty)$. By Taylor's Theorem we have

$$g(t)=g(x)+g'(x)(t-x)+\int_x^t (t-u)g''(u)du.$$

Applying $\widetilde{G}_{n,q}^{\alpha,\beta}$ to the both side of the above equality, we get

$$\begin{aligned}\widetilde{G}_{n,q}^{\alpha,\beta}(g(t);x)-g(x) &= \widetilde{G}_{n,q}^{\alpha,\beta}\left(\int_x^t (t-u)g''(u)du;x\right)\\ &= G_{n,q}^{\alpha,\beta}\left(\int_x^t (t-u)g''(u)du;x\right)\\ &\quad -\left(\int_x^{\frac{[n]}{[n]+\beta}x+\frac{\alpha}{[n]+\beta}}\left(\frac{[n]}{[n]+\beta}x+\frac{\alpha}{[n]+\beta}-u\right)g''(u)du\right)\end{aligned}$$

$$\begin{aligned}\left|\widetilde{G}_{n,q}^{\alpha,\beta}(g;x)-g(x)\right| &\leqslant \left\|g''\right\|\left\{G_{n,q}^{\alpha,\beta}\left(\int_x^t (t-u)du;x\right)\right.\\ &\quad\left.+\left(\int_x^{\frac{[n]}{[n]+\beta}x+\frac{\alpha}{[n]+\beta}}\left|\frac{[n]}{[n]+\beta}x+\frac{\alpha}{[n]+\beta}-u\right|du\right)\right\}\\ &\leqslant \left\|g''\right\|\left\{G_{n,q}^{\alpha,\beta}\left((t-x)^2;x\right)+\left(\frac{[n]}{[n]+\beta}x+\frac{\alpha}{[n]+\beta}-x\right)^2\right\}\\ &= \delta_{n,q}(x)\left\|g''\right\|\end{aligned}$$

On the other hand from (14) and Lemma 2, we have

$$\begin{aligned}\left|\widetilde{G}_{n,q}^{\alpha,\beta}(f;x)\right| &\leqslant \left|G_{n,q}^{\alpha,\beta}(f;x)\right|+2\left\|f\right\|\\ &\leqslant 3\left\|f\right\|.\end{aligned}$$

Thus, from (14) we can write

$$\begin{aligned}\left|G_{n,q}^{\alpha,\beta}(f;x)-f(x)\right| &\leqslant \left|\widetilde{G}_{n,q}^{\alpha,\beta}(f;x)-f(x)\right|+\left|f(x)-f\left(\frac{[n]}{[n]+\beta}x+\frac{\alpha}{[n]+\beta}\right)\right|\\ &\leqslant \left|\widetilde{G}_{n,q}^{\alpha,\beta}(f-g;x)\right|+\left|(f-g)(x)\right|+\left|\widetilde{G}_{n,q}^{\alpha,\beta}(g;x)-g(x)\right|\\ &\quad+\left|f\left(\frac{[n]}{[n]+\beta}x+\frac{\alpha}{[n]+\beta}\right)-f(x)\right|\\ &\leqslant 4\|f-g\|+\delta_{n,q}(x)\left\|g''\right\|+\left|f(x)-f\left(\frac{[n]}{[n]+\beta}x+\frac{\alpha}{[n]+\beta}\right)\right|\end{aligned}$$

Taking infimum on both side of the above inequality over all $g\in C_B^2(0,\infty)$, we get

$$\left|G_{n,q}^{\alpha,\beta}(f;x)-f(x)\right|\leqslant 4K_2(f;\delta_n)+w\left(f;\left|\frac{[n]}{[n]+\beta}x+\frac{\alpha}{[n]+\beta}-x\right|\right)$$

from which we have the desired result by (13). □

Note that if $q=(q_n)$ is a sequence satisfying the conditions given in (12), then we have $\lim_{n\to\infty}\delta_{n,q_n}(x)=0$ and $\lim_{n\to\infty}\mu_{n,q_n}(x)=0$ which gives us the pointwise rate of convergence of the sequence $\left(G_{n,q_n}^{\alpha,\beta}(f;x)\right)$ to $f(x)$ for every $x\in(0,\infty)$ and $f\in C_B(0,\infty)$.

4. Weighted approximation

Let $B_{x^2}(0,\infty)$ be the set of all functions defined on $(0,\infty)$ satisfying $|f(x)|\leqslant M_f(1+x^2)$. Here M_f is a constant depending only on f. We also have the following subspaces of $B_{x^2}(0,\infty)$:

$$\begin{aligned}C_{x^2}(0,\infty)&=\{f\in B_{x^2}(0,\infty): f \text{ is continuous on } (0,\infty)\}\\ C_{x^2}^*(0,\infty)&=\left\{f\in C_{x^2}(0,\infty): \lim_{x\to\infty}\frac{f(x)}{1+x^2}=K_f<\infty\right\}\end{aligned}$$

The norm on $C_{x^2}^*(0,\infty)$ is $\|f\|_{x^2}=\sup_{x>0}\frac{|f(x)|}{1+x^2}$.

In this section we give the weighted approximation theorem for functions f in $C_{x^2}^*(0,\infty)$ using the Korovkin type approximation theorems proved by Gadjiev [5].

THEOREM 3. *Let $q=(q_n)$ be a sequence satisfing the conditions given in (12). Then for each $f\in C_{x^2}^*(0,\infty)$, we have*

$$\lim_{n\to\infty}\left\|G_{n,q_n}^{\alpha,\beta}(f;.)-f(.)\right\|_{x^2}=0.$$

Proof. In order to prove the theorem we need to show that

$$\lim_{n\to\infty}\left\|G_{n,q_n}^{\alpha,\beta}\left(t^k;.\right)-f(.)\right\|_{x^2}=0,\quad \text{for}\quad k=0,1,2. \tag{15}$$

Then from the Korovkin's type Theorem the proof is obvious.

Since $G_{n,q_n}^{\alpha,\beta}(1;x)=1$, condition (15) holds for $k=0$.

For $k=1$, we have

$$\left\|G_{n,q_n}^{\alpha,\beta}(t;.)-x\right\|_{x^2}=\left\|\left(\frac{[n]}{[n]+\beta}-1\right)x+\frac{\alpha}{[n]+\beta}\right\|_{x^2}$$
$$\leqslant\left|\frac{[n]}{[n]+\beta}-1\right|\sup_{x>0}\frac{x}{1+x^2}+\frac{\alpha}{[n]+\beta}\sup_{x>0}\frac{1}{1+x^2}$$
$$\leqslant\left|\frac{[n]}{[n]+\beta}-1\right|+\frac{\alpha}{[n]+\beta}$$

Thus

$$\lim_{n\to\infty}\left\|G_{n,q_n}^{\alpha,\beta}(t;.)-x\right\|_{x^2}=0.$$

Similarly for $k=2$,

$$\left\|G_{n,q_n}^{\alpha,\beta}\left(t^2;.\right)-x^2\right\|_{x^2}=\sup_{x>0}\frac{\left|G_{n,q_n}^{\alpha,\beta}\left(t^2;x\right)-x^2\right|}{1+x^2}$$
$$\leqslant\left[\frac{[n]^2}{([n]+\beta)^2}\left(1+\frac{q^n}{[n]}\right)-1\right]\sup_{x>0}\frac{x^2}{1+x^2}$$
$$+\frac{2[n]\alpha}{([n]+\beta)^2}\sup_{x>0}\frac{x}{1+x^2}+\frac{\alpha^2}{([n]+\beta)^2}$$
$$\leqslant\left|\frac{[n][n+1]}{([n]+\beta)^2}-1\right|+\frac{2[n]\alpha}{([n]+\beta)^2}+\frac{\alpha^2}{([n]+\beta)^2}$$

Since $\lim_{n\to\infty}q_n=1$ we get

$$\lim_{n\to\infty}\left\|G_{n,q_n}^{\alpha,\beta}\left(t^2;.\right)-x^2\right\|_{x^2}=0.$$

Hence the proof is completed from the Korovkin's Theorem given by Gadjiev. □

5. Rate of convergence

Let $w_{x_0,c}(f;\delta)$ denote the modulus of continuity of f on the closed interval $[x_0,c]$, $0<x_0<c$ with

$$w_{x_0,c}(f;\delta)=\sup\{|f(t)-f(x)|:x,t\in[x_0,c],0\leqslant|t-x|\leqslant\delta\},\quad\delta>0.$$

THEOREM 4. *Let $n\in\mathbb{N}$, $q\in(0,1)$ and $0<x_0<c$. For every $f\in C_{x^2}(0,\infty)$,*

$$\left\|G_{n,q}^{\alpha,\beta}(f;.)-f\right\|_{[x_0,c]}\leqslant K\gamma_{n,q}+2w_{x_0,c+1}(f;\sqrt{\gamma_{n,q}}),\tag{16}$$

where $\gamma_{n,q}$ is given by (10) and K is a positive constant depending on f and c. $\|.\|_{[x_0,c]}$ denotes the classical sup-norm on the space $C[x_0,c]$.

Proof. For $x \in [x_0, c]$ and $t > c+1$, since $t - x > 1$

$$\begin{aligned} |f(t) - f(x)| &\leqslant M_f\left(2 + t^2 + x^2\right) \\ &= M_f\left(2 + x^2 + (t - x + x)^2\right) \\ &\leqslant M_f\left(2 + 2x^2 + (t-x)^2 + 2x(t-x)^2\right) \\ &\leqslant M_f\left(\left(2+2x^2\right)(t-x)^2 + (2x+1)(t-x)^2\right) \\ &\leqslant M_f\left(3 + 2c + 2c^2\right)(t-x)^2. \end{aligned} \tag{17}$$

For $x \in [x_0, c]$, $x_0 \leqslant t \leqslant c+1$, and $\delta > 0$

$$|f(t) - f(x)| \leqslant w_{x_0, c+1}(f; |t-x|) \leqslant \left(1 + \frac{|t-x|}{\delta}\right) w_{x_0, c+1}(f; \delta), \quad \delta > 0. \tag{18}$$

From (17) and (18), we get for all $x \in [x_0, c]$ and $t \geqslant x_0$

$$|f(t) - f(x)| \leqslant K(t-x)^2 + \left(1 + \frac{|t-x|}{\delta}\right) w_{x_0, c+1}(f; \delta)$$

where $K = M_f\left(3 + 2c + 2c^2\right)$. Thus we have

$$\left|G_{n,q}^{\alpha,\beta}(f;x) - f(x)\right| \leqslant K G_{n,q}^{\alpha,\beta}\left((t-x)^2; x\right) + w_{x_0,c+1}(f;\delta)\left[1 + \frac{1}{\delta}\left(G_{n,q}^{\alpha,\beta}\left((t-x)^2; x\right)\right)^{1/2}\right]$$

Using the identity for the second central moment of the operator $G_{n,q}^{\alpha,\beta}$ in (10) and taking supremum over the interval $x \in [x_0, c]$, the proof is completed. □

COROLLARY 2. *Let $q = (q_n)$ be a sequence satisfing (12) and $0 < x_0 < c$. For every $f \in C_{x^2}(0, \infty)$, we have*

$$\lim_{n \to \infty} \left\| G_{n,q_n}^{\alpha,\beta}(f;.) - f \right\|_{[x_0, c]} = 0.$$

Proof. Now taking a sequence $q = (q_n)$ satisfing (12) instead of a fixed number $q \in (0,1)$ in Theorem 4, we obtain γ_{n,q_n}, given by (10) tends to zero as $n \to \infty$ which gives us $\lim_{n \to \infty} w_{x_0, c+1}(f; \sqrt{\gamma_{n,q_n}}) = 0$ since f is continuous on $[x_0, c+1]$, $x_0 > 0$. Consequently, it follows from Theorem 4 that for every $f \in C_{x^2}(0, \infty)$, we get $\lim_{n \to \infty} \left\| G_{n,q_n}^{\alpha,\beta}(f;.) - f \right\|_{[x_0, c]} = 0$. □

6. A Voronovskaya type theorem

In this section we give a Voronovskaya Type asymptotic formulas for the operators $G_{n,q}^{\alpha,\beta}$ with the help of the second and fourth central moments. Before the main theorem it is meaningful to give the following Lemma.

LEMMA 3. *Let $G_{n,q}^{\alpha,\beta}$ be the operator defined by (9). Let (q_n) be a sequence such that the conditions in (12) are satisfied. For all $x>0$ we have,*

$$\lim_{n\to\infty}[n]\,G_{n,q_n}^{\alpha,\beta}(t-x;x)=\alpha-\beta x \tag{19}$$

$$\lim_{n\to\infty}[n]\,G_{n,q_n}^{\alpha,\beta}\left((t-x)^2;x\right)=ax^2 \tag{20}$$

and

$$\lim_{n\to\infty}[n]^2\,G_{n,q_n}^{\alpha,\beta}\left((t-x)^4;x\right)=3a^2x^4. \tag{21}$$

Proof. The first equality is obvious. For the second one we have,

$$\begin{aligned}\lim_{n\to\infty}[n]\,G_{n,q_n}^{\alpha,\beta}\left((t-x)^2;x\right)&=\lim_{n\to\infty}\left(\frac{q_n^n[n]^2+\beta^2[n]}{([n]+\beta)^2}x^2-\frac{2\alpha\beta[n]}{([n]+\beta)^2}x+\frac{\alpha^2[n]}{([n]+\beta)^2}\right)\\&=ax^2\end{aligned}$$

For the last one, making some computations, one can easily see that the limit of the coefficients of the terms $x^i,\ i=0,1,2,3$ tends to zero as $n\to\infty$. We have the coefficient of x^4 as

$$q^n(1-q)^2\frac{[n]^5}{([n]+\beta)^4}+\left[(-3[2]+[3]+[2][3])q^{2n}+4\beta q^n(1-q)\right]\frac{[n]^4}{([n]+\beta)^4}. \tag{22}$$

Using the identity $[n]=\frac{1-q^n}{1-q}$, and applying it to the term $[n]^2$, we can rewrite (22) as

$$q^n(1-q^n)^2\frac{[n]^3}{([n]+\beta)^4}+\left[(-3[2]+[3]+[2][3])q^{2n}+4\beta q^n(1-q)\right]\frac{[n]^4}{([n]+\beta)^4},$$

from which, we get the limit $3a^2$ for $n\to\infty$. Consequently we have

$$\lim_{n\to\infty}[n]^2\,G_{n,q_n}^{\alpha,\beta}\left((t-x)^4;x\right)=3a^2x^4$$

as desired. □

Now we present the Voronovskaya type result for the operator (9).

THEOREM 5. *Let $f,f',f''\in C_{x^2}(0,\infty)$ and (q_n) be a sequence such that (12) is satisfied, then we have*

$$\lim_{n\to\infty}[n]\left\{G_{n,q_n}^{\alpha,\beta}(f;x)-f(x)\right\}=(\alpha-\beta x)f'(x)+\frac{ax^2}{2}f''(x),$$

uniformly with respect to $x\in[x_0,c],\ 0<x_0<c$.

Proof. For $f,f',f''\in C_{x^2}(0,\infty)$ and $x>0$, we define a function as

$$r(t)=r(t,x)=\begin{cases}\frac{f(t)-f(x)-(t-x)f'(x)-\frac{1}{2}f''(x)(t-x)^2}{(t-x)^2}, & \text{if } t\neq x\\ 0, & \text{if } t=x.\end{cases}$$

From the definition, $r(x,x)=0$ and the function $r(.,x)\in C_{x^2}(0,\infty)$. So, by the Taylor's formula we write

$$f(t)=f(x)+(t-x)f'(x)+\frac{1}{2}f''(x)(t-x)^2+r(t,x)(t-x)^2 \tag{23}$$

where $r(t,x)$ is the Peano form of the remainder and $\lim_{t\to x} r(t,x)=0$. Applying $G_{n,q_n}^{\alpha,\beta}(f;x)$ to the both side of (23), we have

$$\begin{aligned}[n]\left[G_{n,q_n}^{\alpha,\beta}(f;x)-f(x)\right] &= [n]f'(x)G_{n,q_n}^{\alpha,\beta}(t-x;x) \qquad (24)\\ &+\frac{1}{2}[n]f''(x)G_{n,q_n}^{\alpha,\beta}\left((t-x)^2;x\right)+[n]G_{n,q_n}^{\alpha,\beta}\left((t-x)^2r(t,x);x\right)\end{aligned}$$

If we apply the Cauchy-Schwarz inequality for the last term on the right hand side of the equality (24), we get

$$[n]G_{n,q_n}^{\alpha,\beta}\left((t-x)^2r(t,x);x\right)\leqslant\sqrt{G_{n,q_n}^{\alpha,\beta}(r^2(t,x);x)}\sqrt{[n]^2G_{n,q_n}^{\alpha,\beta}((t-x)^4;x)}. \tag{25}$$

Observe that $r^2(.,x)\in C_{x^2}(0,\infty)$ and $r^2(x,x)=0$. Also, from Corollary 2, we have

$$\lim_{n\to\infty}G_{n,q_n}^{\alpha,\beta}\left(r^2(t,x);x\right)=r^2(x,x)=0 \tag{26}$$

uniformly with respect to $x\in[x_0,c]$. In the view of (25), (26) and the Lemma 3, we obtain

$$\lim_{n\to\infty}[n]G_{n,q_n}^{\alpha,\beta}\left((t-x)^2r(t,x);x\right)=0. \tag{27}$$

Combining (19), (20) and (27) we get the desired result. □

REFERENCES

[1] A. ARAL, V. GUPTA, R. P. AGARWAL, *Applications of q-Calculus in Operator Theory*, Springer, 2013.

[2] D. AYDIN, A. ARAL AND G. BAŞCANBAZ-TUNCA, *A Generalization of Post-Widder Operators Based on q-Integers*, Annals of the Alexandru Ioan Cuza University-Mathematics.

[3] Q. B. CAI AND X. M. ZENG, *On the convergence of a kind of q-gamma operators*, J. Inequal. Appl. 2013.1 (2013), 1–9.

[4] Q. B. CAI, *Properties of Convergence for a Class of Genralized q-Gamma Operators*, J. Math. Study **47**, 4 (2014), 388–395.

[5] A. D. GADJIEV, *A problem on the convergence of a sequence of positive linear operators on unbounded sets and theorems that are analogous to PP Korovkin's theorem*, Dokl. Akad. Nauk SSSR. **218** (1974), 1001–1004.

[6] F. H. JACKSON, *A generalization of the functions* $\Gamma(n)$ *and* x^n, Proc. Roy. Soc. London **74**, (1904), 64–72.
[7] V. G. KAC AND P. CHEUNG, *Quantum Calculus*, Universitext. Springer, New York, 2002.
[8] H. KARSLI, *Rate of convergence of a new gamma type operators for functions with derivatives of bounded variation*, Math. Comput. Model. **45**, 5–6 (2007), 617–624.
[9] H. KARSLI, V. GUPTA AND A. İZGI, *Rate of pointwise convergence of a new kind of gamma operators for functions of bounded variation*, Appl. Math. Letters **22**, 4 (2009), 505–510.
[10] H. KARSLI AND M. A. ÖZARSLAN, *Direct local and global approximation results for operators of gamma type*, Hacet. J. Math. Stat. **39** (2010), 241–253.
[11] H. KARSLI, P. N. AGRAWAL AND M. GOYAL, *General Gamma type operators based on* q*-integers*, Applied Mathematics and Computation **251** (2015), 564–575.
[12] A. LUPAŞ, *A q-analogue of the Bernstein operator*, Seminar on Numerical and Statistical Calculus, University of Cluj-Napoca, **9** (1987), 85–92.
[13] S. M. MAZHAR, *Approximation by positive operators on infinite intervals*, Mathematica Balkanica **5**, 2 (1991), 99–104,
[14] G. M. PHILLIPS, *Bernstein polynomials based on the* q*-integers*, The heritage of P. L. Chebyshev, Ann. Numer. Math. **4** (1997), 511–518.
[15] A. D. SOLE AND V. KAC, *On the integral representation of the* q*-gamma and the* q*-beta functions*, Rendiconti Lincei: Matematica e Applicazioni 9, **16**, 1 (2005), 11–19.
[16] A. D. SOLE AND V. KAC, *On integral representations of* q*-gamma and* q*-beta functions*, arXiv preprint math/0302032 (2003).
[17] D. D. STANCU, *Approximation of functions by a new class of linear polynomial operators*, Rev. Roumaine Math. Pures Appl. **13** (1968), 1173–1194.
[18] J. THOMAE, *Beitrage zur Theorie der durch die Heinesche Reihe*, J. Reine Angew. Math, **70** (1869), 258–281.
[19] Z. ÜNAL, M. A. ÖZARSLAN, O. DUMAN, *Approximation properties of real and complex Post-Widder operators based on* q*-integers*, Miskolc Math. Notes **13** (2012), 581–603.
[20] X. M. ZENG, *Approximation properties of Gamma operators*, J. Math. Anal. Appl., **311** (2005), 389–401.
[21] C. ZHAO, W. T. CHENG, X. M. ZENG, *Some approximation properties of a kind of* q*-Gamma Stancu operators*, J. Inequal. Appl. (94) (2014).

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