Mathematical
Inequalities
& Applications
Volume 19, Number 4 (2016), 1349–1362 doi:10.7153/mia-19-99

CAUCHY'S ERROR REPRESENTATION OF HERMITE INTERPOLATING POLYNOMIAL AND RELATED RESULTS

GORANA ARAS-GAZIĆ, JOSIP PEČARIĆ AND ANA VUKELIĆ

(*Communicated by S. Varošanec*)

Abstract. In this paper we consider convex functions of higher order. Using the Cauchy's error representation of Hermite interpolating polynomial the results concerning to the Hermite-Hadamard inequalities are presented. As a special case, generalizations for the zeros of orthogonal polynomials are considered.

1. Introduction

We follow here notations and terminology about *Hermite interpolating polynomial* from [1, p. 62]:

Let $-\infty < a < b < \infty$, and $a \leqslant a_1 < a_2 \ldots < a_r \leqslant b$, $(r \geqslant 2)$ be given. For $f \in C^n[a,b]$ a unique polynomial $P_H(t)$ of degree $(n-1)$, exists, fulfilling one of the following conditions:

Hermite conditions

$$P_H^{(i)}(a_j) = f^{(i)}(a_j); \quad 0 \leqslant i \leqslant k_j, \quad 1 \leqslant j \leqslant r, \quad \sum_{j=1}^{r} k_j + r = n,$$

in particular:

Simple Hermite or Osculatory conditions ($n = 2m$, $r = m$, $k_j = 1$ for all j)

$$P_O(a_j) = f(a_j), \quad P_O'(a_j) = f'(a_j), \quad 1 \leqslant j \leqslant m,$$

Lagrange conditions ($r = n$, $k_j = 0$ for all j)

$$P_L(a_j) = f(a_j), \quad 1 \leqslant j \leqslant n,$$

Type $(m, n-m)$ *conditions* ($r = 2$, $1 \leqslant m \leqslant n-1$, $k_1 = m-1$, $k_2 = n-m-1$)

$$P_{mn}^{(i)}(a) = f^{(i)}(a), \quad 0 \leqslant i \leqslant m-1,$$
$$P_{mn}^{(i)}(b) = f^{(i)}(b), \quad 0 \leqslant i \leqslant n-m-1,$$

Two-point Taylor conditions ($n = 2m$, $r = 2$, $k_1 = k_2 = m-1$)

$$P_{2T}^{(i)}(a) = f^{(i)}(a), \quad P_{2T}^{(i)}(b) = f^{(i)}(b), \quad 0 \leqslant i \leqslant m-1.$$

Mathematics subject classification (2010): Primary 26D15; Secondary 26D07, 26A51.
Keywords and phrases: n-convex function, Hermite interpolating polynomial, Hermite-Hadamard inequality, orthogonal polynomials.

DEFINITION 1. Let f be a real-valued function defined on the segment $[a,b]$. The *divided difference* of order n of the function f at distinct points $x_0,\dots,x_n \in [a,b]$, is defined recursively (see [7]) by

$$f[x_i] = f(x_i), \quad (i = 0,\dots,n)$$

and

$$f[x_0,\dots,x_n] = \frac{f[x_1,\dots,x_n] - f[x_0,\dots,x_{n-1}]}{x_n - x_0}.$$

The value $f[x_0,\dots,x_n]$ is independent of the order of the points $x_0,\dots,x_n$.

The definition may be extended to include the case that some (or all) of the points coincide. Assuming that $f^{(j-1)}(x)$ exists, we define

$$f[\underbrace{x,\dots,x}_{j-times}] = \frac{f^{(j-1)}(x)}{(j-1)!}.$$

The notion of *n-convexity* goes back to Popoviciu ([8]). We follow the definition given by Karlin ([6]):

DEFINITION 2. A function $f : [a,b] \to \mathbb{R}$ is said to be n-convex on $[a,b]$, $n \geqslant 0$, if for all choices of $(n+1)$ distinct points in $[a,b]$, n-th order divided difference of f satisfies

$$f[x_0,\dots,x_n] \geqslant 0.$$

In fact, Popoviciu proved that each continuous n-convex function on $[0,1]$ is the uniform limit of the sequence of corresponding Bernstein's polynomials (see for example [7, p. 293]). Also, Bernstein's polynomials of continuous n-convex function are also n-convex functions. Therefore, when stating our results for a continuous n-convex function f, without any loss in generality we assume that $f^{(n)}$ exists and is continuous.

In [5] M. Bessenyei and Zs. Páles were investigating the case of higher order convexity. The base points of the Hermite-Hadamard type inequalities turn out to be the zeros of certain orthogonal polynomials. The main tools of the paper are based on some methods of numerical analysis, like Gauss quadrature formula and Hermite interpolation. They considered the following Gauss type quadrature formulae where the coefficients and the base points are to be determined so that be exact when f is a polynomial of degree at most $2n-1$, $2n$, $2n$ and $2n+1$, respectively:

$$\int_a^b \rho(t) f(t)dt = \sum_{k=1}^{n} c_k f(\xi_k), \tag{1}$$

$$\int_a^b \rho(t) f(t)dt = c_0 f(a) + \sum_{k=1}^{n} c_k f(\xi_k), \tag{2}$$

$$\int_a^b \rho(t) f(t)dt = \sum_{k=1}^{n} c_k f(\xi_k) + c_{n+1} f(b), \tag{3}$$

$$\int_a^b \rho(t) f(t)dt = c_0 f(a) + \sum_{k=1}^{n} c_k f(\xi_k) + c_{n+1} f(b). \tag{4}$$

Using this formulae and the remainder term of the Hermite interpolation they proved Hermite-Hadamard type inequalities in cases of odd and even higher order convexity separately in the subsequent theorems:

THEOREM 1. *Let $\rho : [a,b] \to \mathbb{R}$ be a positive integrable function. Denote the zeros of P_m by $\xi_1,\dots,\xi_m$ where P_m is the m-th degree member of the orthogonal polynomial system on $[a,b]$ with respect to the weight function $(x-a)\rho(x)$, furthermore denote the zeros of Q_m by $\eta_1,\dots,\eta_m$ where Q_m is the m-th degree member of the orthogonal polynomial system on $[a,b]$ with respect to the weight function $(b-x)\rho(x)$. Define the coefficients $\alpha_0,\dots,\alpha_m$ and $\beta_1,\dots,\beta_{m+1}$ by the formulae*

$$\alpha_0 := \frac{1}{P_m^2(a)}\int_a^b P_m^2(x)\rho(x)dx, \quad \alpha_k := \frac{1}{\xi_k - a}\int_a^b \frac{(x-a)P_m(x)}{(x-\xi_k)P_m'(\xi_k)}\rho(x)dx$$

and

$$\beta_k := \frac{1}{b-\eta_k}\int_a^b \frac{(b-x)Q_m(x)}{(x-\eta_k)Q_m'(\eta_k)}\rho(x)dx, \quad \beta_{m+1} := \frac{1}{Q_m^2(b)}\int_a^b Q_m^2(x)\rho(x)dx.$$

If a function $f : [a,b] \to \mathbb{R}$ is $(2m+1)$-convex, then it satisfies the following Hermite-Hadamard type inequality

$$\alpha_0 f(a) + \sum_{k=1}^{m}\alpha_k f(\xi_k) \leqslant \int_a^b \rho(x)f(x)dx \leqslant \sum_{k=1}^{m}\beta_k f(\eta_k) + \beta_{m+1}f(b).$$

THEOREM 2. *Let $\rho : [a,b] \to \mathbb{R}$ be a positive integrable function. Denote the zeros of P_m by $\xi_1,\dots,\xi_m$ where P_m is the m-th degree member of the orthogonal polynomial system on $[a,b]$ with respect to the weight function $\rho(x)$, furthermore denote the zeros of Q_{m-1} by $\eta_1,\dots,\eta_{m-1}$ where Q_{m-1} is the $(m-1)$-st degree member of the orthogonal polynomial system on $[a,b]$ with respect to the weight function $(b-x)(x-a)\rho(x)$. Define the coefficients $\alpha_1,\dots,\alpha_m$ and $\beta_0,\dots,\beta_m$ by the formulae*

$$\alpha_k := \int_a^b \frac{P_m(x)}{(x-\xi_k)P_m'(\xi_k)}\rho(x)dx$$

and

$$\beta_0 := \frac{1}{(b-a)Q_{m-1}^2(a)}\int_a^b (b-x)Q_{m-1}^2(x)\rho(x)dx,$$

$$\beta_k := \frac{1}{(b-\eta_k)(\xi_k-a)}\int_a^b \frac{(b-x)(x-a)Q_{m-1}(x)}{(x-\eta_k)Q_{m-1}'(\eta_k)}\rho(x)dx,$$

$$\beta_m := \frac{1}{(b-a)Q_{m-1}^2(b)}\int_a^b (x-a)Q_{m-1}^2(x)\rho(x)dx.$$

If a function $f : [a,b] \to \mathbb{R}$ is $(2m)$-convex, then it satisfies the following Hermite-Hadamard type inequality

$$\sum_{k=1}^{m}\alpha_k f(\xi_k) \leqslant \int_a^b \rho(x)f(x)dx \leqslant \beta_0 f(a) + \sum_{k=1}^{m-1}\beta_k f(\eta_k) + \beta_m f(b).$$

In this paper we obtain generalizations of above inequalities for convex functions of higher order by using the Cauchy's error representation of Hermite interpolating polynomial. As a special case, generalizations of Hermite-Hadamard type inequalities, where the base points turn out to be the zeros of orthogonal polynomials will be considered. Similar results for Lidstone's polynomial can be found in [4]. See also [2] and [3].

2. Cauchy's error representation

In [1, p. 71] the following theorem is proved:

THEOREM 3. *Let* $F(t) \in C^{n-1}([a,b])$ *and suppose that* $F^{(n)}(t)$ *exists at each point of* (a,b)*. Then*

$$F(t) - \sum_{j=1}^{r}\sum_{i=0}^{k_j} H_{ij}(t)F^{(i)}(a_j) = \frac{1}{n!}\omega(t)F^{(n)}(\xi), \tag{5}$$

where $\xi \in (a,b)$ *and* H_{ij} *are fundamental polynomials of the Hermite basis defined by*

$$H_{ij}(t) = \frac{1}{i!}\frac{\omega(t)}{(t-a_j)^{k_j+1-i}}\sum_{k=0}^{k_j-i}\frac{1}{k_j!}\left[\frac{(t-a_j)^{k_j+1}}{\omega(t)}\right]^{(k)}_{t=a_j}(t-a_j)^k, \tag{6}$$

where

$$\omega(t) = \prod_{j=1}^{r}(t-a_j)^{k_j+1}. \tag{7}$$

Motivated by (5) and formulae (1), (2), (3) and (4) we define functionals $\Phi_1(f)$, $\Phi_2(f)$, $\Phi_3(f)$ and $\Phi_4(f)$ respectively, by

$$\Phi_1(F) = F(t) - \sum_{j=1}^{r}\sum_{i=0}^{k_j} H_{ij}(t)F^{(i)}(a_j), \tag{8}$$

$$\Phi_2(F) = F(t) - \sum_{i=0}^{k_1} H_{i1}(t)F^{(i)}(a) - \sum_{j=2}^{r}\sum_{i=0}^{k_j} H_{ij}(t)F^{(i)}(a_j), \tag{9}$$

$$\Phi_3(F) = F(t) - \sum_{j=1}^{r-1}\sum_{i=0}^{k_j} H_{ij}(t)F^{(i)}(a_j) - \sum_{i=0}^{k_r} H_{ir}(t)F^{(i)}(b), \tag{10}$$

$$\Phi_4(F) = F(t) - \sum_{i=0}^{k_1} H_{i1}(t)F^{(i)}(a) - \sum_{j=2}^{r-1}\sum_{i=0}^{k_j} H_{ij}(t)F^{(i)}(a_j) - \sum_{i=0}^{k_r} H_{ir}(t)F^{(i)}(b). \tag{11}$$

Now, using Theorem 3 we get the following corollaries:

COROLLARY 1. *Let $F:[a,b]\to\mathbb{R}$ is n-convex function, and H_{ij} are defined on $[a,b]$ by (6), such that k_j is odd for all $j=1,\dots,r$. Then we have*

$$\Phi_1(F)\geqslant 0. \tag{12}$$

Proof. Since k_j is odd for all $j=1,\dots,r$, then using (7), we get that $\omega(t)\geqslant 0$. By using (5) for n-convex function F, (12) obviously holds. □

REMARK 1. If we put that $n=2m$, $r=m$ and $k_j=1$ for all j we get Hermite interpolating polynomial with simple Hermite or Osculatory conditions and then

$$F(t)-\sum_{j=1}^{m}H_{0j}(t)F(a_j)-\sum_{j=1}^{m}H_{1j}(t)F'(a_j)\geqslant 0.$$

COROLLARY 2. *Let $F:[a,b]\to\mathbb{R}$ is n-convex function, and H_{ij} are defined on $[a,b]$ by (6), such that $a_1=a$ and k_j is odd for all $j=2,\dots,r$. Then we have*

$$\Phi_2(F)\geqslant 0. \tag{13}$$

Proof. Now $\omega(t)=(t-a)^{k_1+1}\prod_{j=2}^{r}(t-a_j)^{k_j+1}$. Since k_j is odd for all $j=2,\dots,r$, we get that $\omega(t)\geqslant 0$. So, by using (5) for n-convex function F, (13) obviously holds. □

COROLLARY 3. *Let $F:[a,b]\to\mathbb{R}$ is n-convex function and H_{ij} are defined on $[a,b]$ by (6), such that $a_r=b$. Then*

(a) If k_j is odd for all $j=1,\dots,r$, we have

$$\Phi_3(F)\geqslant 0. \tag{14}$$

(b) If k_j is odd for all $j=1,\dots,r-1$ and k_r is even, we have

$$\Phi_3(F)\leqslant 0. \tag{15}$$

Proof. Now $\omega(t)=(t-b)^{k_r+1}\prod_{j=1}^{r-1}(t-a_j)^{k_j+1}$.

(a) Since k_j is odd for all $j=1,\dots,r$, we get that $\omega(t)\geqslant 0$.

(b) Since k_j is odd for all $j=1,\dots,r-1$ and k_r is even, we get that $\omega(t)\leqslant 0$.

So, by using (5) for n-convex function F, (14) and (15) obviously hold. □

COROLLARY 4. *Let $F:[a,b]\to\mathbb{R}$ is n-convex function and H_{ij} are defined on $[a,b]$ by (6), such that $a_1=a$ and $a_r=b$. Then*

(a) If k_j is odd for all $j=2,\dots,r$, we have

$$\Phi_4(F)\geqslant 0. \tag{16}$$

(b) If k_j is odd for all $j=2,\dots,r-1$ and k_r is even, we have

$$\Phi_4(F)\leqslant 0. \tag{17}$$

Proof. Now $\omega(t) = (t-a)^{k_1+1}(t-b)^{k_r+1}\prod_{j=2}^{r-1}(t-a_j)^{k_j+1}$.
(a) Since k_j is odd for all $j = 2,\dots,r$, we get that $\omega(t) \geqslant 0$.
(b) Since k_j is odd for all $j = 2,\dots,r-1$ and k_r is even, we get that $\omega(t) \leqslant 0$.
So, by using (5) for n-convex function F, (16) and (17) obviously hold. □

REMARK 2. If we put $r = 2$, $1 \leqslant m \leqslant n-1$, $k_1 = m-1$, $k_2 = n-m-1$ and k_2 is odd then we get Hermite interpolating polynomial with $(m, n-m)$ type conditions and then

$$F(t) - \sum_{i=0}^{m-1} H_{i1}(t)F^{(i)}(a) - \sum_{i=0}^{n-m-1} H_{i2}(t)F^{(i)}(b) \geqslant 0.$$

For k_2 even, the above inequality is reversed.

If we put $n = 2m$, $r = 2$, $k_1 = k_2 = m-1$ and m is even then we get Hermite interpolating polynomial with two-point Taylor conditions and then

$$F(t) - \sum_{i=0}^{m-1} H_{i1}(t)F^{(i)}(a) - \sum_{i=0}^{m-1} H_{i2}(t)F^{(i)}(b) \geqslant 0.$$

For m odd, the above inequality is reversed.

REMARK 3. Similarly as in [3] we can construct new families of exponentially convex function and Cauchy type means by looking at linear functionals (8), (9), (10) and (11). The monotonicity property of the generalized Cauchy means obtained via these functionals can be prove by using the properties of the linear functionals associated with this error representation, such as n-exponential and logarithmic convexity.

3. Generalization of the Hermite-Hadamard type inequalities

The classical Hermite-Hadamard inequality states that for a convex function $F : [a,b] \to \mathbb{R}$ the following estimation holds:

$$F\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b F(t)dt \leqslant \frac{F(a)+F(b)}{2}. \tag{18}$$

As a consequences of our results given in Section 2, here we give the generalized Hermite-Hadamard type inequalities.

THEOREM 4. *Let $F : [a,b] \to \mathbb{R}$ is n-convex function, $\rho : [a,b] \to \mathbb{R}$ is a positive integrable function and H_{ij} and $\bar{H}_{ij}$ are defined on $[a,b]$ by*

$$H_{ij}(t) = \frac{1}{i!}\frac{\omega(t)}{(t-a_j)^{k_j+1-i}}\sum_{k=0}^{k_j-i}\frac{1}{k_j!}\left[\frac{(t-a_j)^{k_j+1}}{\omega(t)}\right]^{(k)}_{t=a_j}(t-a_j)^k, \tag{19}$$

and

$$\bar{H}_{ij}(t) = \frac{1}{i!}\frac{\bar{\omega}(t)}{(t-b_j)^{l_j+1-i}}\sum_{k=0}^{l_j-i}\frac{1}{l_j!}\left[\frac{(t-b_j)^{l_j+1}}{\bar{\omega}(t)}\right]^{(k)}_{t=b_j}(t-b_j)^k, \tag{20}$$

where

$$\omega(t)=\prod_{j=1}^{r}(t-a_j)^{k_j+1},\quad \bar{\omega}(t)=\prod_{j=1}^{\bar{r}}(t-b_j)^{l_j+1}$$

for $a\leqslant a_1<a_2\ldots<a_r\leqslant b,\ a\leqslant b_1<b_2\ldots<b_{\bar{r}}\leqslant b,(r,\bar{r}\geqslant 2)$ *and* $\sum_{j=1}^{r}k_j+r=\sum_{j=1}^{\bar{r}}l_j+\bar{r}=n$.

Then, if $a_1=a,\ b_{\bar{r}}=b,\ k_j$ *odd for all* $j=2,\ldots,r,\ l_j$ *odd for all* $j=1,\ldots,\bar{r}-1$ *and* $l_{\bar{r}}$ *even, we have*

$$\begin{aligned}&\sum_{i=0}^{k_1}F^{(i)}(a)\int_a^b\rho(t)H_{i1}(t)dt+\sum_{j=2}^{r}\sum_{i=0}^{k_j}F^{(i)}(a_j)\int_a^b\rho(t)H_{ij}(t)dt\\ &\leqslant\int_a^b\rho(t)F(t)dt\\ &\leqslant\sum_{j=1}^{\bar{r}-1}\sum_{i=0}^{l_j}F^{(i)}(b_j)\int_a^b\rho(t)\bar{H}_{ij}(t)dt+\sum_{i=0}^{l_{\bar{r}}}F^{(i)}(b)\int_a^b\rho(t)\bar{H}_{i\bar{r}}(t)dt.\end{aligned}$$

If F *is* n*-concave, the inequalities are reversed.*

Proof. We use Corollary 2 and Corollary 3(b). □

COROLLARY 5. *Let* $F:[a,b]\to\mathbb{R}$ *is* $(2r-1)$*-convex function and* $\rho:[a,b]\to\mathbb{R}$ *is a positive integrable function. Then, we have*

$$\begin{aligned}&F(a)\int_a^b\rho(t)H_{01}(t)dt+\sum_{j=2}^{r}F(a_j)\int_a^b\rho(t)H_{0j}(t)dt+\sum_{j=2}^{r}F'(a_j)\int_a^b\rho(t)H_{1j}(t)dt\\ &\leqslant\int_a^b\rho(t)F(t)dt \qquad (21)\\ &\leqslant\sum_{j=1}^{r-1}F(b_j)\int_a^b\rho(t)\bar{H}_{0j}(t)dt+\sum_{j=1}^{r-1}F'(b_j)\int_a^b\rho(t)\bar{H}_{1j}(t)dt+F(b)\int_a^b\rho(t)\bar{H}_{0r}(t)dt,\end{aligned}$$

where

$$H_{01}(t)=\frac{P_{r-1}^2(t)}{P_{r-1}^2(a)},$$

$$H_{0j}(t)=\frac{(t-a)P_{r-1}^2(t)}{(t-a_j)^2\left[P'_{r-1}(a_j)\right]^2(a_j-a)}\left(1-\frac{P'_{r-1}(a_j)+(a_j-a)P''_{r-1}(a_j)}{(a_j-a)P'_{r-1}(a_j)}(t-a_j)\right),$$

$$H_{1j}(t)=\frac{(t-a)P_{r-1}^2(t)}{(t-a_j)(a_j-a)\left[P'_{r-1}(a_j)\right]^2},$$

$$\bar{H}_{0j}(t)=\frac{(b-t)\bar{P}_{r-1}^2(t)}{(t-b_j)^2\left[\bar{P}'_{r-1}(b_j)\right]^2(b-b_j)}\left(1+\frac{\bar{P}'_{r-1}(b_j)-(b-b_j)\bar{P}''_{r-1}(b_j)}{(b-b_j)\bar{P}'_{r-1}(b_j)}(t-b_j)\right),$$

$$\bar{H}_{1j}(t) = \frac{(b-t)\bar{P}^2_{r-1}(t)}{(t-b_j)(b-b_j)\left[\bar{P}'_{r-1}(b_j)\right]^2}, \quad \bar{H}_{0r}(t) = \frac{\bar{P}^2_{r-1}(t)}{\bar{P}^2_{r-1}(b)},$$

and

$$P_{r-1}(t) = \prod_{j=2}^{r}(t-a_j), \quad \bar{P}_{r-1}(t) = \prod_{j=1}^{r-1}(t-b_j)$$

for $a < a_2 \ldots < a_r \leqslant b,\ a \leqslant b_1 < b_2 \ldots < b_{r-1} < b, (r \geqslant 2)$.

If F *is* $(2r-1)$*-concave, the inequalities are reversed.*

Proof. We put $k_1 = 0,\ k_j = 1$ for $j = 2, \ldots, r$ and $l_j = 1$ for $j = 1, \ldots, r-1$, $l_r = 0$ in Theorem 4 and then calculate

$$H_{01}(t) = \frac{(t-a)P^2_{r-1}(t)}{(t-a)} \cdot \left[\frac{(t-a)}{(t-a)P^2_{r-1}(t)}\right]_{t=a} = \frac{P^2_{r-1}(t)}{P^2_{r-1}(a)},$$

$$\begin{aligned} H_{0j}(t) &= \frac{(t-a)P^2_{r-1}(t)}{(t-a_j)^2}\left\{\left[\frac{(t-a_j)^2}{(t-a)P^2_{r-1}(t)}\right]_{t=a_j} + \left[\frac{(t-a_j)^2}{(t-a)P^2_{r-1}(t)}\right]'_{t=a_j}(t-a_j)\right\} \\ &= \frac{(t-a)P^2_{r-1}(t)}{(t-a_j)^2\left[P'_{r-1}(a_j)\right]^2(a_j-a)}\left(1 - \frac{P'_{r-1}(a_j)+(a_j-a)P''_{r-1}(a_j)}{(a_j-a)P'_{r-1}(a_j)}(t-a_j)\right) \end{aligned}$$

and

$$H_{1j}(t) = \frac{(t-a)P^2_{r-1}(t)}{(t-a_j)}\left[\frac{(t-a_j)^2}{(t-a)P^2_{r-1}(t)}\right]_{t=a_j} = \frac{(t-a)P^2_{r-1}(t)}{(t-a_j)(a_j-a)\left[P'_{r-1}(a_j)\right]^2}.$$

Coefficients $\bar{H}_{0j}$, $\bar{H}_{1j}$ and $\bar{H}_{0r}$ we get similarly. □

REMARK 4. If we choose P_{r-1} and $\bar{P}_{r-1}$ such that they are orthogonal with weight $(t-a)\rho(t)$ and $(b-t)\rho(t)$ respectively, to all polynomials of lower degree, i.e.

$$\int_a^b (t-a)\rho(t)P_{r-1}(t)t^k dt = 0, \quad k = 0, 1, \ldots, r-2 \tag{22}$$

and

$$\int_a^b (b-t)\rho(t)\bar{P}_{r-1}(t)t^l dt = 0, \quad l = 0, 1, \ldots, r-2,$$

we get that

$$\int_a^b \rho(t)H_{1j}(t)dt = 0 \ \text{ and } \ \int_a^b \rho(t)\bar{H}_{1j}(t)dt = 0.$$

Now, using the relation for coefficient $H_{1j}(t)$, we get

$$\begin{aligned}&\int_a^b \rho(t)H_{0j}(t)dt\\ &= \int_a^b \frac{\rho(t)(t-a)P_{r-1}^2(t)}{(a_j-a)(t-a_j)^2\left[P_{r-1}'(a_j)\right]^2}dt - \frac{P_{r-1}'(a_j)+(a_j-a)P_{r-1}''(a_j)}{(a_j-a)P_{r-1}'(a_j)}\int_a^b \rho(t)H_{1j}(t)dt.\end{aligned}$$

Now, using (22), we have

$$\int_a^b \frac{\rho(t)(t-a)P_{r-1}(t)}{(t-a_j)P_{r-1}'(a_j)}\left(\frac{P_{r-1}(t)}{(t-a_j)P_{r-1}'(a_j)}-1\right)dt = 0$$

because

$$\frac{P_{r-1}(t)}{(t-a_j)P_{r-1}'(a_j)}-1 = (t-a_j)Q(t),$$

where $Q(t)$ is polynomial of degree $r-3$. So,

$$\int_a^b \frac{\rho(t)(t-a)P_{r-1}^2(t)}{(t-a_j)^2\left[P_{r-1}'(a_j)\right]^2}dt = \int_a^b \frac{\rho(t)(t-a)P_{r-1}(t)}{(t-a_j)P_{r-1}'(a_j)}dt.$$

Similarly, we calculate $\int_a^b \rho(t)\bar{H}_{0j}(t)dt$ and get the following relations for coefficients in (21):

$$\int_a^b \rho(t)H_{01}(t)dt = \frac{1}{P_{r-1}^2(a)}\int_a^b \rho(t)P_{r-1}^2(t)dt,$$

$$\int_a^b \rho(t)H_{0j}(t)dt = \frac{1}{(a_j-a)P_{r-1}'(a_j)}\int_a^b \frac{\rho(t)(t-a)P_{r-1}(t)}{(t-a_j)}dt,$$

$$\int_a^b \rho(t)H_{1j}(t)dt = 0,$$

$$\int_a^b \rho(t)\bar{H}_{0j}(t)dt = \frac{1}{(b-b_j)\bar{P}_{r-1}'(b_j)}\int_a^b \frac{\rho(t)(b-t)\bar{P}_{r-1}(t)}{(t-b_j)}dt,$$

$$\int_a^b \rho(t)\bar{H}_{1j}(t)dt = 0,$$

$$\int_a^b \rho(t)\bar{H}_{0r}(t)dt = \frac{1}{\bar{P}_{r-1}^2(b)}\int_a^b \rho(t)\bar{P}_{r-1}^2(t)dt,$$

which is result proved by M. Bessenyei and Zs. Páles in [5] (see Theorem 1).

THEOREM 5. *Let* $F:[a,b]\to\mathbb{R}$ *is* n*-convex function,* $\rho:[a,b]\to\mathbb{R}$ *is a positive integrable function and* H_{ij} *and* $\bar{H}_{ij}$ *are defined on* $[a,b]$ *by (19) and (20) respectively.*

Then, if $b_1 = a,\ b_{\bar{r}} = b,\ k_j$ *odd for all* $j = 1,\dots,r,\ l_j$ *odd for all* $j = 2,\dots,\bar{r}-1$ *and* $l_{\bar{r}}$ *even, we have*

$$\begin{aligned}
&\sum_{j=1}^{r}\sum_{i=0}^{k_j} F^{(i)}(a_j)\int_a^b \rho(t)H_{ij}(t)dt \\
\leqslant &\int_a^b \rho(t)F(t)dt \\
\leqslant &\sum_{i=0}^{l_1} F^{(i)}(a)\int_a^b \rho(t)\bar{H}_{i1}(t)dt + \sum_{j=2}^{\bar{r}-1}\sum_{i=0}^{l_j} F^{(i)}(b_j)\int_a^b \rho(t)\bar{H}_{ij}(t)dt \\
&+\sum_{i=0}^{l_{\bar{r}}} F^{(i)}(b)\int_a^b \rho(t)\bar{H}_{i\bar{r}}(t)dt.
\end{aligned}$$

If F *is* n*-concave, the inequalities are reversed.*

Proof. We use Corollary 1 and Corollary 4(b). □

COROLLARY 6. *Let* $F : [a,b] \to \mathbb{R}$ *is* $(2r)$*-convex function and* $\rho : [a,b] \to \mathbb{R}$ *is a positive integrable function. Then, we have*

$$\begin{aligned}
&\sum_{j=1}^{r} F(a_j)\int_a^b \rho(t)H_{0j}(t)dt + \sum_{j=1}^{r} F'(a_j)\int_a^b \rho(t)H_{1j}(t)dt \\
\leqslant &\int_a^b \rho(t)F(t)dt \\
\leqslant &F(a)\int_a^b \rho(t)\bar{H}_{01}(t)dt + \sum_{j=2}^{r} F(b_j)\int_a^b \rho(t)\bar{H}_{0j}(t)dt \qquad (23)\\
&+\sum_{j=2}^{r} F'(b_j)\int_a^b \rho(t)\bar{H}_{1j}(t)dt + F(b)\int_a^b \rho(t)\bar{H}_{0(r+1)}(t)dt,
\end{aligned}$$

where

$$H_{0j}(t) = \frac{P_r^2(t)}{(t-a_j)^2\,[P_r'(a_j)]^2}\left(1 - \frac{P_r''(a_j)}{P_r'(a_j)}(t-a_j)\right),$$

$$H_{1j}(t) = \frac{P_r^2(t)}{(t-a_j)\,[P_r'(a_j)]^2}, \quad \bar{H}_{01}(t) = \frac{(b-t)\bar{P}_{r-1}^2(t)}{(b-a)\bar{P}_{r-1}^2(a)},$$

$$\begin{aligned}
\bar{H}_{0j}(t) = &\frac{(t-a)(b-t)\bar{P}_{r-1}^2(t)}{(b_j-a)(b-b_j)(t-b_j)^2\left[\bar{P}_{r-1}'(b_j)\right]^2} \\
&\times\left(1 + \frac{(2b_j-a-b)\bar{P}_{r-1}'(b_j) - (b-b_j)(b_j-a)\bar{P}_{r-1}''(b_j)}{(b-b_j)(b_j-a)\bar{P}_{r-1}'(b_j)}(t-b_j)\right),
\end{aligned}$$

$$\bar{H}_{1j}(t) = \frac{(t-a)(b-t)\bar{P}^2_{r-1}(t)}{(t-b_j)(b_j-a)(b-b_j)\left[\bar{P}'_{r-1}(b_j)\right]^2}, \quad \bar{H}_{0(r+1)}(t) = \frac{(t-a)\bar{P}^2_{r-1}(t)}{(b-a)\bar{P}^2_{r-1}(b)}$$

and

$$P_r(t) = \prod_{j=1}^{r}(t-a_j), \quad \bar{P}_{r-1}(t) = \prod_{j=2}^{r}(t-b_j)$$

for $a \leqslant a_1 < a_2 \ldots < a_r \leqslant b,\ a < b_2 \ldots < b_r < b, (r \geqslant 2)$. *If* F *is* $(2r)$*-concave, the inequalities are reversed.*

Proof. We put $k_j = 1$ for $j = 1, \ldots, r$ and $l_j = 1$ for $j = 2, \ldots, r$, $l_1 = l_{r+1} = 0$ in Theorem 5 and then calculate

$$\begin{aligned} H_{0j} &= \frac{P_r^2(t)}{(t-a_j)^2}\left\{\left[\frac{(t-a_j)^2}{P_r^2(t)}\right]_{t=a_j} + \left[\frac{(t-a_j)^2}{P_r^2(t)}\right]'_{t=a_j}(t-a_j)\right\} \\ &= \frac{P_r^2(t)}{(t-a_j)^2\left[P_r'(a_j)\right]^2}\left(1 - \frac{P_r''(a_j)}{P_r'(a_j)}(t-a_j)\right), \end{aligned}$$

$$H_{1j}(t) = \frac{P_r^2(t)}{t-a_j}\left[\frac{(t-a_j)^2}{P_r^2(t)}\right]_{t=a_j} = \frac{P_r^2(t)}{(t-a_j)\left[P_r'(a_j)\right]^2},$$

$$\bar{H}_{01}(t) = \frac{(t-a)(t-b)\bar{P}^2_{r-1}(t)}{t-a}\left[\frac{t-a}{(t-a)(t-b)\bar{P}^2_{r-1}(t)}\right]_{t=a} = \frac{(b-t)\bar{P}^2_{r-1}(t)}{(b-a)\bar{P}^2_{r-1}(a)},$$

$$\begin{aligned} \bar{H}_{0j}(t) &= \frac{(t-a)(t-b)\bar{P}^2_{r-1}(t)}{(t-b_j)^2} \\ &\quad \times \left\{\left[\frac{(t-b_j)^2}{(t-a)(t-b)\bar{P}^2_{r-1}(t)}\right]_{t=b_j} + \left[\frac{(t-b_j)^2}{(t-a)(t-b)\bar{P}^2_{r-1}(t)}\right]'_{t=b_j}(t-b_j)\right\} \\ &= \frac{(t-a)(b-t)\bar{P}^2_{r-1}(t)}{(b_j-a)(b-b_j)(t-b_j)^2\left[\bar{P}'_{r-1}(b_j)\right]^2} \\ &\quad \times \left(1 + \frac{(2b_j-a-b)\bar{P}'_{r-1}(b_j) - (b-b_j)(b_j-a)\bar{P}''_{r-1}(b_j)}{(b-b_j)(b_j-a)\bar{P}'_{r-1}(b_j)}(t-b_j)\right), \end{aligned}$$

$$\begin{aligned} \bar{H}_{1j}(t) &= \frac{(t-a)(t-b)\bar{P}^2_{r-1}(t)}{(t-b_j)}\left[\frac{(t-b_j)^2}{(t-a)(t-b)\bar{P}^2_{r-1}(t)}\right]_{t=b_j} \\ &= \frac{(t-a)(b-t)\bar{P}^2_{r-1}(t)}{(t-b_j)(b_j-a)(b-b_j)\left[\bar{P}'_{r-1}(b_j)\right]^2}. \end{aligned}$$

Coefficient $\bar{H}_{0(r+1)}$ we get similarly as coefficient $\bar{H}_{01}(t)$. $\square$

REMARK 5. If we choose P_r such that it is orthogonal with weight $\rho(t)$ to all polynomials of lower degree, i.e.

$$\int_a^b \rho(t)P_r(t)t^k dt = 0, \quad k = 0,1,\ldots,r-1$$

we get that

$$\int_a^b \rho(t)H_{1j}(t)dt = 0.$$

Now, similar as in Remark 4 we get

$$\int_a^b \rho(t)H_{0j}(t)dt = \frac{1}{P_r'(a_j)}\int_a^b \frac{\rho(t)P_r(t)}{(t-a_j)}dt.$$

Also, if we choose $\bar{P}_{r-1}$ such that it is orthogonal with weight $(t-a)(b-t)\rho(t)$, to all polynomials of lower degree, i.e.

$$\int_a^b \rho(t)(t-a)(b-t)\bar{P}_{r-1}(t)t^l dt = 0, \quad l = 0,1,\ldots,r-2,$$

we get that

$$\int_a^b \rho(t)\bar{H}_{1j}(t)dt = 0$$

and then

$$\int_a^b \rho(t)\bar{H}_{0j}(t)dt = \frac{1}{(b_j-a)(b-b_j)\bar{P}_{r-1}'(b_j)}\int_a^b \frac{\rho(t)(t-a)(b-t)\bar{P}_{r-1}(t)}{(t-b_j)}dt,$$

which is result proved by M. Bessenyei and Zs. Páles in [5] (see Theorem 2).

REMARK 6. If we put $r=1$ and $\rho(t)=1$ in Corollary 6, we get $n=2$ and for $a_1 = \frac{a+b}{2}$ calculate

$$H_{01}(t) = 1 \Rightarrow \int_a^b H_{01}(t)dt = b-a,$$

$$H_{11}(t) = t - \frac{a+b}{2} \Rightarrow \int_a^b H_{11}(t)dt = 0,$$

$$\bar{H}_{01}(t) = \frac{b-t}{b-a} \Rightarrow \int_a^b \bar{H}_{01}(t)dt = \frac{b-a}{2},$$

$$\bar{H}_{02}(t) = \frac{t-a}{b-a} \Rightarrow \int_a^b \bar{H}_{02}(t)dt = \frac{b-a}{2}.$$

So, using (23) for $n=2$ we get classical Hermite-Hadamard inequality (18).

COROLLARY 7. *Let $F : [a,b] \to \mathbb{R}$ is $(2m)$-convex function and $\rho : [a,b] \to \mathbb{R}$ is a positive integrable function. Then, if m is odd, we have*

$$\begin{aligned}
&\sum_{j=1}^{m} F(a_j) \int_a^b \rho(t) H_{0j}(t)dt + \sum_{j=1}^{m} F'(a_j) \int_a^b \rho(t) H_{1j}(t)dt \\
&\leqslant \int_a^b \rho(t)F(t)dt \\
&\leqslant \sum_{i=0}^{m-1} F^{(i)}(a) \int_a^b \rho(t)\bar{H}_{i1}(t)dt + \sum_{i=0}^{m-1} F^{(i)}(b) \int_a^b \rho(t)\bar{H}_{i2}(t)dt,
\end{aligned}$$

where H_{0j} and H_{1j} as in Corollary 6 with $r = m$,

$$\bar{H}_{i1}(t) = \frac{(t-a)^i (t-b)^m}{i!} \sum_{k=0}^{m-1-i} \frac{(-1)^k (m+k-1)!}{[(m-1)!]^2 (a-b)^{m+k}} (t-a)^k, \text{ and}$$

$$\bar{H}_{i2}(t) = \frac{(t-a)^m (t-b)^i}{i!} \sum_{k=0}^{m-1-i} \frac{(-1)^k (m+k-1)!}{[(m-1)!]^2 (b-a)^{m+k}} (t-b)^k.$$

If F is $(2m)$-concave, the inequalities are reversed.

Proof. We use Remark 1 and 2 and then calculate

$$\begin{aligned}
\bar{H}_{i1}(t) &= \frac{1}{i!} \frac{(t-a)^m (t-b)^m}{(t-a)^{m-i}} \sum_{k=0}^{m-1-i} \frac{1}{(m-1)!} \left[\frac{(t-a)^m}{(t-a)^m (t-b)^m} \right]^{(k)}_{t=a} (t-a)^k \\
&= \frac{(t-a)^i (t-b)^m}{i!} \sum_{k=0}^{m-1-i} \frac{(-1)^k (m+k-1)!}{[(m-1)!]^2 (a-b)^{m+k}} (t-a)^k.
\end{aligned}$$

Coefficient $\bar{H}_{i2}(t)$ we get similarly. □

Acknowledgements. The research of the authors has been fully supported by Croatian Science Foundation under the project 5435.

REFERENCES

[1] R. P. AGARWAL AND P. J. Y. WONG, *Error Inequalities in Polynomial Interpolation and Their Applications*, Kluwer Academic Publishers, Dordrecht/Boston/London, 1993.

[2] G. ARAS-GAZIĆ, V. ČULJAK, J. PEČARIĆ AND A. VUKELIĆ, *Generalization of Jensen's inequality by Lidstone's polynomial and related results*, Math. Ineq. Appl. **16** (4) (2013), 1243–1267.

[3] G. ARAS-GAZIĆ, V. ČULJAK, J. PEČARIĆ AND A. VUKELIĆ, *Generalization of Jensen's inequality by Hermite polynomials and related results*, Math. Rep. **17** (67) (2) (2015), Article 4.

[4] G. ARAS-GAZIĆ, V. ČULJAK, J. PEČARIĆ AND A. VUKELIĆ, *Cauchy's error representation of Lidstone interpolating polynomial and related results*, J. Math. Inequal. **9** (4) (2015), 1207–1225.

[5] M. BESSENYEI AND ZS. PÁLES, *Higher-order generalizations of Hadamard's inequality*, Publ. Math. Debrecen **61** (3–4) (2002), 623–643.

[6] S. KARLIN, *Total Positivity*, Stanford Univ. Press, Stanford, 1968.
[7] J. E. PEČARIĆ, F. PROSCHAN AND Y. L. TONG, *Convex functions, partial orderings, and statistical applications*, Mathematics in science and engineering **187**, Academic Press, 1992.
[8] T. POPOVICIU, *Sur l'approximation des fonctions convexes d'ordre superieur*, Mathematica **10**, (1934), 49–54.

(Received March 30, 2016)

Gorana Aras-Gazić
Faculty of Architecture, University of Zagreb
Fra Andrije Kacica Miosica 26, 10000 Zagreb, Croatia
e-mail: garas@arhitekt.hr

Josip Pečarić
Faculty of Textile Technology, University of Zagreb
Prilaz baruna Filipovica 28a, 10000 Zagreb, Croatia
e-mail: pecaric@element.hr

Ana Vukelić
Faculty of Food Technology and Biotechnology
University of Zagreb
Pierottijeva 6, 10000 Zagreb, Croatia
e-mail: avukelic@pbf.hr