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WEAK AND STRONG TYPE ESTIMATES FOR THE COMMUTATORS OF HAUSDORFF OPERATORS

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Abstract. This paper addresses weak and strong type Lebesgue space estimates for the commutators of n-dimensional Hausdorff operators when the symbol functions belong to Lipschitz space. Strong type estimates for such commutators on classical Morrey spaces are established as well.

1. Introduction

Let us begin our discussion by introducing one dimensional Hausdorff operator:

$$h_\Phi f(x) = \int_0^\infty \frac{\Phi(t)}{t} f\left(\frac{x}{t}\right) dt, \tag{1.1}$$

where Φ is a locally intergrable function on $(0,\infty)$. By changing variables, (1.1) assumes the form:

$$\widetilde{h}_\Phi f(x) = \int_0^\infty \frac{\Phi\left(\frac{x}{t}\right)}{t} f(t)dt. \tag{1.2}$$

Having fundamental importance in analysis the operator has been well studied by many authors. Especially, the boundedness of above operators on various function spaces has been a main focus of study in the past (see, [19, 21, 22] for instance). Among some recent developments the important ones are the multidimensional extensions of h_Φ and $\widetilde{h}_\Phi$. In this regard the seminal work was done in [17] and [20]. However, we consider here the following two extensions of (1.1) and (1.2) discussed in [3]:

$$H_\Phi f(x) = \int_{\mathbb{R}^n} \frac{\Phi(y)}{|y|^n} f\left(\frac{x}{|y|}\right) dy,$$

$$\widetilde{H}_{\Phi,\Omega} f(x) = \int_{\mathbb{R}^n} \frac{\Phi(x/|y|)}{|y|^n} \Omega(y') f(y) dy,$$

where $\Omega(y')$ is an integrable function defined on the unit sphere S^{n-1}. When $\Omega = 1$, we denote $\widetilde{H}_{\Phi,1}$ by $\widetilde{H}_\Phi$.

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Besides its importance in analysis, h_Φ takes many classical operators as its special cases if the kernel function Φ is suitably chosen, see [8] for more details. Since we are mainly concerned with the high-dimensional Hausdorff operator, so, by replacing Φ with $\Phi_1(t) = t^{-n}\chi_{(1,\infty)}(t)$ and $\Phi_2(t) = \chi_{(0,1)}(t)$, in the definition of $\widetilde{H}_\Phi$ we obtain n-dimensional Hardy operator and its adjoint operator [6], respectively. The Hardy operator has also undergone several generalizations and refinements in the past [2, 9]. For the recent account of the boundedness results for high-dimensional Hausdorff operator and its generalizations we refer the interested reader to see [4, 5, 7, 8, 11, 12, 13, 18].

On the other hand, boundedness of commutator operators also play very important role not only in the regularity theory to second order elliptic and parabolic partial differential equation (PDEs) but also in the well posedness problems of solution to many kind of PDEs. Especially, the boundedness of commutators can be used to produce characterization of some function spaces. Keeping these facts in mind, some authors considered the commutators of high-dimensional Hausdorff operator on various function spaces [10, 15, 16]. Here, we define the commutators of $\widetilde{H}_{\Phi,\Omega}$ as:

$$\widetilde{H}^b_{\Phi,\Omega}(f) =: b\widetilde{H}_{\Phi,\Omega}f - \widetilde{H}_{\Phi,\Omega}(bf),$$

where b is a locally integrable function on $\mathbb{R}^n$. When $\Omega = 1$, we obtain the commutator operator $\widetilde{H}_{\Phi,b}$ discussed in [10]. Also, the boundedness of

$$H^b_\Phi(f) =: bH_\Phi f - H_\Phi(bf)$$

on Herz-type spaces was obtained in [15]. Remark (3.1) in the same paper indicates that (L^p, L^q) boundedness of H^b_Φ cannot be deduced from the results presented at there. Therefore, there arise a question regarding (L^p, L^q) boundedness of H^b_Φ. To answer this question, contrary to [15], here we use fractional maximal function to control the commutator of H_Φ. Moreover, this technique allows us to study H^b_Φ on classical Morrey spaces.

One of the major result of harmonic analysis is Marcinkiewicz interpolation theorem (see [14]) which uses weak $L^p(\mathbb{R}^n)$ spaces defined as the set of all measurable functions f such that

$$\|f\|_{L^{p,\infty}(\mathbb{R}^n)} = \sup_{\lambda>0} \lambda |\{x \in \mathbb{R}^n : |f(x)| > \lambda\}|^{\frac{1}{p}},$$

where $0 < p < \infty$. Operators that map L^p to L^q are called of strong type (p,q) while operators that map L^p to $L^{q,\infty}$ are called weak type (p,q).

In this paper we will prove that $\widetilde{H}^b_{\Phi,\Omega}$ is of weak type (p,q) when the symbol function b belongs to homogeneous Lipschitz space which, for $0 < \beta < 1$, can be defined as:

$$\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} =: \sup_{x,h \in \mathbb{R}^n} \frac{|b(x+h) - b(x)|}{|h|^\beta} < \infty.$$

Using Marcinkiewicz interpolation theorem it is asserted that the operator also follows the strong type estimates. In addition, when $b \in \dot{\Lambda}_\beta$, we obtain some size conditions on Φ so that the operator H^b_Φ is bounded from classical Morrey space $L^{p,\lambda}(\mathbb{R}^n)$ to

$L^{q,\lambda}(\mathbb{R}^n)$. The space $L^{p,\lambda}(\mathbb{R}^n)$ was first introduced by Morrey [24] in 1938 to study the local behavior of second order elliptic and parabolic PDEs. For $1 \leqslant p < \infty$, $0 \leqslant \lambda \leqslant n$, we define the Morrey space $L^{p,\lambda}(\mathbb{R}^n)$ as:

$$L^{p,\lambda}(\mathbb{R}^n) = \left\{ f \in L^p_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{L^{p,\lambda}(\mathbb{R}^n)} < \infty \right\},$$

where

$$\|f\|_{L^{p,\lambda}(\mathbb{R}^n)} = \sup_{r>0, x_0 \in \mathbb{R}^n} \left(\frac{1}{r^\lambda} \int_{Q(x_0,r)} |f(x)|^p dx \right)^{1/p},$$

and $Q = Q(x_0, r)$ denotes the cube centered at x_0 with side length r along the coordinate axes. It is easy to see that $L^{p,0}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ and $L^{p,n}(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$. If $n < \lambda$, then we have $L^{p,\lambda}(\mathbb{R}^n) = 0$. Hence, we only consider the case $0 < \lambda < n$.

In the sequel, we shall use the notation $A \preceq B$ to mean that there exist a positive constant C independent of all essential variables such that $A \leqslant CB$.

2. Weak and strong estimates for $\widetilde{H}^b_{\Phi,\Omega}$

We begin this section by stating first the following weak type result for $\widetilde{H}^b_{\Phi,\Omega}$

THEOREM 1. *Let $1 < p, q < \infty$, $0 < \beta < 1$, $n > \beta p$ and $1/p - \beta/n = 1/q$. If Φ is a radial function, $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ and*

$$C_{\beta,p} = \left(\int_0^\infty \frac{|\Phi(t)|^{p'}}{t^{n(1-p')+1}} \max\{1, t^{-\beta p'}\} dt \right)^{1/p'} < \infty,$$

then

$$\|\widetilde{H}^b_{\Phi,\Omega} f\|_{L^{q,\infty}(\mathbb{R}^n)} \leqslant \left(\frac{|S^{n-1}|}{n} \right)^q C_{\beta,p} \|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \|\Omega\|_{L^{p'}(S^{n-1})} \|f\|_{L^p(\mathbb{R}^n)},$$

where $|S^{n-1}|$ denotes the volume of unit sphere S^{n-1}.

Proof. Note that

$$\begin{aligned}
|\widetilde{H}^b_{\Phi,\Omega} f(x)| &\leqslant \left| \int_{\mathbb{R}^n} \frac{\Phi(|x|/|y|)}{|y|^n} \Omega(y') (b(x) - b(0)) f(y) dy \right| \\
&\quad + \left| \int_{\mathbb{R}^n} \frac{\Phi(|x|/|y|)}{|y|^n} \Omega(y') (b(y) - b(0)) f(y) dy \right| \\
&\leqslant \|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} |x|^\beta \int_{\mathbb{R}^n} \left| \frac{\Phi(|x|/|y|)}{|y|^n} \Omega(y') f(y) \right| dy \\
&\quad + \|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \int_{\mathbb{R}^n} \left| \frac{\Phi(|x|/|y|)}{|y|^{n-\beta}} \Omega(y') f(y) \right| dy \\
&=: I_1 + I_2.
\end{aligned}$$

We use Hölder's inequality to approximate I_2 as:

$$I_2 \leqslant \|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \left(\int_{\mathbb{R}^n} \left| \frac{\Phi(|x|/|y|)}{|y|^{n-\beta}} \Omega(y') \right|^{p'} dy \right)^{1/p'} \|f\|_{L^p(\mathbb{R}^n)}.$$

With the help of polar coordinates and change of variables it is easy to see that

$$\left(\int_{\mathbb{R}^n} \left| \frac{\Phi(|x|/|y|)}{|y|^{n-\beta}} \Omega(y') \right|^{p'} dy \right)^{1/p'} = \|\Omega\|_{L^{p'}(S^{n-1})} |x|^{-\frac{n}{p}+\beta} \left(\int_0^\infty \frac{|\Phi(t)|^{p'}}{t^{n(1-p')+1}} t^{-\beta p'} dt \right)^{1/p'}$$

Therefore,

$$I_2 \leqslant \|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \|\Omega\|_{L^{p'}(S^{n-1})} |x|^{-\frac{n}{p}+\beta} \left(\int_0^\infty \frac{|\Phi(t)|^{p'}}{t^{n(1-p')+1}} t^{-\beta p'} dt \right)^{1/p'} \|f\|_{L^p(\mathbb{R}^n)}.$$

By a similar argument we estimate I_1 as:

$$I_1 \leqslant \|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \|\Omega\|_{L^{p'}(S^{n-1})} |x|^{-\frac{n}{p}+\beta} \left(\int_0^\infty \frac{|\Phi(t)|^{p'}}{t^{n(1-p')+1}} dt \right)^{1/p'} \|f\|_{L^p(\mathbb{R}^n)}.$$

From these estimates for I_1 and I_2, one can have

$$|\widetilde{H}^b_{\Phi,\Omega} f(x)| \leqslant C_{\beta,p} \|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \|\Omega\|_{L^{p'}(S^{n-1})} |x|^{-\frac{n}{q}} \|f\|_{L^p(\mathbb{R}^n)},$$

where we have used the condition $1/p - \beta/n = 1/q$. Now for any $\lambda > 0$, we have

$$\begin{aligned}
&|\{x \in \mathbb{R}^n : |\widetilde{H}^b_{\Phi,\Omega} f(x)| > \lambda\}| \\
&\leqslant |\{x \in \mathbb{R}^n : C_{\beta,p} \|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \|\Omega\|_{L^{p'}(S^{n-1})} |x|^{-\frac{n}{q}} \|f\|_{L^p(\mathbb{R}^n)} > \lambda\}| \\
&\leqslant \left| \left\{ x \in \mathbb{R}^n : |x|^n < \frac{C^q_{\beta,p} \|b\|^q_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \|\Omega\|^q_{L^{p'}(S^{n-1})} \|f\|^q_{L^p(\mathbb{R}^n)}}{\lambda^q} \right\} \right|,
\end{aligned}$$

which implies that $\|\widetilde{H}^b_{\Phi,\Omega} f\|_{L^{q,\infty}(\mathbb{R}^n)} \preceq \|f\|_{L^p(\mathbb{R}^n)}$. Thus, we conclude that $\widetilde{H}^b_{\Phi,\Omega} f$ is of weak type (p,q). $\square$

In the next theorem of this section, we are going to prove that strong type estimates also hold for $\widetilde{H}^b_{\Phi,\Omega} f$.

THEOREM 2. *Let* $1 < r,s < \infty$, $0 < \beta < 1$, $n > \beta r$, $1/r - \beta/n = 1/s$. *If* Φ *is a radial function,* $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ *and for small* $\varepsilon > 0$, $C_{\beta,r\pm\varepsilon} < \infty$, *then*

$$\|\widetilde{H}^b_{\Phi,\Omega} f\|_{L^s(\mathbb{R}^n)} \preceq \|f\|_{L^r(\mathbb{R}^n)}.$$

Proof. Since r and s run through open interval, therfore, we can select r_1 and r_2 such that $r_1 = r - \varepsilon$ and $r_2 = r + \varepsilon$ and choose s_1 and s_2 such that $s_1 < s < s_2$ satisfying

$$\frac{1}{r_i} - \frac{\beta}{n} = \frac{1}{s_i}, \quad (i = 1,2),$$

then by Theorem 1, we have

$$\|\widetilde{H}^b_{\Phi,\Omega} f\|_{L^{s_i,\infty}(\mathbb{R}^n)} \preceq \|f\|_{L^{r_i}(\mathbb{R}^n)}, \quad (i = 1,2).$$

Hence, using Marcinkiewicz interpolation theorem, we obtain

$$\|\widetilde{H}^b_{\Phi,\Omega} f\|_{L^{s}(\mathbb{R}^n)} \preceq \|f\|_{L^{r}(\mathbb{R}^n)}.$$

Which is as required. □

REMARK 1. Taking $\Omega = 1$ in Theorem 1, one can deduce the weak type (p,q) estimates for the commutators of Hardy operators (see, [23] for definitions)

3. Boundedness of H^b_Φ on Morrey space

Given a locally integrable function f and β, $0 \leqslant \beta < n$, define the fractional maximal function by

$$M_\beta f(x) = \sup_{Q \ni x} \frac{1}{|Q|^{1-\frac{\beta}{n}}} \int_Q |f(y)| dy,$$

where the supremum is taken over all cubes Q containing x. When $\beta = 0$, then $M_0 f = Mf$ denotes the usual Hardy Littlewood maximal function. For the boundedness of M_β on Morrey space we have the following lemma from [1].

LEMMA 1. *Let $0 < \beta < n$, $1 < p < n/\beta$, $0 < \lambda < n - \beta p$ and $1/q = 1/p - \beta/(n-\lambda)$. Then M_β is bounded from $L^{p,\lambda}(\mathbb{R}^n)$ to $L^{q,\lambda}(\mathbb{R}^n)$.*

The next lemma is very useful in proving our main result for this section.

LEMMA 2. ([25]) *If $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ with $0 < \beta < 1$, then for any cube $Q \subset \mathbb{R}^n$, $\sup_{x \in Q} |b(x) - b_Q| \leqslant C|Q|^{\frac{\beta}{n}} \|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)}$ where $b_Q = \frac{1}{|Q|}\int_Q b$.*

THEOREM 3. *Let $0 < \beta < 1$, $1 < p < n/\beta$, $0 < \lambda < n - \beta p$ and $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$. If $1/q = 1/p - \beta/(n-\lambda)$ and*

$$C_{\beta,\lambda,p} = \int_{\mathbb{R}^n} \frac{|\Phi(y)|}{|y|^n} |y|^{\frac{n+\lambda}{p}} (1 + |y|^\beta) dy < \infty,$$

then we have

$$\|H^b_\Phi f\|_{L^{q,\lambda}(\mathbb{R}^n)} \preceq \|f\|_{L^{p,\lambda}(\mathbb{R}^n)}.$$

Proof. Let $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$. For any ball $Q \subseteq \mathbb{R}^n$ such that $z \in Q$, we have

$$\begin{aligned}
\frac{1}{|Q|}\int_Q |H_\Phi^b f(x)|dx &\leqslant \frac{1}{|Q|}\int_Q\int_{\mathbb{R}^n}\frac{|\Phi(y)|}{|y|^n}|(b(x)-b(x|y|^{-1}))f(x|y|^{-1})|dydx \\
&= \frac{1}{|Q|}\int_{\mathbb{R}^n}\frac{|\Phi(y)|}{|y|^n}\int_Q |(b(x)-b(x|y|^{-1}))f(x|y|^{-1})|dxdy \\
&\leqslant \frac{1}{|Q|}\int_{\mathbb{R}^n}\frac{|\Phi(y)|}{|y|^n}\int_Q |(b(x)-b_Q))f(x|y|^{-1})|dxdy \\
&\quad + \frac{1}{|Q|}\int_{\mathbb{R}^n}\frac{|\Phi(y)|}{|y|^n}\int_Q |(b_Q-b_{|y|^{-1}Q})f(x|y|^{-1})|dxdy \\
&\quad + \frac{1}{|Q|}\int_{\mathbb{R}^n}\frac{|\Phi(y)|}{|y|^n}\int_Q |b(x|y|^{-1})-b_{|y|^{-1}Q})f(x|y|^{-1})|dxdy \\
&= I+II+III.
\end{aligned}$$

Using lemma 2, we approximate I as

$$\begin{aligned}
I &\leqslant C\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)}\int_{\mathbb{R}^n}\frac{|\Phi(y)|}{|y|^n}\left(\frac{1}{|Q|^{1-\beta/n}}\int_Q |f(x|y|^{-1})|dx\right)dy \\
&\leqslant C\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)}\int_{\mathbb{R}^n}\frac{|\Phi(y)|}{|y|^n}|y|^\beta \left(M_\beta f(|y|^{-1}z)\right)dy.
\end{aligned}$$

Similarly, for III, we have

$$III \leqslant C\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)}\int_{\mathbb{R}^n}\frac{|\Phi(y)|}{|y|^n}\left(M_\beta f(|y|^{-1}z)\right)dy.$$

It remain to estimate II. For $0<\beta<1$, we first compute

$$\begin{aligned}
|b_Q-b_{|y|^{-1}Q}| &\leqslant \frac{1}{|Q|}\int_Q |b(x)-b_{|y|^{-1}Q}|dx \\
&\leqslant \frac{1}{|Q|}\frac{1}{|y^{-1}Q|}\int_Q\int_{|y|^{-1}Q}|b(x)-b(t)|dtdx \\
&\leqslant \|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)}\left(\frac{1}{|Q|}\int_Q |x|^\beta dx+\frac{1}{|y^{-1}Q|}\int_{|y|^{-1}Q}|t|^\beta dt\right) \\
&\leqslant C|Q|^{\beta/n}\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)}(1+|y|^{-\beta})
\end{aligned}$$

Therefore.

$$\begin{aligned}
II &\leqslant C\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)}\int_{\mathbb{R}^n}\frac{|\Phi(y)|}{|y|^n}(1+|y|^{-\beta})\left(\frac{1}{|Q|^{1-\beta/n}}\int_Q |f(x|y|^{-1})|dx\right)dy \\
&\leqslant C\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)}\int_{\mathbb{R}^n}\frac{|\Phi(y)|}{|y|^n}(1+|y|^{\beta})\left(M_\beta f(|y|^{-1}z)\right)dy.
\end{aligned}$$

We combine the estimates for I, II and III, to have

$$\frac{1}{|Q|}\int_Q |H_\Phi^b f(x)|dx \leqslant C\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)}\int_{\mathbb{R}^n}\frac{|\Phi(y)|}{|y|^n}(1+|y|^{\beta})\left(M_\beta f(|y|^{-1}z)\right)dy.$$

Taking the supremum over all Q such that $x \in Q$ along with $L^{q,\lambda}(\mathbb{R}^n)$ norm on both sides and using the Minkowski inequality, we obtain

$$\begin{aligned}\|M(H_{\Phi}^{b}f)(\cdot)\|_{L^{q,\lambda}(\mathbb{R}^n)} &\leqslant C\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)}\int_{\mathbb{R}^n}\frac{|\Phi(y)|}{|y|^n}(1+|y|^{\beta})\|M_\beta f(|y|^{-1}\cdot)\|_{L^{q,\lambda}(\mathbb{R}^n)}dy \\ &= C\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)}\int_{\mathbb{R}^n}\frac{|\Phi(y)|}{|y|^n}(1+|y|^{\beta})|y|^{n/p+\lambda/p}dy\|M_\beta f(\cdot)\|_{L^{q,\lambda}(\mathbb{R}^n)}.\end{aligned}$$

Since $H_{\Phi}^{b}f \in L^1_{loc}(\mathbb{R}^n)$, therefore, by Lebesgue differentiation theorem $|H_{\Phi}^{b}f(z)| \leqslant M\left(H_{\Phi}^{b}f\right)(z)$ *a.e.* Finally, an application of Lemma 1 yields the desired result. □

Now the question of (L^p, L^q) boundedness of H_{Φ}^{b} can be answered by stating the following theorem:

THEOREM 4. *Let* $0<\beta<1$, $1<p<n/\beta$, *and* $b\in\dot{\Lambda}_\beta(\mathbb{R}^n)$. *If* $1/q=1/p-\beta/n$ *and*

$$C'_{\beta,p}=\int_{\mathbb{R}^n}\frac{|\Phi(y)|}{|y|^n}|y|^{\frac{n}{p}}(1+|y|^{\beta})dy<\infty,$$

then we have

$$\|H_{\Phi}^{b}f\|_{L^q(\mathbb{R}^n)} \preceq \|f\|_{L^p(\mathbb{R}^n)}.$$

Proof. The proof is similar to the proof of Theorem 3, so we omit the details. □

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