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QUALITATIVE UNCERTAINTY PRINCIPLES FOR THE GENERALIZED HARTLEY TRANSFORM

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Dedicated to the Professor Emeritus Khalifa Trimeche for his 70 birthday

(*Communicated by J. Pečarić*)

Abstract. We consider a new differential-difference operator Λ_s on the real line. We study the harmonic analysis associated with this operator. Next, we prove various mathematical aspects of the qualitative uncertainty principles, including Hardy's, Morgan's, Cowling-Price's and its variants, Beurling's, Gelfand-Shilov's, Miyachi's theorems for the generalized Hartley transform associated to the operator Λ_s.

1. Introduction

The Hartley transform is an integral transform, attributed to Ralph Vinton Lyon Hartley (cf. [2, 13]). This transform permits a function to be decomposed into two independent sets of sinusoidal components, these sets are represented in terms of positive and negative frequency components, respectively. This is in contrast to the complex exponential, $\exp(i\lambda x)$, used in classical Fourier analysis. The Hartley transform has the advantages over the Fourier transform, it transforms real functions to real functions (as opposed to requiring complex numbers), also this transform has complementary symmetry properties with respect to its real and imaginary axis and of being its own inverse. Moreover, is well known that the Hartley transform is in connection with various applications in mathematical physics. The familiar reciprocal pair of the Hartley transforms for f in a suitable functions class, is given by

$$\begin{aligned} \mathscr{H}_C(f)(\lambda) &= \int_{\mathbb{R}} f(x)cas(\lambda x)dx \\ f(x) &= \frac{1}{2\pi}\int_{\mathbb{R}} \mathscr{H}_C(f)(\lambda)cas(\lambda x)d\lambda \end{aligned} \tag{1.1}$$

where

$$\text{cas}(x) := \cos x + \sin x. \tag{1.2}$$

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The cas function (1.2) can be considered as a generalization of the exponential function exp. A simple computation shows that the cas function is the unique C^∞ solution of the differential-reflection problem

$$(Y_r f)(x) = \lambda f(x), \quad f(0) = 1, \tag{1.3}$$

where Y_r is the differential-reflection operator defined by

$$(Y_r f)(x) = \Big(\Big(\frac{d}{dx}\circ r\Big) f\Big)(x) := f'(-x). \tag{1.4}$$

Here r is the reflection, acting on function f as

$$r(f)(x) =: f(-x).$$

Further, the function $\mathrm{cas}(x)$ satisfies the product formula

$$\mathrm{cas}(x)\,\mathrm{cas}(y) = \frac{1}{2}\Big((1-r)\,\mathrm{cas}\Big)(x+y) + \frac{1}{2}\Big((1+r)\,\mathrm{cas}\Big)(x-y).$$

This allows us to define the generalized translation operator related to the differential-reflection operator Y_r by

$$\sigma_x^r f(y) = \frac{1}{2}\Big((1-r)f\Big)(x+y) + \frac{1}{2}\Big((1+r)f\Big)(x-y), \tag{1.5}$$

and the generalized convolution product related to the differential-reflection operator Y_r by

$$f *_r g(x) = \int_{\mathbb{R}} \sigma_x^r f(y) g(y) dy. \tag{1.6}$$

The Hartley transform has the following properties:

$$\begin{aligned} \mathscr{H}_C(Y_r f)(\lambda) &= -\lambda \mathscr{H}_C(f)(\lambda), \\ \mathscr{H}_C(\sigma_x^r f)(\lambda) &= \mathrm{cas}(\lambda x)\mathscr{H}_C(f)(\lambda), \\ \mathscr{H}_C(f *_r g)(\lambda) &= \mathscr{H}_C(f)(\lambda)\mathscr{H}_C(g)(\lambda). \end{aligned}$$

In this paper, we consider the first-order singular differential-difference operator on $\mathbb{R}$

$$\Lambda_s f(x) = Y_r f(x) + \frac{A'(x)}{A(x)}\left(\frac{f(x)-f(-x)}{2}\right) - s\rho f(-x),$$

where $s \in [-1,1]$

$$A(x) = |x|^{2\alpha+1}B(x), \quad \alpha > -\frac{1}{2},$$

B being a positive C^∞ even function on $\mathbb{R}$. We suppose in addition that

i) For all $x \geqslant 0$, $A(x)$ is increasing and $\lim\limits_{x\to\infty} A(x) = \infty$.

ii) For all $x > 0$, $\dfrac{A'(x)}{A(x)}$ is decreasing and $\lim\limits_{x\to\infty} \dfrac{A'(x)}{A(x)} = 2\rho \geqslant 0$.

iii) There exists a constant $\delta > 0$ such that for all $x \in [x_0, \infty)$, $x_0 > 0$, we have

$$\frac{A'(x)}{A(x)} = \begin{cases} 2\rho + e^{-\delta x} D(x), & \text{if } \rho > 0 \\ \frac{2\alpha+1}{x} + e^{-\delta x} D(x) & \text{if } \rho = 0. \end{cases}$$

where D is a C^∞-function, bounded together with its derivatives.

Due to our assumptions on the function A there is a positive constant C such that

$$\forall x \in \mathbb{R}, \quad A(x) \leqslant \begin{cases} C e^{2\rho|x|} & \text{if } \rho > 0 \\ C|x|^{2\alpha+1} & \text{if } \rho = 0. \end{cases} \tag{1.7}$$

The purpose of the present paper is twofold. On one hand, we want to provide a new harmonic analysis on the real line corresponding to the differential-difference operator Λ_s. More precisely, we study the generalized Hartley transform associated to the operator Λ_s, we prove an inversion formula, a Plancherel and a Paley-Wiener theorem, and we show how the intertwining operator V_s can be used to define generalized translation operators and a convolution structure naturally associated to the operator Λ_s. On the other hand we want to study the qualitative uncertainty principles for the generalized Hartley transform.

Classical qualitative uncertainty principles are not inequalities, but are theorems that tell us how a function (and its Fourier transform) behave under certain circumstances. For example: Hardy [12], Morgan [18], Cowling and Price [6], Beurling [3], Miyachi [17] theorems enter within the framework of the qualitative uncertainty principles.

The qualitative uncertainty principles has been studied by many authors for various Fourier transforms, for examples (cf. [10, 15, 25]) and others.

In this paper, we prove Hardy's theorem, Cowling-Price's theorem, Morgan's theorem, Ray-Sarkar's theorem, Miyachi's theorem, Beurling's theorem and Gelfand-Shilov's theorem for the generalized Hartley transform associated to the operator Λ_s.

The outline of this paper is as follow: In §2, we study the harmonic analysis associated with the operator Λ_s. More precisely, we study the eigenfunctions of this operator. In particular we prove the Laplace integral representation for these eigenfunctions. Next, we prove the inversion theorems and Plancherel's formula for the Hartley transform associated to the operator Λ_s noted $\mathscr{H}_{\Lambda_s}$. In §3 we prove an L^p version of Hardy's theorem for the generalized Hartley transform. §4 is devoted to generalize Morgan's theorem for the generalized Hartley transform $\mathscr{H}_{\Lambda_s}$. In §5, we prove the Ray-Sarkar's version of the Cowling-Price's theorem for the generalized Hartley transform $\mathscr{H}_{\Lambda_s}$. §6 is devoted to obtain Beurling's theorem for $\mathscr{H}_{\Lambda_s}$ and in the last section, we establish the Miyachi's theorem for $\mathscr{H}_{\Lambda_s}$.

2. Harmonic analysis associated with the operator Λ_s

In this section we study the harmonic analysis related to the operator Λ_s.

NOTATION. We denote by
$\mathscr{P}_m(\mathbb{R})$ the set of homogeneous polynomials of degree m.
$\mathscr{E}_e(\mathbb{R})$ the space of even C^∞-functions on $\mathbb{R}$.
$\mathscr{E}(\mathbb{R})$ the space of C^∞-functions on $\mathbb{R}$.
$\mathscr{S}(\mathbb{R})$ the Schwartz space of rapidly decreasing functions on $\mathbb{R}$.
$\mathscr{S}_e(\mathbb{R})$ (resp. $\mathscr{S}_o(\mathbb{R})$) the subspace of $\mathscr{S}(\mathbb{R})$ consisting of even (resp. odd) functions.
$D_e(\mathbb{R})$ the space of even C^∞-functions on $\mathbb{R}$ which are of compact support.
$D(\mathbb{R})$ the space of C^∞-functions on $\mathbb{R}$ which are of compact support.
$D'(\mathbb{R})$ the space of distributions with compact support on $\mathbb{R}$.

2.1. The eigenfunctions of the operator Λ_s

To study the eigenfunctions of Λ_s, we consider first those of the second-order singular differential operator on $\mathbb{R}_+$ defined by

$$L = \frac{d^2}{dx^2} + \frac{A'(x)}{A(x)}\frac{d}{dx}.$$

Our basic reference about L will be the papers [22, 24] from which we recall the following result.

LEMMA 1. *(i) For each $\lambda \in \mathbb{C}$ the differential equation*

$$Lu = -(\lambda^2 + \rho^2)u, \quad u(0) = 1, \quad u'(0) = 0, \tag{2.8}$$

admits a unique even C^∞ solution on $\mathbb{R}$, denoted φ_λ.
(ii) For every $x \in \mathbb{R}$, the function $\lambda \mapsto \varphi_\lambda(x)$ is analytic.
(iii) For every $x \in \mathbb{R}$,

$$e^{-\rho|x|} \leqslant \varphi_0(x) \leqslant 1. \tag{2.9}$$

(iv) For all $x > 0$ and $\lambda \in \mathbb{C}$, the function φ_λ possess the Laplace type integral representation

$$\varphi_{\sqrt{\lambda^2-\rho^2}}(x) = \int_0^x \mathscr{K}(x,y)\cos(\lambda y)dy \tag{2.10}$$

where $\mathscr{K}(x,.)$ is a positive continuous function on $(-|x|,|x|)$, with support in $[-|x|,|x|]$.
(v) There is a positive constant C such that for all $\lambda \in \mathbb{C}$, $x \in \mathbb{R}$ and $n \in \mathbb{N}_0$, we have

$$\left|\frac{d^n}{d\lambda^n}\varphi_\lambda(x)\right| \leqslant C|x|^n e^{(|\mathrm{Im}\lambda|-\rho)|x|}. \tag{2.11}$$

REMARK 1. If $A(x)=(\sinh|x|)^{2\alpha+1}(\cosh x)^{2\beta+1}$, $\alpha\geqslant\beta\geqslant\frac{-1}{2}$, $\alpha\neq\frac{-1}{2}$, then the differential operator L reduced to the so-called Jacobi operator. The eigenfunction φ_λ is the Jacobi function of index (α,β) given by

$$\varphi_\lambda(x)=F\Big(\frac{1}{2}(\rho+\lambda),\frac{1}{2}(\rho-\lambda);\ \alpha+1;\ -(\sinh(x))^2\Big), \tag{2.12}$$

where F is the hypergeometric function ${}_2F_1$ of Gauss.

PROPOSITION 1. *For each $\lambda\in\mathbb{C}$ the differential-difference equation*

$$\Lambda_s u=\lambda u,\quad u(0)=1, \tag{2.13}$$

admits a unique C^∞ solution on $\mathbb{R}$, denoted $\Phi_s(\lambda,.)$ and given by

$$\Phi_s(\lambda,x)=\begin{cases}\varphi_{\sqrt{\lambda^2-(s^2+1)\rho^2}}(x)+\frac{1}{\lambda-s\rho}Y_r\varphi_{\sqrt{\lambda^2-(s^2+1)\rho^2}}(x), & \text{if } \lambda\neq s\rho\\ 1+sgn(x)\frac{2s\rho}{A(x)}\int_0^{|x|}A(t)dt, & \text{if } \lambda=s\rho.\end{cases} \tag{2.14}$$

Proof. Write $u=u_e+u_o$ with

$$u_e(x)=\frac{u(x)+u(-x)}{2}\quad\text{and}\quad u_o(x)=\frac{u(x)-u(-x)}{2}.$$

Then (2.13) is equivalent to the system

$$\begin{cases}Y_ru_o+\frac{A'(x)}{A(x)}u_o &=(\lambda+s\rho)u_e\\ Y_ru_e &=(\lambda-s\rho)u_o\\ u_e(0) &=1.\end{cases} \tag{2.15}$$

If $\lambda\neq s\rho$, the identity (2.14) is now immediate from Lemma 1 and the relation (2.15). On the other hand we have

$$Y_r\varphi_{\sqrt{\lambda^2-(s^2+1)\rho^2}}(x)=sgn(x)\frac{(\lambda^2-s^2\rho^2)}{A(x)}\int_0^{|x|}\varphi_{\sqrt{\lambda^2-(s^2+1)\rho^2}}(t)A(t)dt.$$

Then, for $\lambda\neq s\rho$

$$\frac{1}{\lambda-s\rho}Y_r\varphi_{\sqrt{\lambda^2-(s^2+1)\rho^2}}(x)=sgn(x)\frac{(\lambda+s\rho)}{A(x)}\int_0^{|x|}\varphi_{\sqrt{\lambda^2-(s^2+1)\rho^2}}(t)A(t)dt. \tag{2.16}$$

Thus if $\lambda=s\rho$, we have

$$\begin{aligned}u(x)&=u_e(x)+u_o(x)\\ &=\lim_{\lambda\to s\rho}\Big(\varphi_{\sqrt{\lambda^2-(s^2+1)\rho^2}}(x)+\frac{1}{\lambda-s\rho}Y_r\varphi_{\sqrt{\lambda^2-(s^2+1)\rho^2}}(x)\Big)\\ &=\varphi_{i\rho}(x)+sgn(x)\frac{2s\rho}{A(x)}\int_0^{|x|}\varphi_{i\rho}(t)A(t)dt\\ &=1+sgn(x)\frac{2s\rho}{A(x)}\int_0^{|x|}A(t)dt.\quad\square\end{aligned}$$

REMARK 2. For all $x \in \mathbb{R} \setminus \{0\}$ and $\lambda \in \mathbb{C}$, we can write

$$\Phi_s(\lambda,x) = \varphi_{\sqrt{\lambda^2-(s^2+1)\rho^2}}(x) + \operatorname{sgn}(x)\,\frac{\lambda+s\rho}{A(x)}\int_0^{|x|} \varphi_{\sqrt{\lambda^2-(s^2+1)\rho^2}}(t)A(t)dt. \quad (2.17)$$

The eigenfunctions Φ_s possesses the following properties:

i) There is a positive constant C such that for all $\lambda \in \mathbb{C}$, $x \in \mathbb{R}$ and $n \in \mathbb{N}_0$, we have

$$\left|\frac{d^n}{d\lambda^n}\Phi_s(\lambda,x)\right| \leqslant C(1+|\lambda|)|x|^n e^{|\mathrm{Im}\lambda|\,|x|}. \quad (2.18)$$

ii) There exists a positive constant C such that for all $x \in \mathbb{R}$ and $\lambda \in \mathbb{R}$, with $|\lambda| \geqslant \sqrt{s^2+1}\rho$, we have

$$|\Phi_s(\lambda,x)| \leqslant C(1+|x|)(1+|\lambda|+\rho)e^{-\rho|x|}.$$

iii) There exists a positive constant C such that for all $x \in \mathbb{R}$ and $\lambda \in \mathbb{R}$, with $|\lambda| \leqslant \sqrt{s^2+1}\rho$, we have

$$|\Phi_s(\lambda,x)| \leqslant M(1+|x|)(1+|\lambda|+\rho)e^{(\sqrt{(s^2+1)\rho^2-\lambda^2}-\rho)\,|x|}.$$

The following lemma shall be useful.

LEMMA 2. *(i) For all $x > 0$ and $\lambda \in \mathbb{C}$, we have*

$$\varphi_{\sqrt{\lambda^2-(s^2+1)\rho^2}}(x) = \int_0^x \mathscr{K}(x,y)\cos((\sqrt{\lambda^2-s^2\rho^2})y)dy \quad (2.19)$$

where $\mathscr{K}(x,.)$ is the positive continuous function on $(-|x|,|x|)$, with support in $[-|x|,|x|]$ given in (2.10).

(ii) For all $x \in \mathbb{R} \setminus \{0\}$ and $\lambda \in \mathbb{C}$, we have

$$\begin{aligned}&\frac{\lambda+s\rho}{A(x)}\int_0^{|x|} \varphi_{\sqrt{\lambda^2-(s^2+1)\rho^2}}(t)A(t)dt\\ &= \frac{1}{2A(x)}\int_{-|x|}^{|x|}\Big(s\rho\mathscr{G}_{\mathscr{K}}(x,y) - \frac{\partial}{\partial y}\mathscr{G}_{\mathscr{K}}(x,y)\Big)\operatorname{cas}(\sqrt{\lambda^2-s^2\rho^2}\,y)dy, \quad (2.20)\end{aligned}$$

where

$$\mathscr{G}_{\mathscr{K}}(x,y) = \begin{cases}\displaystyle\int_{|y|}^{|x|}\mathscr{K}(t,y)A(t)\,dt & \text{if } |y|<|x|\\ 0 & \text{if } |y| \geqslant |x|.\end{cases}$$

Proof. The first assertion (i) is immediately from (2.10).

Now we want to prove (ii). From the relation (2.19) and Fubini's theorem, we have

$$\begin{aligned}\frac{\lambda+s\rho}{A(x)}\int_0^{|x|}\varphi_{\sqrt{\lambda^2-s^2\rho^2}}(t)A(t)dt &= \frac{\lambda+s\rho}{A(x)}\int_0^{|x|}\Big(\int_0^t\cos(\sqrt{\lambda^2-s^2\rho^2}y)dy\mathscr{K}(t,y)\Big)A(t)dt\\ &= \frac{\lambda+s\rho}{A(x)}\int_0^{|x|}\Big(\int_{|y|}^{|x|}\mathscr{K}(t,y)A(t)dt\Big)\cos(\sqrt{\lambda^2-s^2\rho^2}y)dy\\ &= \frac{\lambda+s\rho}{A(x)}\int_0^{|x|}\mathscr{G}_{\mathscr{K}}(x,y)\cos(\sqrt{\lambda^2-s^2\rho^2}y)dy. \qquad (2.21)\end{aligned}$$

Moreover, by integration by parts, we obtain

$$\frac{\lambda}{A(x)}\int_0^{|x|}\mathscr{G}_{\mathscr{K}}(x,y)\cos(\sqrt{\lambda^2-s^2\rho^2}y)dy=\frac{-1}{2A(x)}\int_{-|x|}^{|x|}\frac{\partial}{\partial y}\mathscr{G}_{\mathscr{K}}(x,y)\sin(\sqrt{\lambda^2-s^2\rho^2}y)dy.$$

On the other hand

$$\frac{s\rho}{A(x)}\int_0^{|x|}\mathscr{G}_{\mathscr{K}}(x,y)\cos(\sqrt{\lambda^2-s^2\rho^2}y)dy=\frac{s\rho}{2A(x)}\int_{-|x|}^{|x|}\mathscr{G}_{\mathscr{K}}(x,y)\cos(\sqrt{\lambda^2-s^2\rho^2}y)dy.$$

Thus

$$\begin{aligned}&\frac{\lambda+s\rho}{A(x)}\int_0^{|x|}\mathscr{G}_{\mathscr{K}}(x,y)\cos(\sqrt{\lambda^2-s^2\rho^2}y)dy\\ &=\frac{1}{2A(x)}\int_{-|x|}^{|x|}\Big(s\rho\mathscr{G}_{\mathscr{K}}(x,y)-\frac{\partial}{\partial y}\mathscr{G}_{\mathscr{K}}(x,y)\Big)\operatorname{cas}(\sqrt{\lambda^2-s^2\rho^2}y)dy.\end{aligned}$$

This finishes the proof. □

THEOREM 1. *The eigenfunction $\Phi_s(\lambda,x)$ have the following Laplace integral representation*

$$\forall x\in\mathbb{R}\setminus\{0\},\forall\lambda\in\mathbb{C},\quad \Phi_s(\lambda,x)=\int_{-|x|}^{|x|}K_s(x,y)\operatorname{cas}((\sqrt{\lambda^2-s^2\rho^2})y)\,dy \qquad (2.22)$$

where $K_s(x,y)$ is a continuous function on $(-|x|,|x|)$, with support in $[-|x|,|x|]$ given by

$$K_s(x,y)=\frac{1}{2}\mathscr{K}(x,y)+s\rho\frac{sgn(x)}{2A(x)}\mathscr{G}_{\mathscr{K}}(x,y)-\frac{sgn(x)}{2A(x)}\frac{\partial}{\partial y}\mathscr{G}_{\mathscr{K}}(x,y). \qquad (2.23)$$

Proof. For all $x\in\mathbb{R}\setminus\{0\}$ and $\lambda\in\mathbb{C}$, the relations (2.17), (2.19) and (2.20), give that

$$\begin{aligned}\Phi_s(\lambda,x) &= \varphi_{\sqrt{\lambda^2-(s^2+1)\rho^2}}(x)+\operatorname{sgn}(x)\,\tfrac{\lambda+s\rho}{A(x)}\int_0^{|x|}\varphi_{\sqrt{\lambda^2-(s^2+1)\rho^2}}(y)A(y)dy\\ &= \int_0^{|x|}\mathscr{K}(x,y)\cos((\sqrt{\lambda^2-s^2\rho^2})y)dy\\ &\quad +sgn(x)\tfrac{\lambda+s\rho}{A(x)}\int_0^{|x|}\varphi_{\sqrt{\lambda^2-(s^2+1)\rho^2}}(y)A(y)dy\\ &= \int_{-|x|}^{|x|}\Big(\frac{1}{2}\mathscr{K}(x,y)+\frac{s\rho sgn(x)}{2A(x)}\mathscr{G}_{\mathscr{K}}(x,y)-\frac{sgn(x)}{2A(x)}\frac{\partial}{\partial y}\mathscr{G}_{\mathscr{K}}(x,y)\Big)\\ &\quad\times\operatorname{cas}(\sqrt{\lambda^2-s^2\rho^2}y)A(y)dy.\end{aligned}$$

Thus

$$\Phi_s(\lambda,x)=\int_{-|x|}^{|x|}K_s(x,y)\operatorname{cas}(\sqrt{\lambda^2-s^2\rho^2}\,y)\,dy. \quad \square$$

We consider the generalized intertwining operator V_s defined on $\mathscr{E}(\mathbb{R})$ by

$$V_s f(x)=\begin{cases}\int_{-|x|}^{|x|}K_s(x,y)f(y)\,dy & \text{if } x\in\mathbb{R}\setminus\{0\},\\ f(0) & \text{if } x=0,\end{cases} \tag{2.24}$$

where $K_s(x,y)$ is the continuous function on $(-|x|,|x|)$, with support in $[-|x|,|x|]$ defined by relation (2.23).

We have

$$\forall\lambda\in\mathbb{C},\ \forall x\in\mathbb{R},\quad \Phi_s(\lambda,x)=V_s\left(\operatorname{cas}(\sqrt{\lambda^2-s^2\rho^2})\right)(x). \tag{2.25}$$

Using the similar method presented in [22, 23, 24], we prove the following results.

The operator V_s is a topological automorphism of $\mathscr{E}(\mathbb{R})$ satisfying

$$\forall f\in\mathscr{E}(\mathbb{R}),\quad \Lambda_s\left(V_s f\right)=V_s\left(Y_r f\right). \tag{2.26}$$

The operator ${}^tV_s$ is defined on $D(\mathbb{R})$ by

$$\forall y\in\mathbb{R},\quad {}^tV_s\left(f\right)(y)=\int_{|x|\geqslant|y|}K_s(x,y)f(x)A\left(x\right)dx. \tag{2.27}$$

The operator ${}^tV_s$ is a topological automorphism of $D(\mathbb{R})$. The operators V_s and ${}^tV_s$ possess the following properties:

(i) For all $f\in D(\mathbb{R})$ and $g\in\mathscr{E}(\mathbb{R})$, we have

$$\int_{\mathbb{R}}{}^tV_s\left(f\right)(y)g(y)dy=\int_{\mathbb{R}}f(x)V_s\,g(x)A\left(x\right)dx. \tag{2.28}$$

(ii) The inverse operator ${}^tV_s^{-1}$ is given by

$${}^tV_s^{-1}f={}^t\chi^{-1}(f_e)+(s\rho I-Y_r)\,{}^t\chi^{-1}(Jf_o), \tag{2.29}$$

where ${}^t\chi^{-1}$ is the inverse of the transmutation operator associated with operator L and J is the integral operator defined by

$$Jf(x)=\int_{-\infty}^{x}f(t)dt,\quad x\in\mathbb{R}. \tag{2.30}$$

(iii) The generalized intertwining operator V_s and its dual ${}^tV_s$ are positive.

COROLLARY 1. *For all* $\lambda,x\in\mathbb{R}$, *and* $|\lambda|\geqslant s\rho$ *we have*

$$|\Phi_s(\lambda,x)|\leqslant 2. \tag{2.31}$$

Proof. The relations (2.24), (2.25) and the previous result (iii), give the result. $\square$

2.2. Generalized Hartley transform associated to the operator Λ_s

NOTATIONS. We denote by $L^p_A(\mathbb{R})$, $1 \leqslant p \leqslant \infty$, the space of measurable functions f on $\mathbb{R}$ satisfying

$$\|f\|_{L^p_A(\mathbb{R})} = \left(\int_{\mathbb{R}} |f(x)|^p A(x)dx\right)^{1/p} < \infty, \quad \text{if } 1 \leqslant p < \infty$$
$$\|f\|_{L^\infty_A(\mathbb{R})} = ess \sup_{x\in\mathbb{R}} |f(x)| < \infty.$$

$\mathscr{S}^2_s(\mathbb{R})$, $s \in [-1,1]$, the space of C^∞-functions on $\mathbb{R}$ such that for all $m,n \in \mathbb{N}$

$$q_{n,m}(f) := \sup_{x\in\mathbb{R}} e^{\rho(1+\sqrt{1-s^2})|x|}(1+x^2)^m \left|\frac{d^n}{dx^n} f(x)\right| < \infty.$$

$\mathscr{S}^2_{s,e}(\mathbb{R})$ (resp. $\mathscr{S}^2_{s,o}(\mathbb{R})$) the subspace of $\mathscr{S}^2_s(\mathbb{R})$ consisting of even (resp. odd) functions.

DEFINITION 1. The generalized Hartley transform, associated to Λ_s, of a function $f \in L^1_A(\mathbb{R})$ is defined by

$$\mathscr{H}_{\Lambda_s}(f)(\lambda) = \int_{\mathbb{R}} f(x)\Phi_s(\lambda,x)A(x)dx, \quad \text{for all } \lambda \in \mathbb{R}. \tag{2.32}$$

PROPOSITION 2. *For all $f \in D(\mathbb{R})$ we have*

$$\mathscr{H}_{\Lambda_s}(f) = \mathscr{H}_m \circ {}^tV_s(f), \tag{2.33}$$

where $\mathscr{H}_m$ is the modified Hartley transform defined on $D(\mathbb{R})$ by

$$\forall \lambda \in \mathbb{C}, \quad \mathscr{H}_m(f)(\lambda) = \int_{\mathbb{R}} f(x)\,\mathrm{cas}(\sqrt{\lambda^2 - s^2\rho^2}x)dx = \mathscr{H}_C(f)(\sqrt{\lambda^2 - s^2\rho^2}). \tag{2.34}$$

Proof. The result is immediately from the relations (2.25) and (2.28). □

PROPOSITION 3. *Let $f \in L^1_A(\mathbb{R})$. For almost all y, the function*

$$y \longmapsto {}^tV_s(f)(y) = \int_{|x|\geqslant|y|} K_s(x,y)f(x)A(x)\,dx, \tag{2.35}$$

is defined almost everywhere on $\mathbb{R}$ and belongs to $L^1(\mathbb{R})$. Moreover, for all bounded continuous function g on $\mathbb{R}$, we have the following formula:

$$\int_{\mathbb{R}} {}^tV_s(f)(y)g(y)dy = \int_{\mathbb{R}} f(x)V_s g(x)A(x)dx. \tag{2.36}$$

Proof. The functions $(x,y) \mapsto K_s(x,y)f(x)A(x)$ and $(x,y) \mapsto K_s(x,y)f(x)g(y)A(x)$ are Lebesgue integrable on $\mathbb{R}^2$. Then by using Fubini's theorem, we get the result. □

PROPOSITION 4. *For all* $f \in \mathscr{S}_s^2(\mathbb{R})$*, we have the decomposition*

$$\mathscr{H}_{\Lambda_s} f(\lambda) = 2\mathscr{F}_L(f_e)(\sqrt{\lambda^2-(s^2+1)\rho^2}) - 2(\lambda+s\rho)\mathscr{F}_L J(f_o)(\sqrt{\lambda^2-(s^2+1)\rho^2}), \tag{2.37}$$

where J *is the integral operator defined by (2.30) and* $\mathscr{F}_L$ *stands for the Fourier transform related to the differential operator* L*, defined on* $\mathscr{S}_{s,e}^2(\mathbb{R})$ *by*

$$\mathscr{F}_L(f)(\lambda) = \int_0^\infty f(x)\varphi_\lambda(x)A(x)dx, \quad \lambda \in \mathbb{R},$$

φ_λ *being the eigenfunction of* L *as defined by (2.8).*

Proof. If $f \in \mathscr{S}_{s,e}^2(\mathbb{R})$, identity (2.37) is obvious. Assume $f \in \mathscr{S}_{s,o}^2(\mathbb{R})$. By using (2.32), (2.14), (2.8), and by integrating by parts we obtain

$$\begin{aligned}
\mathscr{H}_{\Lambda_s} f(\lambda) &= \frac{1}{\lambda - s\rho}\int_{\mathbb{R}} f(x) Y_r \varphi_{\sqrt{\lambda^2-(s^2+1)\rho^2}}(x)A(x)dx \\
&= \frac{1}{\lambda - s\rho}\int_{\mathbb{R}} \frac{d}{dx}\Big(A(x)\frac{d}{dx}\varphi_{\sqrt{\lambda^2-(s^2+1)\rho^2}}(x)\Big)J(f)(x)dx \\
&= \frac{1}{\lambda - s\rho}\int_{\mathbb{R}} L\varphi_{\sqrt{\lambda^2-(s^2+1)\rho^2}}(x)J(f)(x)A(x)dx \\
&= -(\lambda+s\rho)\int_{\mathbb{R}} \varphi_{\sqrt{\lambda^2-(s^2+1)\rho^2}}(x)J(f)(x)A(x)dx \\
&= -2(\lambda+s\rho)\mathscr{F}_L(Jf)(\sqrt{\lambda^2-(s^2+1)\rho^2}),
\end{aligned}$$

which completes the proof. □

Using Bloom-Xu's approach [4], we prove.

THEOREM 2. *The transform* $\mathscr{H}_{\Lambda_s}$ *is a bijection from* $\mathscr{S}_s^2(\mathbb{R})$ *to* $\mathscr{S}(\mathbb{R})$.

We shall need the following properties.

PROPOSITION 5. (Transmutation formula)

(i) Let $f \in \mathscr{S}_s^2(\mathbb{R})$ *and* g *a nice function. Then*

$$\int_{\mathbb{R}} \Lambda_s f(x)g(x)A(x)dx = \int_{\mathbb{R}} f(x)\Lambda_s g(x)A(x)dx. \tag{2.38}$$

(ii) For $f \in \mathscr{S}_s^2(\mathbb{R})$

$$\mathscr{H}_{\Lambda_s}(\Lambda_s f)(\xi) = \xi\mathscr{H}_{\Lambda_s} f(\xi), \quad \xi \in \mathbb{R}. \tag{2.39}$$

Proof. Let $f \in \mathscr{S}_s^2(\mathbb{R})$ and g a nice function, and consider the bracket

$$\langle f,g\rangle = \int_{\mathbb{R}} f(x)g(x)A(x)dx.$$

First, we have

$$\begin{aligned}
\langle Y_r f,g\rangle &= \int_{\mathbb{R}} f'(-x)g(x)A(x)dx = -\int_{\mathbb{R}} f(x)\frac{d}{dx}[g(-x)A(x)]dx \\
&= -\int_{\mathbb{R}} f(x)g(-x)A'(x)dx + \int_{\mathbb{R}} f(x)g'(-x)A(x)dx \\
&= \langle f,Y_r g\rangle - \left\langle f,\breve{g}\frac{A'}{A}\right\rangle
\end{aligned}$$

where $\breve{h}(x) = h(-x)$.

Second, we have

$$\begin{aligned}
\left\langle \frac{1}{2}\frac{A'}{A}(f-\breve{f}),g\right\rangle &= \int_{\mathbb{R}} \frac{A'(x)}{A(x)}\frac{f(x)-f(-x)}{2}g(x)A(x)dx \\
&= \int_{\mathbb{R}} \Big(\frac{f(x)-f(-x)}{2}\Big)g(x)A'(x)dx \\
&= \frac{1}{2}\int_{\mathbb{R}} (A'(x)f(x)g(x) - A'(x)f(-x)g(x))dx \\
&= \frac{1}{2}\int_{\mathbb{R}} (A'(x)f(x)g(x) - A'(-x)f(x)g(-x))dx \\
&= \frac{1}{2}\int_{\mathbb{R}} (A'(x)f(x)g(x) + A'(x)f(x)g(-x))dx \\
&= \frac{1}{2}\int_{\mathbb{R}} A'(x)f(x)(g(-x)+g(x))dx \\
&= \int_{\mathbb{R}} \frac{A'(x)}{A(x)}f(x)\Big(\frac{g(x)+g(-x)}{2}\Big)A(x)dx \\
&= \left\langle f,\frac{1}{2}\frac{A'}{A}(g+\breve{g})\right\rangle.
\end{aligned}$$

Finally,

$$\begin{aligned}
\langle(-s\rho\breve{f}),g\rangle &= -s\rho\int_{\mathbb{R}} f(-x)g(x)A(x)dx = -s\rho\int_{\mathbb{R}} f(x)g(-x)A(x)dx \\
&= \langle f,(-s\rho\breve{g})\rangle.
\end{aligned}$$

All together, this gives

$$\begin{aligned}
\langle\Lambda_s f,g\rangle &= \left\langle Y_r f + \frac{1}{2}\frac{A'}{A}(f-\breve{f}) - s\rho\breve{f},g\right\rangle \\
&= \left\langle f,Y_r g - \breve{g}\frac{A'}{A} + \frac{1}{2}\frac{A'}{A}(g+\breve{g}) - s\rho\breve{g}\right\rangle \\
&= \left\langle f,Y_r g + \frac{1}{2}\frac{A'}{A}(g-\breve{g}) - s\rho\breve{g}\right\rangle \\
&= \langle f,\Lambda_s g\rangle.
\end{aligned}$$

Assertion (ii) follows by substituting in (2.38) g by Φ_s. □

2.3. Inversion formula and Plancherel's theorem for $\mathscr{H}_{\Lambda_s}$

DEFINITION 2. The generalized translation operators τ_x^s, $x \in \mathbb{R}$ are defined on $L_A^2(\mathbb{R})$, by

$$\forall \lambda \in \mathbb{C}, \quad \mathscr{H}_{\Lambda_s}(\tau_x^s f)(\lambda) = \Phi_s(\lambda, x)\mathscr{H}_{\Lambda_s}(f)(\lambda). \tag{2.40}$$

Using the generalized translation operator, we define the generalized convolution product of functions as follows.

DEFINITION 3. For $f, g \in D(\mathbb{R})$, the generalized convolution product $f *_{\Lambda_s} g$ is defined by

$$f *_{\Lambda_s} g(x) = \int_{\mathbb{R}} \tau_x^s f(y) g(y) A(y) dy, \quad \text{for all } x \in \mathbb{R}. \tag{2.41}$$

PROPOSITION 6. *(i) Let f and g be in $D(\mathbb{R})$. Then the generalized convolution product $f *_{\Lambda_s} g$ belongs to $D(\mathbb{R})$ and we have*

$$\mathscr{H}_{\Lambda_s}(f *_{\Lambda_s} g)(\xi) = \mathscr{H}_{\Lambda_s}(f)(\xi)\mathscr{H}_{\Lambda_s}(g)(\xi), \quad \text{for all } \xi \in \mathbb{R}. \tag{2.42}$$

(ii) We have the relation

$$\forall f, g \in D(\mathbb{R}), \quad {}^tV_s(f *_{\Lambda_s} g) = {}^tV_s(f) *_r {}^tV_s(g), \tag{2.43}$$

where $*_r$ is the convolution product on $\mathbb{R}$ given by (1.6).

Proof. We obtain the result by the similar ideas as on the context of Chébli-Trimèche hypergroup (cf. [22]). □

DEFINITION 4. Let $f \in L_A^\infty(\mathbb{R})$, f is called $\mathscr{H}_{\Lambda_s}$-function of positive type if for all $\varphi \in D(\mathbb{R})$, we have

$$\int_{\mathbb{R}} f(x) \varphi *_{\Lambda_s} \varphi(x) A(x) dx \geqslant 0.$$

LEMMA 3. *Let u in $D'(\mathbb{R})$. The following assertions are equivalents:*
(i) For all $\varphi \in D(\mathbb{R})$, we have $\langle u, \varphi^2 \rangle \geqslant 0$.
(ii) u is a positive distribution.
(iii) u is a positive measure.

Proof. By using the proof of Theorem XVIII of [21] p. 276–277, we obtain the result. □

THEOREM 3. (Böckner's theorem) *Let $f \in L_A^\infty(\mathbb{R})$, if f is $\mathscr{H}_{\Lambda_s}$-function of positive type, there exists a positive bounded measure μ on $\mathbb{R}$, such that $f(x) = \mathscr{H}_{\Lambda_s}^{-1}(\mu)(x)$, for all $x \in \mathbb{R}$.*

Proof. Let $f \in L^\infty_A(\mathbb{R})$, of positive type and putting $u = \mathscr{H}_{\Lambda_s}(f)$. For all $\varphi \in D(\mathbb{R})$, we have

$$\begin{aligned}\langle u, \varphi^2\rangle &= \langle \mathscr{H}_{\Lambda_s}(f), \varphi^2\rangle \\ &= \langle f, \mathscr{H}^{-1}_{\Lambda_s}(\varphi^2)\rangle \\ &= \langle f, \mathscr{H}_{\Lambda_s}{}^{-1}(\varphi) *_{\Lambda_s} \mathscr{H}_{\Lambda_s}{}^{-1}(\varphi)\rangle \geqslant 0.\end{aligned}$$

Hence u is a positive distribution. Again, by using lemma 3 it is a measure of positive type. As $u \in \mathscr{H}_{\Lambda_s}(L^\infty_A(\mathbb{R}))$, by a standard analysis, we prove this measure is bounded and we deduce the result. □

DEFINITION 5. Let $u \in D'(\mathbb{R})$, u is called $\mathscr{H}_{\Lambda_s}$-distribution of positive type if for all $\varphi \in D(\mathbb{R})$, we have

$$\langle u, \varphi *_{\Lambda_s} \varphi\rangle \geqslant 0.$$

THEOREM 4. (Böckner-Schwartz's theorem)
Let $u \in D'(\mathbb{R})$. The following assertions are equivalents:
(i) u is of positive type.
(ii) u is a tempered distribution, and there exists a tempered positive measure μ on $\mathbb{R}$, such that $u = \mathscr{H}^{-1}_{\Lambda_s}(\mu)$.

Proof. We want to prove that (i) $\Rightarrow$ (ii).

It is easy to see that for all $\varphi \in D(\mathbb{R})$, the function $\varphi \mapsto u *_{\Lambda_s} \varphi *_{\Lambda_s} \varphi$ is of positive type. Then by the Theorem 3, there exists a bounded positive measure μ_φ such that $\mu_\varphi = \mathscr{H}_{\Lambda_s}(u *_{\Lambda_s} \varphi *_{\Lambda_s} \varphi)$. Let $\chi \in D(\mathbb{R})$ such that $\mathscr{H}_{\Lambda_s}(\chi)(\lambda) \neq 0$, for all $\lambda \in \mathbb{R}$. We consider the measure μ defined by $\mu = \frac{\mu_\chi}{|\mathscr{H}_{\Lambda_s}(\chi)(\lambda)|^2}$. It is clear that μ is a positive measure, we can write:

$$\mathscr{H}_{\Lambda_s}(u *_{\Lambda_s} \varphi *_{\Lambda_s} \varphi *_{\Lambda_s} \chi *_{\Lambda_s} \chi) = \big(\mathscr{H}_{\Lambda_s}(\chi)\big)^2 \mu_\varphi = \big(\mathscr{H}_{\Lambda_s}(\varphi)\big)^2 \mu_\chi.$$

Thus

$$\mu_\varphi = \big(\mathscr{H}_{\Lambda_s}(\varphi)\big)^2 \mu, \quad \text{for all } \varphi \in D(\mathbb{R}).$$

Then, we have

$$\begin{aligned}\langle u, \varphi *_{\Lambda_s} \varphi\rangle &= \langle u *_{\Lambda_s} \varphi *_{\Lambda_s} \varphi, \delta\rangle \\ &= \mathscr{H}^{-1}_{\Lambda_s}(\mu_\varphi)(0) \\ &= \int_{\mathbb{R}} \Big(\mathscr{H}_{\Lambda_s}(\varphi)\Big)^2(\lambda) d\mu(\lambda).\end{aligned}$$

Thus, we deduce that

$$\langle u, \psi\rangle = \int_{\mathbb{R}} \mathscr{H}_{\Lambda_s}(\psi)(\lambda) d\mu(\lambda) = \langle \mathscr{H}^{-1}_{\Lambda_s}(\mu), \psi\rangle.$$

So the result follows.

Now we want to prove that (ii) $\Rightarrow$ (i).

If $\mu = \mathscr{H}_{\Lambda_s}(u)$, we have for all $\varphi \in D(\mathbb{R})$,

$$\begin{aligned}\langle \mathscr{H}^{-1}_{\Lambda_s}(\mu), \varphi *_{\Lambda_s} \varphi\rangle &= \langle \mu, \mathscr{H}_{\Lambda_s}(\varphi *_{\Lambda_s} \varphi)\rangle \\ &= \langle \mu, \big(\mathscr{H}_{\Lambda_s}(\varphi)\big)^2\rangle \geqslant 0.\end{aligned}$$

The theorem is then proved. □

REMARK 3. As above, we prove the Böckner and Böckner-Schwartz's theorems for the classical Hartley transform $\mathscr{H}_C$ defined by (1.1).

NOTATION. We denote by $L^p_c(\mathbb{R}_+)$, $1 \leqslant p \leqslant \infty$, the space of measurable functions f on $\mathbb{R}_+$ satisfying

$$\|f\|_{L^p_c(\mathbb{R}_+)} = \left(\int_{\mathbb{R}_+} |f(x)|^p \frac{dx}{|c(x)|^2}\right)^{1/p} < \infty, \quad \text{if } 1 \leqslant p < \infty$$
$$\|f\|_{L^\infty_c(\mathbb{R}_+)} = ess \sup_{x\in\mathbb{R}_+} |f(x)| < \infty,$$

where $c(s)$ is a continuous function on $(0,\infty)$ such that

$$c^{-1}(s) \sim k_1\, s^{\alpha+\frac{1}{2}}, \quad \text{as } s \longrightarrow \infty, \tag{2.44}$$

$$c^{-1}(s) \sim \begin{cases} k_2\, s, & \text{as } s \longrightarrow 0 \text{ if } \rho > 0 \\ k_3\, s^{2\alpha+1}, & \text{as } s \longrightarrow 0 \text{ if } \rho = 0 \end{cases} \tag{2.45}$$

for some $k_1 k_2, k_3 \in \mathbb{C}$.

$L^p_{\nu_s}(\mathbb{R})$, $1 \leqslant p \leqslant \infty$, the space of measurable functions f on $\mathbb{R}$ satisfying

$$\|f\|_{L^p_{\nu_s}(\mathbb{R})} = \left(\int_{\mathbb{R}} |f(x)|^p d\nu_s(x)\right)^{1/p} < \infty, \quad \text{if } 1 \leqslant p < \infty$$
$$\|f\|_{L^\infty_{\nu_s}(\mathbb{R})} = ess \sup_{x\in\mathbb{R}} |f(x)| < \infty,$$

where $d\nu_s$ is the measure given by

$$d\nu_s(\lambda) = \frac{|\lambda|}{4\sqrt{\lambda^2-(s^2+1)\rho^2}|c(\sqrt{\lambda^2-(s^2+1)\rho^2})|^2} 1_{\mathbb{R}\setminus(-\sqrt{s^2+1}\rho,\sqrt{s^2+1}\rho)} d\lambda, \tag{2.46}$$

with $1_{\mathbb{R}\setminus(-\sqrt{(s^2+1)}\rho,\sqrt{(s^2+1)}\rho)}$ is the characteristic function of

$$\mathbb{R}\setminus(-\sqrt{(s^2+1)}\rho, \sqrt{(s^2+1)}\rho).$$

THEOREM 5. *For all $f \in D(\mathbb{R})$, we have*

$$f(x) = \int_{\mathbb{R}} \mathscr{H}_{\Lambda_s}(f)(\lambda)\Phi_s(\lambda,x)d\nu_s(\lambda),$$

where $d\nu_s$ is given by (2.46).

Proof. By using Proposition 6 we see that the functional

$$\langle S, f\rangle = {}^tV_s^{-1}(f)(0), \quad f \in D(\mathbb{R})$$

is $\mathscr{H}_C$- positive type distribution in $D'(\mathbb{R})$. From Remark 3, we have the Böchner-Schwartz theorem for $\mathscr{H}_C$. Thus, we deduce that there exists a positive measure ν_s, such that for all f in $D(\mathbb{R})$, we have

$$f(0)=\int_{\mathbb{R}}\mathscr{H}_m({}^t V_s(f))(\lambda)d\nu_s(\lambda).$$

Using the relation (2.33) we obtain

$$f(0)=\int_{\mathbb{R}}\mathscr{H}_{\Lambda_s}(f)(\lambda)d\nu_s(\lambda).$$

By substituting in this relation f by $\tau^s_x f$, $x\in\mathbb{R}$, and by using (2.40) we obtain

$$f(x)=\int_{\mathbb{R}}\mathscr{H}_{\Lambda_s}(\tau^s_x f)(\lambda)d\nu_s(\lambda)=\int_{\mathbb{R}}\mathscr{H}_{\Lambda_s}(f)(\lambda)\Phi_s(\lambda,x)d\nu_s(\lambda). \tag{2.47}$$

If we suppose that f is even, then by the inversion formula for the transform $\mathscr{F}_L$ (see [22]), we have

$$f(x)=\int_{\mathbb{R}}\mathscr{F}_L(f)(\mu)\varphi_\mu(x)\frac{d\mu}{2|c(\mu)|^2}. \tag{2.48}$$

But the parametrization $\mu^2=\lambda^2-(s^2+1)\rho^2$, shows that $\mu\in[0,\infty)$ if and only if $\lambda\in\mathbb{R}\setminus(-\sqrt{(s^2+1)}\rho,\sqrt{(s^2+1)}\rho)$, thus from (2.37) we deduce that (2.48) can also be written in the form

$$f(x)=\int_{\mathbb{R}\setminus(-\sqrt{(s^2+1)}\rho,\sqrt{(s^2+1)}\rho)}\mathscr{H}_{\Lambda_s}(f)(\lambda)\Phi_s(\lambda,x)$$
$$\times\frac{|\lambda|d\lambda}{4\sqrt{\lambda^2-(s^2+1)\rho^2}|c(\sqrt{\lambda^2-(s^2+1)\rho^2})|^2}.$$

Using the relation (2.47) we deduce that the measure $d\nu_s$ is supported by

$$\mathbb{R}\setminus(-\sqrt{(s^2+1)}\rho,\sqrt{(s^2+1)}\rho)$$

and given by (2.46). $\square$

THEOREM 6. *Let $f\in L^1_A(\mathbb{R})$ such that $\mathscr{H}_{\Lambda_s}(f)$ belongs to $L^1_{\nu_s}(\mathbb{R})$. Then we have the following inversion formula*

$$f(x)=\int_{\mathbb{R}}\mathscr{H}_{\Lambda_s}(f)(\lambda)\Phi_s(\lambda,x)d\nu_s(\lambda)\quad a.e.\, x\in\mathbb{R}.$$

Proof. It is easy to see that if $\mathscr{H}_{\Lambda_s}(f)$ belongs to $L^1_{\nu_s}(\mathbb{R})$ then $\mathscr{F}_L(f_e)$ belongs to $L^1_c(\mathbb{R}_+)$, where f_e is the even part of f. For $|\lambda|\geqslant\sqrt{(s^2+1)}\rho$ the function $\lambda\mapsto\frac{\mathscr{H}_{\Lambda_s}(f)(\lambda)-\mathscr{H}_{\Lambda_s}(f)(-\lambda)}{\lambda}$ belongs to $L^1_{\nu_s}(\mathbb{R})$, then $\mathscr{F}_L(Jf_o)\in L^1_c(\mathbb{R}_+)$, where f_o is the odd part of f and J the transform given by (2.30). On the other hand using (2.31) we obtain for $\mu^2=\lambda^2-(s^2+1)\rho^2$

$$|Y_r\varphi_\mu(x)|\leqslant|(\lambda-s\rho)\frac{\Phi_s(\lambda,x)-\Phi_\lambda(-x)}{2}|\leqslant 2(|\lambda|+|s|\rho).$$

Furthermore, the function $\mu\mapsto\sqrt{\mu^2+(s^2+1)\rho^2}\mathscr{F}_L(Jf_o)(\mu)$ is in $L^1_c(\mathbb{R}_+)$, so by the dominated convergence theorem we obtain

$$\int_{\mathbb{R}_+}\mathscr{F}_L(Jf_o)(\lambda)Y_r\varphi_\mu(x)\frac{d\mu}{|c(\mu)|^2}=Y_r\Big(\int_{\mathbb{R}_+}\mathscr{F}_L(Jf_o)(\lambda)\varphi_\mu(x)\frac{d\mu}{|c(\mu)|^2}\Big).$$

We get the result by using the relation

$$\begin{aligned}&\int_{\mathbb{R}}\mathscr{H}_{\Lambda_s}(f)(\lambda)\Phi_s(\lambda,x)d\nu_s(\lambda)\\&=\int_{\mathbb{R}_+}\mathscr{F}_L(f_e)(\mu)\varphi_\mu(x)\frac{d\mu}{|c(\mu)|^2}-Y_r\Big(\int_{\mathbb{R}_+}\mathscr{F}_L(Jf_o)(\mu)\varphi_\mu(x)\frac{d\mu}{|c(\mu)|^2}\Big)\end{aligned}$$

and the analogous theorem corresponding to $\mathscr{F}_L$. $\square$

THEOREM 7. (Plancherel formula) *For all $f\in\mathscr{S}^2(\mathbb{R})$, we have*

$$\begin{aligned}&\int_{\mathbb{R}}|\mathscr{H}_{\Lambda_s}(f)(\lambda)|^2d\nu_s(\lambda)\\&=\int_{\mathbb{R}}|f(x)|^2A(x)dx+2s^2\rho^2\int_{\mathbb{R}}|Jf_o(x)|^2A(x)dx-2s\rho\mathrm{Re}\Big(\int_{\mathbb{R}}f_e(x)\overline{Jf_o(x)}A(x)dx\Big),\end{aligned}\tag{2.49}$$

where $d\nu_s$ is the measure given by (2.46).

Proof. By (2.37)

$$\begin{aligned}&\int_{\mathbb{R}}|\mathscr{H}_{\Lambda_s}(f)(\lambda)|^2d\nu_s(\lambda)\\&=4\int_{\mathbb{R}}|\mathscr{F}_L(f_e)(\sqrt{\lambda^2-(s^2+1)\rho^2})|^2d\nu_s(\lambda)\\&\quad+4\int_{\mathbb{R}}(\lambda+s\rho)^2|\mathscr{F}_L(Jf_o)(\sqrt{\lambda^2-(s^2+1)\rho^2})|^2d\nu_s(\lambda)\\&\quad-8s\rho\mathrm{Re}\Big(\int_{\mathbb{R}}\mathscr{F}_L(f_e)(\sqrt{\lambda^2-(s^2+1)\rho^2})\overline{\mathscr{F}_L(Jf_o)(\sqrt{\lambda^2-(s^2+1)\rho^2})}d\nu_s(\lambda)\Big)\\&=I_1+I_2+I_3.\end{aligned}$$

By a Plancherel formula for the Fourier transform $\mathscr{F}_L$

$$I_1=\int_{\mathbb{R}}|f_e(x)|^2A(x)dx,\quad\text{and}\quad I_3=-2s\rho\mathrm{Re}\Big(\int_{\mathbb{R}}f_e(x)\overline{Jf_o(x)}A(x)dx\Big).$$

Moreover

$$\begin{aligned}I_2&=4\int_{\mathbb{R}}(\lambda+s\rho)^2|\mathscr{F}_L(Jf_o)(\sqrt{\lambda^2-(s^2+1)\rho^2})|^2d\nu_s(\lambda)\\&=4\int_{\mathbb{R}}(\lambda^2+s^2\rho^2)|\mathscr{F}_L(Jf_o)(\sqrt{\lambda^2-(s^2+1)\rho^2})|^2d\nu_s(\lambda)\\&=\int_{\mathbb{R}}(\mu^2+\rho^2)|\mathscr{F}_L(Jf_o)(\mu)|^2\frac{d\mu}{|c(\mu)|^2}+2s^2\rho^2\int_{\mathbb{R}}|\mathscr{F}_L(Jf_o)(\mu)|^2\frac{d\mu}{|c(\mu)|^2}.\end{aligned}$$

Using now the identity

$$\mathscr{F}_L(Lh)(\mu) = -(\mu^2+\rho^2)\mathscr{F}_L(h)(\mu), \quad h \in \mathscr{S}_e^2(\mathbb{R})$$

and, by integration by parts, we deduce that

$$\begin{aligned} I_2 &= -\int_{\mathbb{R}} \mathscr{F}_L(LJf_o)(\mu)\overline{\mathscr{F}_L(Jf_o)(\mu)}\frac{d\mu}{|c(\mu)|^2} + 2s^2\rho^2\int_{\mathbb{R}} |\mathscr{F}_L(Jf_o)(\mu)|^2\frac{d\mu}{|c(\mu)|^2} \\ &= -\int_{\mathbb{R}} LJf_o(x)\overline{Jf_o(x)}A(x)dx + 2s^2\rho^2\int_{\mathbb{R}} |Jf_o(x)|^2A(x)dx \\ &= -\int_{\mathbb{R}} \frac{d}{dx}\Big(A(x)\frac{d}{dx}Jf_o(x)\Big)\overline{Jf_o(x)}dx + 2s^2\rho^2\int_{\mathbb{R}} |Jf_o(x)|^2A(x)dx \\ &= \int_{\mathbb{R}} |f_o(x))|^2A(x)dx + 2s^2\rho^2\int_{\mathbb{R}} |Jf_o(x)|^2A(x)dx. \end{aligned}$$

Hence

$$\begin{aligned} I_1+I_2+I_3 &= \int_{\mathbb{R}} |f_e(x)|^2A(x)dx + \int_{\mathbb{R}} |f_o(x))|^2A(x)dx + 2s^2\rho^2\int_{\mathbb{R}} |Jf_o(x)|^2A(x)dx \\ &\quad -2s\rho\mathrm{Re}\Big(\int_{\mathbb{R}} f_e(x)\overline{Jf_o(x)}A(x)dx\Big) \\ &= \int_{\mathbb{R}} |f(x)|^2A(x)dx + 2s^2\rho^2\int_{\mathbb{R}} |Jf_o(x)|^2 - 2s\rho\mathrm{Re}\Big(\int_{\mathbb{R}} f_e(x)\overline{Jf_o(x)}A(x)dx\Big). \end{aligned}$$

This completes the proof. □

In the rest of this article, we assume that $s = 0$.

2.4. The generalized heat kernel

DEFINITION 6. Let $t > 0$. The heat kernel E_t associated with the operator Λ_s is defined by

$$\forall x \in \mathbb{R}, \quad E_t(x) = \mathscr{H}_{\Lambda_s}^{-1}(e^{-t\lambda^2})(x). \tag{2.50}$$

REMARK 4. As the function $\lambda \mapsto e^{-t\lambda^2}$ is an even function on $\mathbb{R}$, then from the relation (2.37), we deduce that

$$\forall x \in \mathbb{R}, \quad E_t(x) = \frac{1}{2}\mathscr{F}_L^{-1}(e^{-t(\lambda^2+\rho^2)})(x). \tag{2.51}$$

We introduce also the generalized heat functions $N_n(t,.)$, $n \in \mathbb{N}$ are defined on $\mathbb{R}$ by

$$N_n(t,x) = (2\pi)^{-1}\int_{\mathbb{R}} \lambda^n e^{-t\lambda^2}\Phi_s(\lambda,x)d\nu_s(\lambda). \tag{2.52}$$

These functions satisfies the following properties.

i) For all $t > 0$, $N_n(t,.)$ is an C^∞-function on $\mathbb{R}$.
ii) For all $t > 0$ and $\forall x \in \mathbb{R}$, $N_0(t,x) = E_t(x) > 0$.
iii) For all $t > 0$, $\forall \lambda \in \mathbb{R}$, $\mathscr{H}_{\Lambda_s}\Big(N_n(t,.)\Big)(\lambda) = \lambda^n e^{-t\lambda^2}$.

PROPOSITION 7. *Let $t > 0$. We have*

$$\forall y \in \mathbb{R}, \quad {}^tV_s(E_t)(y) = \frac{1}{2\sqrt{\pi t}} e^{-\frac{y^2}{4t}}. \tag{2.53}$$

Proof. From the relations (2.50) and (2.33), we have

$$\forall y \in \mathbb{R}, \quad {}^tV_s(E_t)(y) = \mathscr{H}_C^{-1}(e^{-t\lambda^2})(y) = \frac{1}{2\sqrt{\pi t}} e^{-\frac{y^2}{4t}}. \quad \square$$

PROPOSITION 8. *Let $p \in [1, \infty)$. There exists a positive constant $C(p,t)$ such that*

$$\forall x \in \mathbb{R}, \quad (E_t(x))^p \leqslant C(p,t) E_{\frac{t}{p}}(x). \tag{2.54}$$

Proof. From [9], p. 251, there exist two real numbers μ_1 and μ_2, such that

$$\forall x \in \mathbb{R}, \quad \frac{e^{\mu_1 t}}{2^{2\alpha+1}\Gamma(\alpha+1)t^{\alpha+1}} \frac{e^{-\frac{x^2}{4t}}}{\sqrt{B(x)}} \leqslant E_t(x) \leqslant \frac{e^{\mu_2 t}}{2^{2\alpha+1}\Gamma(\alpha+1)t^{\alpha+1}} \frac{e^{-\frac{x^2}{4t}}}{\sqrt{B(x)}}. \tag{2.55}$$

Using the hypothesis on the function A, there exist $C > 0$ such that for all $x \in \mathbb{R}$, $B(x) \geqslant C$. Thus, according (2.55), we obtain (2.54). $\square$

3. An L^p version of Hardy's theorem

PROPOSITION 9. *Let $p \in [1, \infty]$ and f a measurable function on $\mathbb{R}$ such that $e^{ax^2} f$ belongs to $L^p_A(\mathbb{R})$ for some $a > 0$. Then the function $\mathscr{H}_{\Lambda_s}(f)$ given for all $\lambda \in \mathbb{C}$ by*

$$\mathscr{H}_{\Lambda_s}(f)(\lambda) = \int_{\mathbb{R}} f(x)\Phi_s(\lambda, x)A(x)dx,$$

is well defined, entire on $\mathbb{C}$, and there exists a positive constant C such that

$$\forall \xi, \eta \in \mathbb{R}, \quad |\mathscr{H}_{\Lambda_s}(f)(\xi + i\eta)| \leqslant C e^{\frac{\eta^2}{4a}}. \tag{3.56}$$

Proof. The first assertion follows from Hölder's inequality, the relation (2.18), and the derivation theorem under the integral sign.

If $p = 1$, we deduce from Remark 2 that for all $\xi, \eta \in \mathbb{R}$

$$\begin{aligned} |\mathscr{H}_{\Lambda_s}(f)(\xi + i\eta)| &\leqslant \int_{\mathbb{R}} |f(x)||\Phi_s(\xi + i\eta, x)|A(x)dx \\ &\leqslant C \int_{\mathbb{R}} e^{ax^2}|f(x)|e^{-ax^2} e^{|\eta||x|} A(x)dx \\ &\leqslant C e^{\frac{|\eta|^2}{4a}} \int_{\mathbb{R}} e^{-a(x-\frac{|\eta|}{2a})^2} |f(x)|A(x)dx \leqslant e^{\frac{|\eta|^2}{4a}} ||e^{ax^2} f||_{L^1_A(\mathbb{R})}. \end{aligned} \tag{3.57}$$

If $p \in (1,\infty]$, we deduce from Remark 2 and Hölder's inequality that for all $\xi, \eta \in \mathbb{R}$

$$\begin{aligned}|\mathscr{H}_{\Lambda_s}(f)(\xi+i\eta)| &\leqslant \int_{\mathbb{R}} |f(x)||\Phi_s(\xi+i\eta,x)|A(x)dx \\ &\leqslant C\int_{\mathbb{R}} e^{ax^2}|f(x)|e^{-ax^2}e^{|\eta||x|}A(x)dx \\ &\leqslant Ce^{\frac{|\eta|^2}{4a}}\Big(\int_{\mathbb{R}} e^{-ap'(x-\frac{|\eta|}{2a})^2}A(x)dx\Big)^{\frac{1}{p'}}||e^{ax^2}f||_{L^p_A(\mathbb{R})},\end{aligned}$$

where p' is the conjugate exponent of p. By using the properties of the function A, the integral of the right member is finite. So, as the function $e^{ax^2}f$ belongs to $L^p_A(\mathbb{R})$, we obtain (3.56). □

THEOREM 8. *Let f be a measurable function on $\mathbb{R}$ such that*

$$e^{ax^2}f \in L^p_A(\mathbb{R}) \text{ and } e^{b\lambda^2}\mathscr{H}_{\Lambda_s}(f) \in L^q_{\nu_s}(\mathbb{R}), \tag{3.58}$$

for some constants $a,b>0$, $1 \leqslant p,q \leqslant \infty$, and at least one of p and q is finite. Then

- *If $ab \geqslant \frac{1}{4}$, we have $f=0$, almost everywhere.*
- *If $ab < \frac{1}{4}$, for all $t \in (b, \frac{1}{4a})$, the functions $f = E_t$, satisfy the relations (3.58).*

For prove this theorem we need the following lemmas.

LEMMA 4. ([10]) *Let h be an entire function on $\mathbb{C}$ such that*

$$\forall z \in \mathbb{C}, \quad |h(z)| \leqslant C(1+|z|)^m e^{a(\mathrm{Re} z)^2} \tag{3.59}$$

and

$$\forall x \in \mathbb{R}, \quad |h(x)| \leqslant C, \tag{3.60}$$

for some $m \in \mathbb{N}$, $a, C > 0$. Then h is constant on $\mathbb{C}$.

LEMMA 5. ([10]) *Let $q \in [1,\infty)$ and h an entire function on $\mathbb{C}$ such that*

$$\forall z \in \mathbb{C}, \quad |h(z)| \leqslant M(1+|z|)^m e^{a(\mathrm{Re} z)^2} \tag{3.61}$$

and

$$||h_{|\mathbb{R}}||_{L^q_{\nu_s}(\mathbb{R})} < \infty, \tag{3.62}$$

for some $m \in \mathbb{N}$, $a, M > 0$. Then $h \equiv 0$.

Proof of Theorem 8. We will divide the proof in several steps.
1^{st} *step*: If $ab > \frac{1}{4}$. We consider the function h defined on $\mathbb{C}$ by

$$h(\lambda) = e^{\frac{\lambda^2}{4a}}\mathscr{H}_{\Lambda_s}(f)(\lambda). \tag{3.63}$$

From Proposition 9, there exist a positive constant C such that for all $\xi,\ \eta \in \mathbb{R}$, we have $|h(\xi+i\eta)| \leqslant Ce^{\frac{\xi^2}{4a}}$.

i) If $1 \leqslant q < \infty$, we have

$$\|h_{|\mathbb{R}}\|^q_{L^q_{\nu_s}(\mathbb{R})} = \int_{\mathbb{R}} |e^{b\lambda^2}\mathscr{H}_{\Lambda_s}(f)(\lambda)|^q e^{q(\frac{1}{4a}-b)\lambda^2}\, d\nu_s(\lambda).$$

The inequality $ab > \frac{1}{4}$ implies

$$\|h_{|\mathbb{R}}\|_{L^q_{\nu_s}(\mathbb{R})} \leqslant \|e^{b\lambda^2}\mathscr{H}_{\Lambda_s}(f)\|_{L^q_{\nu_s}(\mathbb{R})} < \infty.$$

We deduce from Lemma 5 that for all $\lambda \in \mathbb{C},\ h(\lambda) = 0$.

It follows that for all $\lambda \in \mathbb{R}$, $\mathscr{H}_{\Lambda_s}(f)(\lambda) = 0$ and then from the injectivity of the transform $\mathscr{H}_{\Lambda_s}$, we have

$$f = 0,\ a.e.,\ \text{on } \mathbb{R}.$$

ii) If $q = \infty$, we have

$$\|h_{|\mathbb{R}}\|_{L^\infty_\nu(\mathbb{R})} = \|e^{b\lambda^2}\mathscr{H}_{\Lambda_s}(f)\ e^{(\frac{1}{4a}-b)\lambda^2}\|_{L^\infty_\nu(\mathbb{R})} \leqslant \|e^{b\lambda^2}\mathscr{H}_{\Lambda_s}(f)\|_{L^\infty_\nu(\mathbb{R})} < \infty.$$

From Lemma 4, there exists a constant C such that for all $\lambda \in \mathbb{C},\ h(\lambda) = C$.

It follows that for all $\lambda \in \mathbb{R}$, $\quad \mathscr{H}_{\Lambda_s}(f)(\lambda) = Ce^{-\frac{\lambda^2}{4a}}$. The assumption on $\mathscr{H}_{\Lambda_s}(f)$ is expressed as

$$|\mathscr{H}_{\Lambda_s}(f)(\lambda)| \leqslant Me^{-b\lambda^2},\ a.e.\ \lambda \in \mathbb{R},$$

for some constant $M > 0$.

The continuity of $\mathscr{H}_{\Lambda_s}(f)$ on $\mathbb{R}$ shows that for all $\lambda \in \mathbb{R}$, $\quad |\mathscr{H}_{\Lambda_s}(f)(\lambda)| \leqslant Me^{-b\lambda^2}$. Then for all $\lambda \in \mathbb{R}$, $\quad |C| \leqslant Me^{(\frac{1}{4a}-b)\lambda^2}$. It follows from the inequality $ab > \frac{1}{4}$, that $C = 0$. Therefore

$$f = 0,\ a.e.,\ \text{on } \mathbb{R}.$$

2^{nd} *step*: If $ab = \frac{1}{4}$, we have

i) If $1 \leqslant q < \infty$. With the same proof as for the point *i*) of the first step, we deduce that

$$f = 0,\ a.e.,\ \text{on } \mathbb{R}.$$

ii) If $q = \infty$. We have $\|h_{|\mathbb{R}}\|_{L^q_{\nu_s}(\mathbb{R})} < \infty$. Then by Lemma 4, the relation (3.63) and the property (iii) of the generalized heat function $N_0(\frac{1}{4a},.)$, we deduce that

$$\forall \xi \in \mathbb{R}, \quad \mathscr{H}_{\Lambda_s}(f)(\xi) = Ce^{\frac{-\xi^2}{4a}} = C\mathscr{H}_{\Lambda_s}(E_{\frac{1}{4a}})(\xi) \tag{3.64}$$

for some constant C. Thus from the injectivity of the transform $\mathscr{H}_{\Lambda_s}$, we obtain

$$\forall x \in \mathbb{R}, \quad f(x) = CE_{\frac{1}{4a}}(x)\ a.e. \tag{3.65}$$

By using this relation and (2.55), we have

$$\forall x \in \mathbb{R}, \quad \frac{Ce^{\frac{\mu_1}{4a}} 2a^{\alpha+1}}{\Gamma(\alpha+1)\sqrt{B(x)}} \leqslant e^{ax^2} f(x).$$

From the properties of the functions A and B, we see that for finite p, $||\frac{1}{\sqrt{B(x)}}||_{L^p_A(\mathbb{R})} = \infty$. In contrast from equation (3.58) we have $||e^{ax^2} f||_{L^p_A(\mathbb{R})} < \infty$, this is impossible unless $C = 0$. Then, we deduce from equation (3.65) that $f = 0$ a.e.

3^{rd} *step*: If $ab < \frac{1}{4}$. Let $t \in (b, \frac{1}{4a})$ and $f = E_t$. From the relation (2.55), we get

$$\forall x \in \mathbb{R}, \quad K_1 e^{-(\frac{1}{4t}-a)x^2} \leqslant e^{ax^2} f(x) \leqslant K_2 e^{-(\frac{1}{4t}-a)x^2},$$

for some constants K_1, $K_2 > 0$. As $t < \frac{1}{4a}$, we deduce that $e^{ax^2} f \in L^p_A(\mathbb{R})$. Using the relation (2.50), we get

$$\forall \lambda \in \mathbb{R}, \quad e^{b\lambda^2} \mathscr{H}_{\Lambda_s}(f)(\lambda) = e^{-(t-b)\lambda^2}.$$

The condition $t > b$ and the relations (2.44) and (2.45), imply that $e^{b\lambda^2} \mathscr{H}_{\Lambda_s}(f) \in L^q_{\nu_s}(\mathbb{R})$. This completes the proof of the theorem. □

We determine, in this section, the functions f satisfying the relations (3.58) in the special case $p = q = \infty$. The result obtained for the generalized Hartley transform is an analogue of the classical Hardy's theorem.

THEOREM 9. *Let f be a measurable function on $\mathbb{R}$ such that*

$$|f(x)| \leqslant Me^{-ax^2}, \ a.e.\ x \in \mathbb{R} \text{ and } |\mathscr{H}_{\Lambda_s}(f)(\lambda)| \leqslant Me^{-b\lambda^2}, \text{ for all } \lambda \in \mathbb{R}, \tag{3.66}$$

for some constants $a, b, M > 0$. Then

- *If $ab > \frac{1}{4}$, we have $f = 0$, almost everywhere.*
- *If $ab = \frac{1}{4}$, the function f is of the form $f = C_0 E_{\frac{1}{4a}}$, for some real constant C_0.*
- *If $ab < \frac{1}{4}$, there are infinitely many nonzero functions f satisfying the conditions (3.66).*

Proof. 1^{st} *step*: If $ab > \frac{1}{4}$, the point $ii)$ of the first step of the proof of Theorem 8 gives the result.

2^{nd} *step*: If $ab = \frac{1}{4}$, we obtain $||e^{b\lambda^2} \mathscr{H}_{\Lambda_s}(f)||_{L^\infty_\nu(\mathbb{R})} < \infty$ from the relation (3.64). In contrast, as $B(x) \geqslant 1$, we get from (2.55) that

$$\forall x \in \mathbb{R}, \quad e^{ax^2} |f(x)| \leqslant \frac{Ce^{\frac{\mu_2}{4a}} 2a^{\alpha+1}}{\Gamma(\alpha+1)\sqrt{B(x)}} \ a.e.$$

Thus $||e^{ax^2} f||_{L^\infty_A(\mathbb{R})} < \infty$. This completes the proof of the theorem.

3^{rd} *step*: If $ab < \frac{1}{4}$, the functions $f = E_t$, $t \in (b, \frac{1}{4a})$, satisfy the conditions (3.66). This completes the proof of the theorem. □

4. Morgan's theorem for the generalized Hartley transform

Let $\theta_1, \theta_2 \in [0, 2\pi]$, Ω be the open angle in $\mathbb{C}$ defined by

$$\Omega := \left\{ re^{i\theta},\ r > 0,\ \theta_1 < \theta < \theta_2 \right\}$$

and $\overline{\Omega}$ be the closure of Ω. Let $g : \overline{\Omega} \to \mathbb{C}$ be a continuous holomorphic on $\mathbb{C}$.

The order of g is defined by

$$r(g) = \inf\left\{ \gamma > 0,\ g(z) = O(e^{|z|^\gamma}),\ |z| \to \infty,\ z \in \Omega \right\}.$$

Suppose that $0 < r(g) < \infty$. The type of g on Ω is defined by

$$\tau(g) = \inf\left\{ \gamma > 0,\ g(z) = O(e^{\gamma |z|^{r(g)}}),\ |z| \to \infty,\ z \in \Omega \right\}.$$

Before stating the main result of this section, we give the following lemma which we prove by the same way as Lemma 2.3 in [1].

LEMMA 6. *Suppose $r \in (1,2)$, $q \in [1, \infty]$, $\tau > 0$ and $B > \tau \sin(\frac{\pi}{2}(r-1))$. If g is an entire function on $\mathbb{C}$ satisfying the following conditions*

$$|g(x+iy)| \leqslant Ce^{\tau |y|^r}, \quad for\, any\, x, y\, \mathbb{R} \tag{4.67}$$

and

$$e^{B|x|^r} g_{|\mathbb{R}} \in L^q_A(\mathbb{R}), \tag{4.68}$$

then $g = 0$.

PROPOSITION 10. *Let $p \in [1, \infty]$, $a > 0$ and $\gamma > 2$ and f a measurable function on $\mathbb{R}$ such that*

$$e^{a|x|^\gamma} f \in L^p_A(\mathbb{R}). \tag{4.69}$$

Then $\mathscr{H}_{\Lambda_s}(f)$ is well defined on $\mathbb{C}$, entire and we have

$$\forall \lambda \in \mathbb{C}, \quad \mathscr{H}_{\Lambda_s} f(\lambda) = \int_{\mathbb{R}} f(x) \Phi_s(\lambda, x) A(x) dx, \quad \lambda \in \mathbb{R}. \tag{4.70}$$

Proof. The first assertion follows from Hölder's inequality, the relation (2.18), and the derivation theorem under the integral sign. □

THEOREM 10. *Let f be a measurable function on $\mathbb{R}$ such that*

$$e^{a|x|^\gamma} f \in L^p_A(\mathbb{R}) \text{ and } e^{b\lambda^\delta} \mathscr{H}_{\Lambda_s}(f) \in L^q_{\nu_s}(\mathbb{R}), \tag{4.71}$$

for some constants $a, b > 0$, $1 \leqslant p, q \leqslant \infty$, and γ, δ be positive real numbers satisfying $\gamma > 2$ and $\frac{1}{\gamma} + \frac{1}{\delta} = 1$. If $(a\gamma)^{\frac{1}{\gamma}} (b\delta)^{\frac{1}{\delta}} > \left(\sin(\frac{\pi}{2}(\delta - 1)) \right)^{\frac{1}{\delta}}$, then $f = 0$.

Proof. From the relations (2.18) and (4.70), we have

$$\forall \lambda \in \mathbb{C}, \quad |\mathscr{H}_{\Lambda_s} f(\lambda)| \leqslant \int_{\mathbb{R}} |f(x)| e^{|\mathrm{Im}\,\lambda|\,|x|} A(x)dx. \tag{4.72}$$

First case $p = 1$

Let $R \in I := \left((b\delta)^{\frac{-1}{\delta}}\left(\sin(\frac{\pi}{2}(\delta-1))\right)^{\frac{1}{\delta}}, (a\gamma)^{\frac{1}{\gamma}}\right)$. The inequality (4.72) and the convexity inequality

$$|\mathrm{Im}\,\lambda|\,|x| \leqslant \frac{R^\gamma}{\gamma}|x|^\gamma + \frac{1}{\delta R^\delta}|\mathrm{Im}\,\lambda|^\delta,$$

imply that for all $\lambda \in \mathbb{C}$, we have

$$\begin{aligned} |\mathscr{H}_{\Lambda_s} f(\lambda)| &\leqslant \int_{\mathbb{R}} e^{a|x|^\gamma} |f(x)| e^{-a|x|^\gamma} e^{|\mathrm{Im}\,\lambda|\,|x|} A(x)dx \\ &\leqslant e^{\frac{1}{\delta R^\delta}|\mathrm{Im}\,\lambda|^\delta} \int_{\mathbb{R}} e^{a|x|^\gamma} |f(x)| e^{(\frac{R^\gamma}{\gamma}-a)|x|^\gamma} A(x)dx. \end{aligned}$$

As $a > \frac{R^\gamma}{\gamma}$, we obtain

$$\forall \lambda \in \mathbb{C}, \quad |\mathscr{H}_{\Lambda_s} f(\lambda)| \leqslant C e^{\frac{1}{\delta R^\delta}|\mathrm{Im}\,\lambda|^\delta}. \tag{4.73}$$

Second case $p \in (1, \infty]$

By applying Hölder's inequality to the relation (4.72) we get

$$\begin{aligned} \forall \lambda \in \mathbb{C}, \quad |\mathscr{H}_{\Lambda_s} f(\lambda)| &\leqslant \int_{\mathbb{R}} e^{a|x|^\gamma} |f(x)| e^{-a|x|^\gamma} e^{|\mathrm{Im}\,\lambda|\,|x|} A(x)dx \\ &\leqslant \left(\int_{\mathbb{R}} e^{-ap'|x|^\gamma} e^{p'|\mathrm{Im}\,\lambda|\,|x|} A(x)dx\right)^{\frac{1}{p'}} \left(\int_{\mathbb{R}} e^{ap|x|^\gamma} |f(x)|^p A(x)dx\right)^{\frac{1}{p}}, \end{aligned}$$

where p' is the conjugate exponent of p.

As above, let $R \in I := \left((b\delta)^{\frac{-1}{\delta}}\left(\sin(\frac{\pi}{2}(\delta-1))\right)^{\frac{1}{\delta}}, (a\gamma)^{\frac{1}{\gamma}}\right)$. The convexity inequality

$$|\mathrm{Im}\,\lambda|\,|x| \leqslant \frac{R^\gamma}{\gamma}|x|^\gamma + \frac{1}{\delta R^\delta}|\mathrm{Im}\,\lambda|^\delta,$$

imply that

$$\int_{\mathbb{R}} e^{-ap'|x|^\gamma} e^{p'|\mathrm{Im}\,\lambda|\,|x|} A(x)dx \leqslant C e^{\frac{p'}{\delta R^\delta}|\mathrm{Im}\,\lambda|^\delta} \int_{\mathbb{R}} e^{p'(\frac{R^\gamma}{\gamma}-a)|x|^\gamma} A(x)dx. \tag{4.74}$$

As $a > \frac{R^\gamma}{\gamma}$, we obtain

$$\forall \lambda \in \mathbb{C}, \quad |\mathscr{H}_{\Lambda_s} f(\lambda)| \leqslant C e^{\frac{1}{\delta R^\delta}|\mathrm{Im}\,\lambda|^\delta}. \tag{4.75}$$

Condition (4.71), inequalities (4.73) and (4.75) imply that the function $\mathscr{H}_{\Lambda_s}(f)$ satisfies the assumptions (4.67) and (4.68) of Lemma 6 with $r = \delta$, $\tau = \frac{1}{\delta R^\delta}$ and $B = b$. The condition $K \in I$ implies that

$$b > \frac{1}{\delta R^\delta} \sin\left(\frac{\pi}{2}(\delta-1)\right),$$

which gives $\mathscr{H}_{\Lambda_s}(f) = 0$ by Lemma 6. □

5. Generalized Cowling-Price theorem for the generalized Hartley transform

In this section we assume that $\rho = 0$.

THEOREM 11. *Let f be a measurable function on $\mathbb{R}$ such that*

$$\int_{\mathbb{R}} \frac{\left(E_{\frac{1}{4a}}(x)\right)^{-p} |f(x)|^p}{(1+|x|)^n} A(x)dx < \infty \tag{5.76}$$

and

$$\int_{\mathbb{R}} \frac{e^{bq\xi^2} |\mathscr{H}_{\Lambda_s}(f)(\xi)|^q}{(1+|\xi|)^m} d\xi < \infty, \tag{5.77}$$

for some constants $a,b,n > 0$, $m > 1$ and $1 \leqslant p,q < \infty$. Then

i) *If $ab > \frac{1}{4}$, we have $f = 0$ almost everywhere.*

ii) *If $ab = \frac{1}{4}$, then f is of the form $f = \sum_{j=0}^{d} C_j N_j(b,.)$ where $d \leqslant \min(\frac{n}{p} + \frac{2\alpha+1}{2p'}, \frac{m-1}{q})$, where p' is the conjugate of p. Especially, if*

$$n \leqslant 1 + p\min(\frac{n}{p} + \frac{2\alpha+1}{2p'}, \frac{m-1}{q}),$$

then $f = 0$ almost everywhere. Furthermore, if $n > 1$ and $m \in (1, q+1]$, then f is a constant multiple of E_b.

iii) *If $ab < \frac{1}{4}$, for all $\delta \in (b, \frac{1}{4a})$, the functions of the form $f = \sum_{j=0}^{d} C_j N_j(\delta,.)$, $d \in \mathbb{N}$, satisfy (5.76) and (5.77).*

Proof. We shall show that $\mathscr{H}_{\Lambda_s}(f)(z)$ exists and is an entire function in $z \in \mathbb{C}$ and

$$|\mathscr{H}_{\Lambda_s}(f)(z)| \leqslant Ce^{\frac{1}{4a}|\mathrm{Im} z|^2}(1+|\mathrm{Im} z|)^s, \quad \text{for all}\, z \in \mathbb{C}, \quad \text{for some} \quad s > 0. \tag{5.78}$$

The first assertion follows from the hypothesis on the function f and Hölder's inequality using (5.76) and the derivation theorem under the integral sign. We want to prove (5.78). Actually, it follows from (2.18) that for all $z = \xi + i\eta \in \mathbb{C}$,

$$\begin{aligned}
|\mathscr{H}_{\Lambda_s}(f)(\xi+i\eta)| &\leqslant \int_{\mathbb{R}} |f(x)||\Phi_s(\xi+i\eta,x)|A(x)dx \\
&\leqslant C\int_{\mathbb{R}} \frac{\left(E_{\frac{1}{4a}}(x)\right)^{-1}|f(x)|}{(1+|x|)^{\frac{n}{p}}}(1+|x|)^{\frac{n}{p}} E_{\frac{1}{4a}}(x)e^{|\eta||x|}A(x)dx \\
&\leqslant Ce^{\frac{|\eta|^2}{4a}} \int_{\mathbb{R}} \frac{\left(E_{\frac{1}{4a}}(x)\right)^{-1}|f(x)|}{(1+|x|)^{\frac{n}{p}}}(1+|x|)^{\frac{n}{p}} e^{-a(|x|-\frac{|\eta|}{2a})^2}A(x)dx.
\end{aligned}$$

Then by using the Hölder inequality, and the relations (5.76) and (1.7), we can obtain

$$\begin{aligned}
|\mathscr{H}_{\Lambda_s}(f)(\xi+i\eta)| &\leqslant Ce^{\frac{|\eta|^2}{4a}}\Big(\int_{\mathbb{R}}(1+|x|)^{\frac{np'}{p}}e^{-ap'(|x|-\frac{|\eta|}{2a})^2}A(x)dx\Big)^{\frac{1}{p'}}\\
&\leqslant Ce^{\frac{|\eta|^2}{4a}}\Big(\int_0^{\infty}(1+t)^{\frac{np'}{p}+2\alpha+1}e^{-ap'(t-\frac{|\eta|}{2a})^2}dt\Big)^{\frac{1}{p'}}\\
&\leqslant Ce^{\frac{1}{4a}|\mathrm{Im}z|^2}(1+|\mathrm{Im}z|)^{\frac{n}{p}+\frac{2\alpha+1}{2p'}}.
\end{aligned}$$

Thus (5.78) is proved.

If $ab=\frac{1}{4}$, then

$$|\mathscr{H}_{\Lambda_s}(f)(\xi+i\eta)| \leqslant Ce^{b|\mathrm{Im}z|^2}(1+|\mathrm{Im}z|)^{\frac{n}{p}+\frac{2\alpha+1}{2p'}}.$$

Therefore, if we let $g(z)=e^{bz^2}\mathscr{H}_{\Lambda_s}(f)(z)$, then

$$|g(z)| \leqslant Ce^{b(\mathrm{Re}z)^2}(1+|\mathrm{Im}z|)^{\frac{n}{p}+\frac{2\alpha+1}{2p'}}.$$

Hence it follows from (5.77) that

$$\int_{\mathbb{R}}\frac{|g(\xi)|^q}{(1+|\xi|)^m}d\xi<\infty.$$

Here we use the following lemma.

LEMMA 7. ([20]) *Let h be an entire function on $\mathbb{C}$ such that*

$$|h(z)| \leqslant Ce^{a|\mathrm{Re}z|^2}(1+|\mathrm{Im}z|)^m$$

for some $m>0$, $a>0$ and

$$\int_{\mathbb{R}}\frac{|h(x)|^q}{(1+|x|)^s}|Q(x)|dx<\infty$$

for some $q\geqslant 1$, $s>1$ and $Q\in\mathscr{P}_M(\mathbb{R})$. Then h is a polynomial with $\deg h\leqslant \min\{m,\frac{s-M-1}{q}\}$ *and, if $s\leqslant q+M+1$, then h is a constant.*

Hence by this lemma g is a polynomial, we say P_b, with $\deg P_b := d \leqslant \min\{\frac{n}{p}+\frac{2\alpha+1}{2p'},\frac{m-1}{q}\}$. Then

$$\mathscr{H}_{\Lambda_s}(f)(x)=P_b(x)e^{-bx^2}$$

and thus,

$$f(x)=\sum_{j=0}^{d}C_jN_j(b,.) \quad \text{for all } x\in\mathbb{R}.$$

Therefore, nonzero f satisfies (5.76) provided that

$$n>1+p\min\Big\{\frac{n}{p}+\frac{2\alpha+1}{2p'},\frac{m-1}{q}\Big\}.$$

Furthermore, if $m \leqslant q+1$, then g is a constant by the Lemma 7 and thus

$$\mathscr{H}_{\Lambda_s}(f)(x) = Ce^{-bx^2} \quad \text{and} \quad f(x) = C_b E_b(x).$$

When $n > 1$ and $m > 1$, these functions satisfy (5.77) and (5.76) respectively. This proves ii).

If $ab > \frac{1}{4}$, then we can choose positive constants, a_1, b_1 such that $a > a_1 = \frac{1}{4b_1} > \frac{1}{4b}$. Then f and $\mathscr{H}_{\Lambda_s}(f)$ also satisfy (5.76) and (5.77) with a and b replaced by a_1 and b_1 respectively. Therefore, it follows that $\mathscr{H}_{\Lambda_s}(f)(x) = P_{b_1}(x)e^{-b_1x^2}$. But then $\mathscr{H}_{\Lambda_s}(f)$ cannot satisfy (5.77) unless $P_{b_1} \equiv 0$, which implies $f \equiv 0$. This proves i).

If $ab < \frac{1}{4}$, then for all $\delta \in (b, \frac{1}{4a})$, the functions of the form $f(x) = \sum_{j=0}^{d} C_j N_j(\delta, .)$, where $d \in \mathbb{N}$, satisfy (5.76) and (5.77). This proves iii). $\square$

6. Beurling's theorem for the generalized Hartley transform

Beurling's theorem and Bonami, Demange, and Jaming's extension are generalized for the generalized Hartley transform as follows.

THEOREM 12. *Let $N \in \mathbb{N}$, $\delta > 0$ and $f \in L^2_A(\mathbb{R})$ satisfy*

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|f(x)||\mathscr{H}_{\Lambda_s}(f)(y)||R(y)|^{\delta}}{(1+|x|+|y|)^N} e^{|x|\,|y|} A(x) dx d\nu_s(y) < \infty, \tag{6.79}$$

where R is a polynomial of degree m. If $N \geqslant m\delta + 3$, then

$$f(x) = \sum_{j < \frac{N-m\delta-1}{2}} a_j N_j(r,x) \ a.e., \tag{6.80}$$

where $r > 0$, $a_j \in \mathbb{C}$. Otherwise, $f(x) = 0$ almost everywhere.

Proof. We start the following lemma.

LEMMA 8. *We suppose that $f \in L^2_A(\mathbb{R})$ satisfies (6.79). Then $f \in L^1_A(\mathbb{R})$.*

Proof. We may suppose that f is not negligible. (6.79) and the Fubini theorem imply that for almost every $(t,y) \in \mathbb{R}$,

$$\frac{|\mathscr{H}_{\Lambda_s}(f)(y)||R(y)|^{\delta}}{(1+|y|)^N} \int_{\mathbb{R}} \frac{|f(x)|}{(1+|x|)^N} e^{|x|\,|y|} A(x) dx < \infty.$$

Since f and thus, $\mathscr{H}_{\Lambda_s}(f)$ are not negligible, there exist $y_0 \in \mathbb{R}$, $y_0 \neq 0$, such that

$$\mathscr{H}_{\Lambda_s}(f)(y_0) R(y_0) \neq 0.$$

Therefore,

$$\int_{\mathbb{R}} \frac{|f(x)|}{(1+|x|)^N} e^{|x|\,|y_0|} A(x)dx < \infty.$$

Since $\dfrac{e^{|x|\,|y_0|}}{(1+|x|)^N} \geqslant 1$ for large x, it follows that $\displaystyle\int_{\mathbb{R}} |f(x)|A(x)dx < \infty$. $\square$

This lemma and Proposition 3 imply that ${}^tV_s(f)$ is well-defined almost everywhere on $\mathbb{R}$. By the same techniques used in [15], we can deduce that

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{e^{|x|\,|y|}|{}^tV_s(f)(x)||\mathscr{H}_C({}^tV_s)(f)(y)||R(y)|^{\delta}}{(1+|x|+|y|)^N} dxdy < \infty.$$

According to Theorem 2.3 in [19], we conclude that for all $x \in \mathbb{R}$,

$${}^tV_s(f)(x) = P(x)e^{-\frac{x^2}{4r}},$$

where $r > 0$ and P a polynomial of degree strictly lower than $\frac{N-m\delta-1}{2}$. Then by (2.33),

$$\mathscr{H}_{\Lambda_s}(f)(y) = \mathscr{H}_C \circ {}^tV_s(f)(y) = \mathscr{H}_C\left(P(x)e^{-\frac{x^2}{4r}}\right)(y) = Q(y)e^{-ry^2},$$

where Q is a polynomial of degree $\deg P$. Then by using properties of the generalized heat kernels functions we can find constants a_j such that

$$\mathscr{H}_{\Lambda_s}(f)(y) = \mathscr{H}_{\Lambda_s}\Big(\sum_{j<\frac{N-m\delta-1}{2}} a_j N_j(r,.)\Big)(y).$$

By the injectivity of $\mathscr{H}_{\Lambda_s}$ the desired result follows. $\square$

As an application of Theorem 12, we want to prove the following Gelfand-Shilov type theorem for the generalized Hartley transform.

COROLLARY 2. *Let $N,m \in \mathbb{N}$, $\delta > 0$, $a,b > 0$ with $ab \geqslant \frac{1}{4}$, and $1 < p,q < \infty$ with $\frac{1}{p}+\frac{1}{q} = 1$. Let $f \in L^2_A(\mathbb{R})$ satisfy*

$$\int_{\mathbb{R}} \frac{|f(x)|e^{\frac{(2a)^p}{p}|x|^p}}{(1+|x|)^N} A(x)dx < \infty \tag{6.81}$$

and

$$\int_{\mathbb{R}} \frac{|\mathscr{H}_{\Lambda_s}(f)(y)|e^{\frac{(2b)^q}{q}|y|^q}|R(y)|^{\delta}}{(1+|y|)^N} d\nu_s(y) < \infty \tag{6.82}$$

for some $R \in \mathscr{P}_m$.

i) If $ab > \frac{1}{4}$ or $(p,q) \neq (2,2)$, then $f(x) = 0$ almost everywhere.

ii) If $ab = \frac{1}{4}$ and $(p,q) = (2,2)$, then f is of the form (6.80) whenever $N \geqslant \frac{m\delta+3}{2}$ and $r = 2b^2$. Otherwise, $f(x) = 0$ almost everywhere.

Proof. Since

$$4ab|x||y| \leqslant \frac{(2a)^p}{p}|x|^p + \frac{(2b)^q}{q}|y|^q,$$

it follows from (6.81) and (6.82) that

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|f(x)||\mathscr{H}_{\Lambda_s}(f)(y)||R(y)|^\delta}{(1+|x|+|y|)^{2N}} e^{4ab|x|\,|y|} A(x)dxd\nu_s(y) < \infty.$$

Then (6.79) is satisfied, because $4ab \geqslant 1$. Therefore, according to the proof of Theorem 12, we can deduce that

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{e^{4ab|x|\,|y|}|{}^tV_s(f)(x)||\mathscr{H}_C({}^tV_s)(f)(y)||R(y)|^\delta}{(1+|x|+|y|)^{2N}} dxdy < \infty,$$

and ${}^tV_s(f)$ and f are of the forms

$${}^tV_s(f)(x) = P(x)e^{-\frac{x^2}{4r}} \text{ and } \mathscr{H}_{\Lambda_s}(f)(y) = Q(y)e^{-ry^2},$$

where $r > 0$ and P, Q are polynomials of the same degree strictly lower than $\frac{2N-m\delta-1}{2}$. Therefore, substituting these from, we can deduce that

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{e^{-(\sqrt{r}|y|-\frac{1}{2\sqrt{r}}|x|)^2} e^{(4ab-1)|x|\,|y|}|P(x)||Q(x)||R(y)|^\delta}{(1+|x|+|y|)^{2N}} dxdy < \infty.$$

When $4ab > 1$, this integral is not finite unless $f = 0$ almost everywhere. Moreover, it follows from (6.81) and (6.82) that

$$\int_{\mathbb{R}} \frac{|P(x)|e^{-\frac{1}{4r}x^2} e^{\frac{(2a)^p}{p}|x|^p}}{(1+|x|)^N} A(x)dx < \infty$$

and

$$\int_{\mathbb{R}} \frac{|Q(y)|e^{-ry^2} e^{\frac{(2b)^q}{q}|y|^q}|R(y)|^\delta}{(1+|y|)^N} d\nu_s(y) < \infty.$$

Hence, one of these integrals is not finite unless $(p,q) = (2,2)$. When $4ab = 1$ and $(p,q) = (2,2)$, the finiteness of above integrals implies that $r = 2b^2$ and the rest follows from Theorem 12. $\square$

7. Miyachi's theorem for the generalized Hartley transform

THEOREM 13. *Let f be a measurable function on $\mathbb{R}$ such that*

$$\left(E_{\frac{1}{4a}}\right)^{-1} f \in L^p_A(\mathbb{R}) + L^q_A(\mathbb{R}) \tag{7.83}$$

and

$$\int_{\mathbb{R}} \log^+ \frac{e^{b\xi^2}|\mathscr{H}_{\Lambda_s}(f)(\xi)|}{\lambda} d\xi < \infty, \tag{7.84}$$

for some constants $a,b,\lambda > 0$, $1 \leqslant p,q \leqslant \infty$. *Then*

i) If $ab > \frac{1}{4}$, *we have* $f = 0$ *almost everywhere.*

ii) If $ab = \frac{1}{4}$, *we have* $f = CE_b$ *with* $|C| \leqslant \lambda$.

iii) If $ab < \frac{1}{4}$, *for all* $\delta \in (b, \frac{1}{4a})$, *the functions of the form* $f = \sum_{j=0}^{d} C_j N_j(\delta,.)$, $d \in \mathbb{N}$, *satisfy* (7.83) *and* (7.84).

To prove this result we need the following lemmas.

LEMMA 9. ([16]) *Let* h *be an entire on* $\mathbb{C}$ *function such that*

$$|h(z)| \leqslant Ae^{B|\mathrm{Re}z|^2} \text{ and } \int_{\mathbb{R}} \log^+ |h(y)| dy < \infty, \tag{7.85}$$

for some positive constants A,B. *Then* h *is a constant on* $\mathbb{C}$.

LEMMA 10. *Let* $p \in [1,\infty]$ *and* f *a measurable function on* $\mathbb{R}$ *such that* $\left(E_{\frac{1}{4a}}\right)^{-1} f$ *belongs to* $L^p_A(\mathbb{R})$ *for some* $a > 0$. *Then*

$$e^{ay^2}\left({}^tV_s(f)\right) \in L^p(\mathbb{R}).$$

Proof. We consider two cases.

1^{st} *case*: If $p \in [1,\infty[$, from (2.27), we have

$$\left\|e^{ay^2}\left({}^tV_s(f)\right)\right\|^p_{L^p(\mathbb{R})} \leqslant \int_{\mathbb{R}} e^{apy^2}\left(\int_{|x|\geqslant|y|} K_s(x,y)\left[\left(E_{\frac{1}{4a}}\right)^{-1}(x)|f(x)|\right] E_{\frac{1}{4a}}(x)A(x)\,dx\right)^p dy.$$

By applying Hölder's inequality to the middle integral, we obtain

$$\left\|e^{ay^2}\left({}^tV_s(f)\right)\right\|^p_{L^p(\mathbb{R})} \leqslant \int_{\mathbb{R}} e^{apy^2}\, {}^tV_s\left(|\left(E_{\frac{1}{4a}}\right)^{-1} f|^p\right)(y)\left[{}^tV_s\left[\left(E_{\frac{1}{4a}}\right)^{p'}\right](y)\right]^{\frac{p}{p'}} dy,$$

where p' is the conjugate exponent of p. By the relations (2.54), (2.53), and (2.36), we deduce that

$$\left\|e^{ay^2}\left({}^tV_s(f)\right)\right\|_{L^p(\mathbb{R})} \leqslant M\left\|\left(E_{\frac{1}{4a}}\right)^{-1} f\right\|_{L^p_A(\mathbb{R})} < \infty,$$

where $M = \left(C\left(p', \frac{1}{4a}\right)\sqrt{\frac{p'a}{\pi}}\right)^{\frac{1}{p'}}$.

2^{nd} *case*: If $p = \infty$, using (2.27), we obtain for almost all y in $\mathbb{R}$:

$$\begin{aligned}|{}^tV_s(f)(y)| &\leqslant \int_{|x|\geqslant|y|} K_s(x,y)\left(\left(E_{\frac{1}{4a}}\right)^{-1}(x)|f(x)|\right) E_{\frac{1}{4a}}(x)A(x)\,dx \\ &\leqslant \|\left(E_{\frac{1}{4a}}\right)^{-1} f\|_{L^\infty_A(\mathbb{R})}\, {}^tV(E_{\frac{1}{4a}})(y).\end{aligned}$$

By the relation (2.53), we deduce that

$$\|e^{ay^2}\,{}^tV_s(f)(y)\|_{L^\infty_A(\mathbb{R})} \leqslant M_0\|\left(E_{\frac{1}{4a}}\right)^{-1}f\|_{L^\infty_A(\mathbb{R})} < \infty,$$

where $M_0 = \sqrt{\frac{a}{\pi}}$. This completes the proof. □

LEMMA 11. *Let r be in $[1,\infty]$. We consider a function g in $L^r_A(\mathbb{R})$. Then there exists a positive constant C such that:*

$$||e^{ax^2}\,{}^tV_s(E_{\frac{1}{4a}}g)||_{L^r(\mathbb{R})} \leqslant C||g||_{L^r_A(\mathbb{R}))},$$

where $||\cdot||_{L^r(\mathbb{R})}$ is the norm of the usual Lebesgue space $L^r(\mathbb{R})$ and $a > 0$.

Proof. The proof is immediately from Lemma 10. □

LEMMA 12. *Let p, q in $[1,\infty]$ and f a measurable function on $\mathbb{R}$ such that*

$$\left(E_{\frac{1}{4a}}\right)^{-1}f \in L^p_A(\mathbb{R}) + L^q_A(\mathbb{R}), \tag{7.86}$$

for some $a > 0$. Then the function defined on $\mathbb{C}$ by

$$\mathscr{H}_{\Lambda_s}(f)(\lambda) = \int_{\mathbb{R}} f(x)\Phi_s(\lambda,x)A(x)dx, \tag{7.87}$$

is well defined and entire on $\mathbb{C}$. Moreover there exists a positive constant C such that for all ξ,η in $\mathbb{R}$ we have

$$|\mathscr{H}_{\Lambda_s}(f)(\xi+i\eta)| \leqslant Ce^{\frac{\eta^2}{4a}}. \tag{7.88}$$

Proof. The first assertion follows from the hypothesis on the function f and Hölder's inequality using (7.86) and the derivation theorem under the integral sign. We want to prove (7.88).

The condition (7.86) implies that the function f belongs to $L^1_A(\mathbb{R})$. Hence we deduce from (2.33) that for all ξ,η in $\mathbb{R}$ we have

$$\begin{aligned}|\mathscr{H}_{\Lambda_s}(f)(\xi+i\eta)| &= |\int_{\mathbb{R}} {}^tV_s(f)(y)\,\mathrm{cas}(y(\xi+i\eta))dy|.\\ &\leqslant \int_{\mathbb{R}}\left|{}^tV_s(f)(y)\right|(e^{y\eta}+e^{-y\eta})dy.\end{aligned}$$

The integral of the second member can also be estimate in the form

$$e^{\frac{\eta^2}{4a}}\int_{\mathbb{R}} e^{ay^2}|{}^tV_s(f)(y)|(e^{-a(y-\frac{\eta}{2a})^2}+e^{-a(y+\frac{\eta}{2a})^2})dy.$$

Indeed from (7.86) there exists u in $L^p_A(\mathbb{R})$ and v in $L^q_A(\mathbb{R})$ such that

$$f = E_{\frac{1}{4a}}(u+v).$$

Thus using the Lemma 11 and Hölder's inequality we obtain

$$\int_{\mathbb{R}} e^{ay^2}|{}^tV_s(f)(y)|(e^{-a\left(y-\frac{\eta}{2a}\right)^2}+e^{-a\left(y+\frac{\eta}{2a}\right)^2})\,dy \leqslant C(||u||_{L^p_A(\mathbb{R})}+||v||_{L^q_A(\mathbb{R})})<\infty.$$

Therefore, the desired result follows. □

Proof of Theorem 13. We will divide the proof in several cases.

First case: $ab>\frac{1}{4}$.

Consider the function h defined on $\mathbb{C}$ by

$$h(z)=e^{\frac{z^2}{4a}}\mathscr{H}_{\Lambda_s}(f)(z). \tag{7.89}$$

This function is entire on $\mathbb{C}$ and using (7.88) we obtain:

$$|h(\xi+i\eta)|\leqslant Ce^{\frac{\xi^2}{4a}}, \tag{7.90}$$

for all $\xi,\eta\in\mathbb{R}$. On the other hand we have

$$\begin{aligned}\int_{\mathbb{R}_+}\log^+|h(y)|dy &= \int_{\mathbb{R}_+}\log^+|e^{\frac{y^2}{4a}}\mathscr{H}_{\Lambda_s}(f)(y)|dy,\\ &= \int_{\mathbb{R}}\log^+|\lambda e^{(\frac{1}{4a}-b)y^2}\frac{e^{by^2}\mathscr{H}_{\Lambda_s}(f)(y)}{\lambda}|dy\\ &\leqslant \int_{\mathbb{R}}\log^+\left|\frac{e^{by^2}\mathscr{H}_{\Lambda_s}(f)(y)}{\lambda}\right|dy+\int_{\mathbb{R}}e^{(\frac{1}{4a}-b)y^2}dy\end{aligned}$$

because $\log^+(cd)\leqslant\log^+(c)+d$ for all $c,d>0$. Since $ab>\frac{1}{4}$, (7.84) implies that

$$\int_{\mathbb{R}}\log^+|h(y)|dy<\infty. \tag{7.91}$$

From the relations (7.90) and (7.91), it follows from Lemma 9 that there exists a constant C such that

$$h(\xi+i\eta)=C,\ \ \xi,\eta\in\mathbb{R}.$$

Thus

$$\mathscr{H}_{\Lambda_s}(f)(y)=Ce^{-\frac{y^2}{4a}}.$$

Using now the condition (7.84) and that $ab>\frac{1}{4}$, we deduce that $C=0$ and hence from the injectivity of $\mathscr{H}_{\Lambda_s}(f)$ we deduce that $f=0$.

Second case: $ab=\frac{1}{4}$.

The same proof as for the the first step give that

$$\mathscr{H}_{\Lambda_s}(f)(y)=Ce^{-\frac{y^2}{4a}}.$$

Thus (7.84) holds whenever $|C|\leqslant\lambda$. Hence

$$f=Ce^{-\frac{y^2}{4a}},\quad \text{with } |C|\leqslant\lambda.$$

Third case: $ab < \frac{1}{4}$.
If f is a given form, then

$$\mathscr{H}_{\Lambda_s}(f)(y) = Q(y)e^{-\frac{y^2}{4a}}$$

for some $Q \in \mathscr{P}$. These functions clearly satisfy the conditions (7.83),(7.84) for all $\delta \in (b, \frac{1}{4a})$. The proof of the Theorem is complete. $\square$

The following is an immediate corollary of Theorem 13.

COROLLARY 3. *Let f be a measurable function on $\mathbb{R}$ such that*

$$\left(E_{\frac{1}{4a}}\right)^{-1} f \in L^p_A(\mathbb{R}) + L^q_A(\mathbb{R}) \tag{7.92}$$

and

$$\int_{\mathbb{R}} |\mathscr{H}_{\Lambda_s}(f)(\xi)|^r e^{br\xi^2} d\xi < \infty, \tag{7.93}$$

for some constants $a, b, r > 0$ and $1 \leqslant p, q \leqslant \infty$. Then

i) If $ab \geqslant \frac{1}{4}$, we have $f = 0$ almost everywhere.

ii) If $ab < \frac{1}{4}$, then for all $\delta \in (b, \frac{1}{4a})$, all the functions of the form $f = \sum_{j=0}^{d} C_j N_j(\delta, .)$, $d \in \mathbb{N}$, satisfy (7.92) and (7.93).

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REFERENCES

[1] S. BEN FARAH AND K. MOKNI, *Uncertainty principle and $L^p - L^q$-sufficient pairs on non compact real symmetric spaces*, Comptes Rendus de I'Academie des Sciences Paris, Série I, **336**, (2003), 889–892.

[2] R. N. BRACEWELL, *The Hartley Transform*, Oxford University Press, New York, 1986.

[3] A. BEURLING, *The collect works of Arne Beurling*, Birkhäuser, Boston, 1989, 1–2.

[4] W. R. BLOOM AND Z. XU, *Fourier transforms of Schwartz functions on Chébli-Trimèche hypergroups*, Mh. Math. **125** 1998, 89–109.

[5] A. BONAMI, B. DEMANGE AND P. JAMING, *Hermite functions and uncertainty principles for the Fourier and the windowed Fourier transforms*, Rev. Mat. Iberoamericana, **19** (2002), 22–35.

[6] M. G. COWLING AND J. F. PRICE, *Generalizations of Heisenberg inequality*, Lecture Notes in Math., **992**, Springer, Berlin (1983), 443–449.

[7] R. DAHER, T. KAWAZOE AND H. MEJJAOLI, *A generalization of Miyachi's theorem*, J. Math. Soc. Japon. V., **61**, no. 2 (2009), 551–558.

[8] M. EGUICHI, S. KOIZUMI AND K. KUMAHARA, *An L^p version of Hardy theorem for the motion group*, J. Austral. Math. Soc. Serie A., **68**, no. 2 (2000), 55–67.

[9] A. FITOUHI, *Heat polynomials for a singular differential operator on $(0, \infty)$*, J. Constructive approximation, V., **5**, no. 2 (1989), 241–270.

[10] L. GALLARDO AND K. TRIMÈCHE, *An L^p version of Hardy's theorem for the Dunkl transform*, J. Austr. Math. Soc. Volume **77**, Issue 03, (2004), 371–386.

[11] J. M. GELFAND AND N. YA. VILENKIN, *Les distributions, tome 4*, Application de l'analyse harmonique, Dunod, Paris, 1967.
[12] G. H. HARDY, *A theorem concerning Fourier transform*, J. London Math. Soc., **8** (1933), 227–231.
[13] R. V. L. HARTLEY, *A more symmetrical Fourier analysis applied to transmission problems*, Proc IRE. **30**, (1942) 144–150.
[14] L. HÖRMANDER, *A uniqueness theorem of Beurling for Fourier transform pairs*, Ark. För Math., **2** (1991), 237–240.
[15] H. MEJJAOLI, *An analogue of Beurling-Hörmander's theorem associated with Dunkl-Bessel operator*, Fract. Calc. Appl. Anal. **9** (2006), no. 3, 247–264.
[16] H. MEJJAOLI AND M. SALHI, *Uncertainty principles for the Weinstein transform*, Czechoslovak Mathematical Journal (2011), Volume **61**, Issue 4, 941–974.
[17] A. MIYACHI, *A generalization of theorem of Hardy*, Harmonic Analysis Seminar held at Izunagaoka, Shizuoka-Ken, Japon 1997, 44–51.
[18] G. W. MORGAN, *A note on Fourier transforms*, J. London Math. Soc., **9** (1934), 188–192.
[19] S. PARUI AND R. P. SARKAR, *Beurling's theorem and L^p-L^q Morgan's theorem for step two nilpotent Lie groups*, Publ. Res. Inst. Math. Sci **44**, (2008), 1027–1056.
[20] S. K. RAY AND R. P. SARKAR, *Cowling-Price theorem and characterization of heat kernel on symmetric spaces*, Proc. Indian Acad. Sci. (Math. Sci.), **114** (2004), 159–180.
[21] L. SCHWARTZ, *Théorie des distributions*, Hermann, Paris, 1966.
[22] K. TRIMÈCHE, *Inversion of the J.L. Lions transmutation operators using generalized wavelets*, Applied and Computational Harmonic Analysis, **4** (1997), 97–112.
[23] K. TRIMÈCHE, *Positivity of the transmutation operators associated with a Cherednik type operator on the real line*, Advances in Pure and Applied Mathematics, Volume **3**, Issue 4, (2013), 361–376.
[24] K. TRIMÈCHE, *The transmutation operators relating to a Dunkl Type operator on $\mathbb{R}$ and their positivity*, Mediterr. J. Math. May 2015, Volume **12**, Issue 2, 349–369.
[25] S. B. YAKUBOVICH, *Uncertainty principles for the Kontorovich-Lebedev transform*, Math. Model. Anal., **13** (2) (2008), 289–302.

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