CONVOLUTION PRESERVES PARTIAL SYNCHRONICITY OF LOG–CONCAVE SEQUENCES

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Abstract. In a recent proof of the log-concavity of genus polynomials of some families of graphs, Gross et al. defined the weak synchronicity relation between log-concave sequences, and conjectured that the convolution operation by any log-concave sequence preserves weak synchronicity. In this paper we disprove it by providing a counterexample. Furthermore, we introduce the so-called partial synchronicity relation between log-concave sequences, which is proved to be (i) weaker than synchronicity, (ii) stronger than weak synchronicity, and (iii) preserved by the convolution operation.

1. Introduction

The log-concavity of sequences of nonnegative numbers has been paid extensive and intensive attention during the past thirty years, see Stanley [10] and Brenti [3,4]. In the late 1980s, Gross et al. [6] posed the LCGD conjecture that the genus polynomial of every graph is log-concave, which firstly connected the log-concavity of sequences with topological graph theory, or more precisely, with the surface embedding of graphs. For survey books of topological graph theory, see [7, 1]. In their recent work [5], Gross et al. established a criterion determining the log-concavity of sum of products of ratio-dominance polynomials. With aid of that criterion, they confirmed the LCGD conjecture for several families of graphs generated by vertex- or edge-amalgamations, including the graphs called iterated 4-wheels.

The criterion is considered to have its own interest, since it deals with the intrinsic arithmetic relations between log-concave polynomials. See [2, 9, 8] for related works. The idea of the criterion consists of three key parts, the synchronicity, the ratio-dominance, and the lexicographicity. It is the synchronicity part, which originally arises from common facts observed from topological embeddings of graphs into surfaces, starts the whole development of the new log-concave results.

Though the synchronicity relation is sufficient to ensure the log-concavity of positive linear combination of log-concave polynomials, Gross et al. managed to weaken it to certain weak synchronicity relation. Such a weaker relation was supposed to be preserved by sequence convolution, which was posed as the following conjecture; see [5, Conjecture 2.13].

CONJECTURE 1.1. *Let A, B, C be three log-concave nonnegative sequences without internal zeros. If $A \sim_{\mathrm{w}} B$, then the convolution sequences $A * C$ and $B * C$ are weakly synchronized as well.*

We disprove Conjecture 1.1 by providing an explicit counterexample. This example actually helps us to find a relation in Definition 2.4, called partial synchronicity, between log-concave sequences, to achieve the original motivation. Namely, the partial synchronicity relation is (i) weaker than synchronicity, (ii) stronger than the weak synchronicity, and (iii) preserved by the convolution operation. See Theorems 3.3 and 3.6.

2. Preliminaries and the counterexample

All sequences concerned in this paper consist of nonnegative numbers. For any finite sequence $A = (a_k)_{k=0}^n$ of nonnegative numbers, we identify the sequence A with the infinite sequence $(a_k')_{k \in \mathbb{N}}$, where $a_k' = a_k$ for $0 \leqslant k \leqslant n$, and $a_k' = 0$ otherwise. Under this convenience, one may denote by $A = (a_k)$ for simplicity. Furthermore we use uA to denote the scalar multiple sequence (ua_k), for any constant $u \geqslant 0$. Suppose $B = (b_k)$ is another sequence of nonnegative numbers. The notation $A + B$ stands for the sequence $(a_k + b_k)$.

Without loss of generality, we may assume that the first term of the sequence is always positive for all sequences concerned in this paper. The sequence A is said to be *log-concave* if $a_k^2 \geqslant a_{k-1}a_{k+1}$ for all integers k. It is said to *have no internal zeros* if for any integers $i < j$ with $a_i a_j > 0$, one has $\prod_{k=i}^j a_k > 0$. Denote by $\mathscr{L}$ the set of log-concave sequences without internal zeros. We call the sequence consisting of only zeros the *zero sequence*, denoted (0). Define

$$\mathscr{L}^* = \mathscr{L} \setminus \{(0)\}.$$

DEFINITION 2.1. *Let $A = (a_k) \in \mathscr{L}$ and $B = (b_k) \in \mathscr{L}$. We say that the sequences A and B are synchronized, denoted as $A \sim B$, if*

$$a_{k-1}b_{k+1} \leqslant a_k b_k \qquad \text{and} \qquad a_{k+1}b_{k-1} \leqslant a_k b_k$$

for all k.

It is obvious that scalar multiplications preserve synchronicity. Moreover, the synchronicity relation is reflexive, symmetric and non-transitive; see [5].

DEFINITION 2.2. *Let $A = (a_k) \in \mathscr{L}$ and $B = (b_k) \in \mathscr{L}$. We say that the sequences A and B are weakly synchronized, denoted $A \sim_{\mathrm{w}} B$, if*

$$a_{k-1}b_{k+1} + a_{k+1}b_{k-1} \leqslant 2a_k b_k, \qquad (2.1)$$

for all k.

For example, consider the sequences $A = (1,3,5)$ and $B = (1,4,13)$. It is easy to verify that $A \sim_w B$ while $A \not\sim B$.

Recall that if $A = (a_k)$ and $B = (b_k)$, the convolution sequence $A * B$ is defined to be the coefficient sequence of the polynomial product

$$\left(\sum_{i \geq 0} a_i x^i \right) \left(\sum_{i \geq 0} b_j x^j \right).$$

The convolution preserves log-concavity; see Stanley [10].

The following example disproves Conjecture 1.1.

EXAMPLE 2.3. Let

$$A = (1, 20, 200, 1800),$$
$$B = (1, 6, 30, 60),$$
$$C = (40, 60, 10, 1).$$

It is direct to verify $A \sim_w B$ from Definition 2.2, and to compute that

$$A * C = (40, 860, 9210, 84201, 110020, 18200, 1800),$$
$$B * C = (40, 300, 1570, 4261, 3906, 630, 60).$$

Then, for the convolution sequences $A * C$ and $B * C$, Ineq. (2.1) does not hold for $k = 2$:

$$(A * C)_1 (B * C)_3 + (A * C)_3 (B * C)_1 - 2(A * C)_2 (B * C)_2$$
$$= 860 \times 4261 + 84201 \times 300 - 2 \times 9210 \times 1570$$
$$= 5360$$
$$> 0.$$

Next we introduce the partial synchronicity relation between log-concave sequences, which is expected to serve the original motivation of Gross et al. in [5].

DEFINITION 2.4. Let $A, B \in \mathcal{L}$. We say that the sequences A and B are *partially synchronized*, denoted by $A \sim_p B$, if

$$a_m b_n + a_n b_m \geq a_{m+1} b_{n-1} + a_{n-1} b_{m+1}, \tag{2.2}$$

holds for all integers m and n such that $m \geq n$.

EXAMPLE 2.5. Note that in the previous example $A \not\sim_p B$ either. Thus we modify the sequence A slightly by revising its vert last entry,

$$A' = (1, 20, 200, 1740).$$

The sequences B and C are still intact. It is direct to verify $A \sim_p B$ from Definition 2.4, and to compute that

$$A' * C = (40, 860, 9210, 81801, 106420, 17600, 1740),$$
$$B * C = (40, 300, 1570, 4261, 3906, 630, 60).$$

Then, for the convolution sequences $A' * C$ and $B * C$, Ineq. (2.2) holds after a fair amount of calculations. In particular, for the case $m = n = 2$:

$$(A' * C)_1 (B * C)_3 + (A' * C)_3 (B * C)_1 - 2(A' * C)_2 (B * C)_2$$
$$= 860 \times 4261 + 81801 \times 300 - 2 \times 9210 \times 1570$$
$$= -714640$$
$$< 0.$$

3. The partial synchronicity relation

Let $A = (a_k)$ and $B = (b_k)$ be two sequences of numbers.

LEMMA 3.1. *Suppose that*

$$a_m b_n + a_n b_m \geqslant a_{m+1} b_{n-1} + a_{n-1} b_{m+1}, \tag{3.1}$$

for all integers m and n such that $m \geqslant n$. Then we have

$$a_p b_q + a_q b_p \geqslant a_r b_s + a_s b_r, \tag{3.2}$$

for any integers p, q, r, s such that

$$p + q = r + s, \qquad and \tag{3.3}$$
$$|p - q| < |r - s|. \tag{3.4}$$

Proof. Let p, q, r, s be integers satisfying Eq. (3.3) and Ineq. (3.4). In order to prove Ineq. (3.2), by symmetry we may assume that that $p \geqslant q$ and $r \geqslant s$. Then Ineq. (3.4) reduces to

$$p - q < r - s. \tag{3.5}$$

Summing up Eq. (3.3) and Ineq. (3.5), one obtains that

$$p < r. \tag{3.6}$$

Substituting $m = p$ and $n = q$ in the premise Ineq. (3.1), one finds that

$$a_p b_q + a_q b_p \geqslant a_{p+1} b_{q-1} + a_{q-1} b_{p+1}. \tag{3.7}$$

Since $p \geqslant q$, we have $p + 1 \geqslant q - 1$. Therefore, in Ineq. (3.7), by replacing the number p by $p + 1$, and replacing q by $q - 1$, we obtain that

$$a_{p+1} b_{q-1} + a_{q-1} b_{p+1} \geqslant a_{p+2} b_{q-2} + a_{q-2} b_{p+2}. \tag{3.8}$$

The same substitution for Ineq. (3.8) gives that

$$a_{p+2}b_{q-2} + a_{q-2}b_{p+2} \geqslant a_{p+3}b_{q-3} + a_{q-3}b_{p+3}.$$

Continuing in this way, one finds

$$a_{p+i-1}b_{q-i+1} + a_{q-i+1}b_{p+i-1} \geqslant a_{p+i}b_{q-i} + a_{q-i}b_{p+i} \tag{3.9}$$

for all positive integers i. Since $p < r$ from Ineq. (3.6), we can sum up Ineq. (3.9) over $i \in \{1, 2, \ldots, r-p\}$, which yields that

$$a_p b_q + a_q b_p \geqslant a_r b_{q-r+p} + a_{q-r+p} b_r.$$

Hence, we obtain the desired Ineq. (3.2), by noticing $s = q - r + p$ from Eq. (3.3). □

It is clear that scalar multiplications preserve partial synchronicity. Moreover, the partial synchronicity relation is reflexive, symmetric, and non-transitive. The non-transitivity can be seen from the example

$$A = (1, 2, 3), \qquad B = (1, 3, 8), \qquad C = (1, 4, 15),$$

where $A \sim_{\mathrm{p}} B$, $B \sim_{\mathrm{p}} C$, but $A \not\sim_{\mathrm{p}} C$. In fact, the above example has been used to exemplify the non-transitivity of the synchronicity relation in [5].

The next proposition helps check quickly the weak synchronicity of two sequences, which is also of help in the proof of Theorem 3.3.

We name the first positive term of the sequence $A \in \mathscr{L}^*$ the *head* of A, and the last positive term the *tail* of A. In other words, the term a_h is said to be the head of A either $h = 0$ and $a_0 > 0$ or $h > 0$ and $a_{h-1} = 0 < a_h$. We call the integer h the *head index* of A, denoted by $h(A) = h$. Similarly, one may define the *tail index*, denoted by $t(A)$. It is clear that $h(A) \leqslant t(A)$.

PROPOSITION 3.2. *Let* $A, B \in \mathscr{L}^*$. *Then* $A \sim_{\mathrm{p}} B$ *holds iff*

(i) $|h(A) - h(B)| \leqslant 1$;

(ii) $|t(A) - t(B)| \leqslant 1$; *and*

(iii) *Ineq. (2.2) holds for all integers* m *and* n *satisfying the following restrictions:*

$$\max\{h(A), h(B)\} \leqslant m \leqslant \max\{t(A), t(B)\} - 1,$$
$$\min\{h(A), h(B)\} + 1 \leqslant n \leqslant \min\{t(A), t(B)\},$$

and that $m \geqslant n$.

Proof. Let $A = (a_k)$ and $B = (b_k)$ be sequences such that $A, B \in \mathscr{L}^*$.
Necessity. Suppose that $A \sim_{\mathrm{p}} B$, i.e., Ineq. (2.2) holds for all integers m and n such that $m \geqslant n$.

In order to show (i), one may suppose that $h(A) \leqslant h(B)$ without loss of generality. Assume to the contrary that $|h(A) - h(B)| \geqslant 2$. Take

$$m = h(B) - 1 \qquad \text{and} \qquad n = h(A) + 1.$$

It follows that $m \geqslant n$, and that Ineq. (2.2) becomes

$$a_{h(B)-1}b_{h(A)+1} + a_{h(A)+1}b_{h(B)-1} \geqslant a_{h(B)}b_{h(A)} + a_{h(A)}b_{h(B)}. \tag{3.10}$$

From definition of the head $h(B)$, we have

$$b_{h(B)-1} = 0. \tag{3.11}$$

On the other hand, since $h(B) - h(A) \geqslant 2$, we have

$$b_{h(A)} = 0 \qquad \text{and} \qquad b_{h(A)+1} = 0. \tag{3.12}$$

Substituting Eqs. (3.11) and (3.12) into Ineq. (3.10), one obtains

$$0 \geqslant a_{h(A)}b_{h(B)}. \tag{3.13}$$

From the definition of the head function h, one sees that $a_{h(A)} > 0$ and $b_{h(B)} > 0$, contradicting Ineq. (3.13).

Condition (ii) can be shown along the same lines. The necessity of (iii) is obvious from the premise $A \sim_{\mathrm{p}} B$.

Sufficiency. For convenience, we denote

$$mh = \min\{h(A), h(B)\}, \qquad mt = \min\{t(A), t(B)\},$$
$$Mh = \max\{h(A), h(B)\}, \qquad Mt = \max\{t(A), t(B)\}.$$

If $m \geqslant Mt$, then $a_{m+1} = b_{m+1} = 0$, and thus

$$a_{m+1}b_{n-1} + a_{n-1}b_{m+1} = 0. \tag{3.14}$$

Since $a_m b_n + a_n b_m \geqslant 0$, Ineq. (3.1) holds. Below we can suppose that

$$m \leqslant Mt - 1. \tag{3.15}$$

In another case that $n \leqslant mh$, we have $a_{n-1} = b_{n-1} = 0$. Therefore, we infer Eq. (3.14), which allows us to suppose without loss of generality that

$$n \geqslant mh + 1. \tag{3.16}$$

In view of (iii), Ineqs. (3.15) and (3.16), it suffices to prove Ineq. (3.1) for all integers m and n such that $m \geqslant n$, and that either $m \leqslant Mh - 1$ or $n \geqslant mt + 1$.

When $m \leqslant Mh - 1$, by using Ineq. (3.16), one may deduce that

$$Mh - 1 \geqslant m \geqslant n \geqslant mh + 1,$$

contradicting Condition (i), which implies that $Mh - mh \leqslant 1$. When $n \geqslant mt + 1$, by using Ineq. (3.15), we can derive that

$$mt + 1 \leqslant n \leqslant m \leqslant Mt + 1,$$

contradicting Condition (ii), which implies that $Mt - mt \leqslant 1$. This completes the proof. □

Now we are ready to clarify the relations among the synchronicity, the weak synchronicity, and the partial synchronicity.

THEOREM 3.3. *The synchronicity relation $\sim$ is stronger than the partial synchronicity relation $\sim_p$, and $\sim_p$ yields the weak synchronicity $\sim_w$. In other words, any two synchronized log-concave sequences without internal zeros must be partially synchronized, and any two partially synchronized log-concave sequences without internal zeros are weakly synchronized.*

Proof. Taking $m = n = k$ in Ineq. (2.2) gives Ineq. (2.1), which immediately shows that the partial synchronicity $\sim_p$ is stronger than weak synchronicity $\sim_w$.

Now, take $A = (a_k) \in \mathscr{L}$, $B = (b_k) \in \mathscr{L}$ with $A \sim B$. Suppose $m \geqslant n$. We are going to show Ineq. (2.2). By Proposition 3.2 (iii), we may suppose that

$$m \leqslant \max\{t(A), t(B)\} - 1 \qquad \text{and} \qquad n \geqslant \min\{h(A), h(B)\} + 1.$$

Thus we have $a_{m+1}b_{m+1} \neq 0$ and $a_n b_n \neq 0$. Since $A, B \in \mathscr{L}$, neither of the sequences A and B has internal zeros. It follows that

$$\prod_{i=n}^{m+1} (a_i b_i) \neq 0.$$

By dividing Ineq. (2.2) by the factor $a_{m+1}b_{m+1}$, we see that it is equivalent to prove

$$\frac{a_m b_n}{a_{m+1}b_{m+1}} + \frac{a_n b_m}{a_{m+1}b_{m+1}} \geqslant \frac{b_{n-1}}{b_{m+1}} + \frac{a_{n-1}}{a_{m+1}}. \qquad (3.17)$$

Borrowing the notation in [5], we let

$$\alpha_k = \frac{a_k}{a_{k-1}} \qquad \text{and} \qquad \beta_h = \frac{b_h}{b_{h-1}},$$

when $a_{k-1} \neq 0$ and $b_{h-1} \neq 0$. Then the desired Ineq. (3.17) can be recast as

$$\frac{1}{\alpha_{m+1}} \prod_{i=n+1}^{m+1} \frac{1}{\beta_i} + \frac{1}{\beta_{m+1}} \prod_{i=n+1}^{m+1} \frac{1}{\alpha_i} \geqslant \prod_{i=n}^{m+1} \frac{1}{\beta_i} + \prod_{i=n}^{m+1} \frac{1}{\alpha_i}. \qquad (3.18)$$

Multiplying Ineq. (3.18) by the product $\prod_{i=n}^{m+1} (\alpha_i \beta_i)$, we find Ineq. (3.18) is equivalent to:

$$\beta_n \prod_{i=n}^{m} \alpha_i + \alpha_n \prod_{i=n}^{m} \beta_i \geqslant \prod_{i=n}^{m+1} \alpha_i + \prod_{i=n}^{m+1} \beta_i.$$

Hence, it suffices to show that

$$(\beta_n - \alpha_{m+1}) \prod_{i=n}^{m} \alpha_i + (\alpha_n - \beta_{m+1}) \prod_{i=n}^{m} \beta_i \geqslant 0. \tag{3.19}$$

On the other hand, the synchronicity relation $A \sim B$ implies that

$$\alpha_n \geqslant \beta_{n+1} \qquad \text{and} \qquad \beta_n \geqslant \alpha_{n+1}.$$

By the log-concavity of the sequence B, the sequence β_k is decreasing. Thus we have

$$\alpha_n \geqslant \beta_{n+1} \geqslant \beta_{m+1}. \tag{3.20}$$

For the same reason, we have

$$\beta_n \geqslant \alpha_{m+1}. \tag{3.21}$$

In view of Ineqs. (3.20) and (3.21), the desired Ineq. (3.19) follows. This completes the proof. $\square$

Gross et al. [5, Theorems 2.10, 2.11] showed that any collection of pairwise synchronized sequences is closed under linear combinations with nonnegative coefficients, and that the same property holds for the weak synchronicity relation. Next we illustrate that partial synchronicity behaves in the same manner.

LEMMA 3.4. *Let* $A, B \in \mathscr{L}$ *such that* $A \sim_{\mathrm{p}} B$. *Then we have* $uA + vB \in \mathscr{L}$ *for all nonnegative numbers* u *and* v.

Proof. Let $A = (a_k)$ and $B = (b_k)$ be log-concave sequences such that $A \sim_{\mathrm{p}} B$. Let $u, v \geqslant 0$. It is clear that the sequence $uA + vB$ has no internal zeros. Since the sequence A is log-concave, we have

$$u^2 a_k^2 \geqslant u^2 a_{k-1} a_{k+1}. \tag{3.22}$$

For the same reason, the log-concavity of the sequence B implies that

$$v^2 b_k^2 \geqslant v^2 b_{k-1} b_{k+1}. \tag{3.23}$$

Since $A \sim_{\mathrm{p}} B$, one may take $m = n = k$ in Ineq. (2.2), which yields

$$uv(a_k b_k + a_k b_k) \geqslant uv(a_{k+1} b_{k-1} + a_{k-1} b_{k+1}). \tag{3.24}$$

Adding Ineqs. (3.22) and (3.23) to (3.24) up, we obtain that

$$(ua_k + vb_k)^2 \geqslant (ua_{k+1} + vb_{k+1})(ua_{k-1} + vb_{k-1}).$$

In other words, the sequence $uA + vB$ is log-concave. $\square$

THEOREM 3.5. *Suppose that the sequences* $A_1, A_2, \ldots, A_n$ *are pairwise partially synchronized. Then for any nonnegative numbers* u_1, v_1, u_2, v_2, $\ldots$, u_n, v_n, *we have* $\sum_{i=1}^{n} u_i A_i \sim_{\mathrm{p}} \sum_{i=1}^{n} v_i A_i$.

Proof. Since scalars preserve the partial synchronicity relation, we see that the $2n$ sequences $u_i A_i$ and $v_i A_i$ are pairwise partially synchronized. By iterative application, it suffices to show that summation preserves partial synchronicity. Namely, given $A, B, C \in \mathscr{L}^*$, we only need to show that $A + B \sim_{\mathrm{p}} C$ if $A \sim_{\mathrm{p}} C$ and $B \sim_{\mathrm{p}} C$.

Let m and n be integers such that $m \geqslant n$. The condition $A \sim_{\mathrm{p}} C$ implies that

$$a_m c_n + a_n c_m \geqslant a_{m+1} c_{n-1} + a_{n-1} c_{m+1}. \tag{3.25}$$

The condition $B \sim_{\mathrm{p}} C$ implies that

$$b_m c_n + b_n c_m \geqslant b_{m+1} c_{n-1} + b_{n-1} c_{m+1}. \tag{3.26}$$

Adding Ineqs. (3.25) and (3.26) up, one obtains that

$$(a_m + b_m) c_n + (a_n + b_n) c_m \geqslant (a_{m+1} + b_{m+1}) c_{n-1} + (a_{n-1} + b_{n-1}) c_{m+1}.$$

On the other hand, the sequence $A + B$ is log-concave by Lemma 3.4. Hence, we find $A + B \sim_{\mathrm{p}} C$. This completes the proof. $\quad\square$

In [5, Theorem 2.12], Gross et al. also showed that the synchronicity relation is preserved by the sequence convolution operation. Example 2.3 illustrates that this property does not hold for weak synchronicity. Below we demonstrate that, like synchronicity, partial synchronicity nonetheless is preserved by convolution.

THEOREM 3.6. *Let $A, B, C \in \mathscr{L}^*$. If $A \sim_{\mathrm{p}} B$, then $A * C \sim_{\mathrm{p}} B * C$.*

Proof. Suppose that $A = (a_k)$, $B = (b_k)$, and $C = (c_k)$. Since all the sequences A, B and C are log-concave without internal zeros, so are the sequences $A * C$ and $B * C$. Let m and n be integers such that $m \geqslant n$. From Definition 2.4 of weak synchronization, it suffices to show that

$$
\begin{aligned}
(A * C)_m (B * C)_n &+ (A * C)_n (B * C)_m \\
&\geqslant (A * C)_{m+1} (B * C)_{n-1} + (A * C)_{n-1} (B * C)_{m+1}, \quad (3.27)
\end{aligned}
$$

where as usual the notation S_n for a sequence S denotes the nth term of S.

We consider each summand in Ineq. (3.27) as a linear combination of the products of form $c_k c_l$, where $k < l$ are integers. Then the coefficient of $c_k c_l$ in the expansion of the first summand $(A * C)_m (B * C)_n$ is

$$a_{m-k} b_{n-l} + a_{m-l} b_{n-k}.$$

Dealing with the second summand $(A * C)_n (B * C)_m$ by exchanging the numbers m and n in the above expression, we find the coefficient of $c_k c_l$ of the left hand side of Ineq. (3.27) is

$$(a_{m-k} b_{n-l} + a_{m-l} b_{n-k}) + (a_{n-k} b_{m-l} + a_{n-l} b_{m-k}). \tag{3.28}$$

In Eq. (3.28), replacing m by $m+1$, and replacing n by $n-1$, we find that the coefficient of $c_k c_l$ of the right hand side of Ineq. (3.27) is

$$(a_{m+1-k}b_{n-1-l}+a_{n-1-l}b_{m+1-k})+(a_{m+1-l}b_{n-1-k}+a_{n-1-k}b_{m+1-l}).$$

In the same way, one may check that the coefficients of c_k^2 in the two sides of Ineq. (3.27) are respectively

$$a_{m-k}b_{n-k}+a_{n-k}b_{m-k} \qquad \text{and} \qquad a_{m+1-k}b_{n-1-k}+a_{n-1-k}b_{m+1-k}.$$

To sum up, we can recast the desired Ineq. (3.27) in terms of the function f as

$$\sum_{k<l} [(a_{m-k}b_{n-l}+a_{m-l}b_{n-k})+(a_{n-k}b_{m-l}+a_{n-l}b_{m-k})] \cdot c_k c_l \tag{3.29}$$

$$-\sum_{k<l} [(a_{m+1-k}b_{n-1-l}+a_{n-1-l}b_{m+1-k})+(a_{m+1-l}b_{n-1-k}+a_{n-1-k}b_{m+1-l})] \cdot c_k c_l$$

$$+\sum_{k} [a_{m-k}b_{n-k}+a_{n-k}b_{m-k}-a_{m+1-k}b_{n-1-k}-a_{n-1-k}b_{m+1-k}] \cdot c_k^2 \geqslant 0,$$

where the indices of every summation run over, in fact, a finite number of integers (since the number of non-zero terms in the sequence C is finite). We omit the range of such indices, and adopt this simplicity convention throughout the rest of this paper.

Define

$$g(k,l) = (a_{m-k}b_{n-l}+a_{n-l}b_{m-k}) - (a_{m+1-l}b_{n-1-k}+a_{n-1-k}b_{m+1-l}). \tag{3.30}$$

Then the desired Ineq. (3.29) can be written simply as

$$\sum_{k<l} [g(k,l)+g(l,k)]c_k c_l + \sum_k g(k,k)c_k^2 \geqslant 0,$$

that is,

$$\sum_{k,l} g(k,l)c_k c_l \geqslant 0. \tag{3.31}$$

Let s be a nonnegative integer, indicating the sum $k+l$ of the indices. For notation simplicity, we define

$$h_{\mathrm{e}}(k) = g(k, 2s-k), \tag{3.32}$$

$$h_{\mathrm{o}}(k) = g(k, 2s+1-k), \tag{3.33}$$

where the subscript letter "e" indicates that the sum $2s$ of the two variates k and $2s-k$ in Def. (3.32) is an even integer, and the subscript letter "o" indicates "odd". By virtue of these notation, the desired Ineq. (3.31) can be recast as

$$\sum_{s,k} \big(h_{\mathrm{e}}(k)c_k c_{2s-k}+h_{\mathrm{o}}(k)c_k c_{2s+1-k}\big) \geqslant 0.$$

Thus, it suffices to show, for all integers $s \geqslant 0$, that

$$\sum_k h_{\mathrm{e}}(k)c_k c_{2s-k} \geqslant 0 \tag{3.34}$$

and that

$$\sum_k h_{\mathrm{o}}(k) c_k c_{2s+1-k} \geqslant 0. \tag{3.35}$$

We shall show them individually. Let $s \geqslant 0$.

We transform the left hand sides of the desired Ineq. (3.34) to

$$\sum_k h_{\mathrm{e}}(k) c_k c_{2s-k} = \sum_{k=0}^{s} (c_{s-k} c_{s+k} - c_{s-k-1} c_{s+k+1}) \sum_{i=s-k-1}^{s+k+1} h_{\mathrm{e}}(i). \tag{3.36}$$

Since the sequence C is log-concave, we infer that

$$c_{s-k} c_{s+k} - c_{s-k-1} c_{s+k+1} \geqslant 0.$$

Thus, in view of Eq. (3.36), the desired Ineq. (3.34) holds if

$$\sum_{i=s-k-1}^{s+k+1} h_{\mathrm{e}}(i) \geqslant 0 \qquad \text{for all } 0 \leqslant k \leqslant s. \tag{3.37}$$

Now, let us reduce the left hand side $\sum_{i=s-k-1}^{s+k+1} h_{\mathrm{e}}(i)$. From Def. (3.30) of the function g, it is straightforward to verify that

$$g(k,l) + g(l-1, k+1) = 0. \tag{3.38}$$

Taking $k = s - i$ and $l = s + i$, Eq. (3.38) becomes

$$g(s-i, s+i) + g(s+i-1, s-i+1) = 0. \tag{3.39}$$

By Def. (3.32) of the function h_{e}, Eq. (3.39) can be rewritten as

$$h_{\mathrm{e}}(s-i) + h_{\mathrm{e}}(s+i-1) = 0. \tag{3.40}$$

By using Eq. (3.40), the left hand side of Ineq. (3.37) is simplified to

$$\sum_{i=s-k-1}^{s+k+1} h_{\mathrm{e}}(i) = h_{\mathrm{e}}(s+k+1).$$

Thereby to confirm the desired Ineq. (3.34), it suffices to show $h_{\mathrm{e}}(s+k+1) \geqslant 0$, that is, $g(s+k+1, s-k-1) \geqslant 0$. To do this, we will prove a stronger result that

$$g(k,l) \geqslant 0 \qquad \text{for all } k \geqslant l. \tag{3.41}$$

On the way using Lemma 3.1, one needs to check three conditions. First, the sequences A and B are partially synchronized as in the premise. Second, the sum of variates of the functions

$$a_{m-k} b_{n-l} + a_{n-l} b_{m-k} \qquad \text{and} \qquad a_{m+1-l} b_{n-1-k} + a_{n-1-k} b_{m+1-l}$$

are equal, i.e.,
$$(m-k)+(n-l) = (m+1-l)+(n-1-k).$$

Third, since $m \geqslant n$ and $k \geqslant l$, the distances of variates are comparative as

$$|(m-k)-(n-l)| \leqslant m-n < |(m+1-l)-(n-1-k)|.$$

By Lemma 3.1, we deduce the claimed Ineq. (3.41).

When $k \geqslant 0$, we have $s+k+1 \geqslant s-k-1$, and hence by Ineq. (3.41),

$$h_e(s+k+1) = g(s+k+1, s-k-1) \geqslant 0.$$

This completes the proof of Ineq. (3.34).

Inequality (3.35) can be shown along the same lines. In fact,

$$\sum_k h_o(k) c_k c_{2s+1-k} = \sum_{k=0}^{s} (c_k c_{2s+1-k} - c_{k-1} c_{2s+2-k}) \sum_{i=k}^{2s+1-k} h_o(i).$$

Since the sequence C is log-concave, we have

$$c_k c_{2s+1-k} - c_{k-1} c_{2s+2-k} \geqslant 0.$$

Thus the desired Ineq. (3.34) holds if

$$\sum_{i=k}^{2s+1-k} h_o(i) \geqslant 0 \qquad \text{for all } 0 \leqslant k \leqslant s. \tag{3.42}$$

On the other hand, from Def. (3.33) of the function h_o, Eq. (3.38) implies that

$$h_o(i) + h_o(2s-i) = 0. \tag{3.43}$$

In particular, taking $i = s$ in Eq. (3.43), one obtains that

$$h_o(s) = 0. \tag{3.44}$$

By using Eqs. (3.43) and (3.44), the left hand side of Ineq. (3.42) is simplified to

$$\sum_{i=k}^{2s+1-k} h_o(i) = h_o(2s+1-k).$$

When $0 \leqslant k \leqslant s$, we have $2s+1-k \geqslant k$. Hence, Def. (3.33) and Ineq. (3.41) imply that

$$h_o(2s+1-k) = g(2s+1-k, k) \geqslant 0.$$

This completes the proof. □

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