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SOME INEQUALITIES ASSOCIATED TO THE RATIO OF POCHHAMMER k–SYMBOL

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Abstract. In this paper, we study some ratios of the Pochhammer k-symbol. We prove some sharp inequalities for the ratio of Pochhammer k-symbol by Multiple-correction method.

1. Introduction

Diaz and Pariguan [8] introduced the generalized k-gamma function as

$$\Gamma_k(x) = \lim_{n\to\infty} \frac{n!k^n(nk)^{\frac{x}{k}} - 1}{(x)_{n,k}}, \quad k > 0, \quad x \in \mathbb{C} \setminus k\mathbb{Z}^-,$$

where the Pochhammer k-symbol is given for $k \in \mathbb{R}$ by

$$(x)_{n,k} = x(x+k)(x+2k)\cdots(x+(n-1)k), \quad x \in \mathbb{C}, \quad n \in \mathbb{N},$$

which appears repeatedly in a variety of contexts, such as, the combinatorics of creation and annihilation operators and the perturbative computation of Feynman integrals. See, e.g. Diaz and Pariguan [9] and Deligne etal. [7]. It has been shown that the Mellin transform of the exponential function $e^{-\frac{t^k}{k}}$ is the k-gamma function, given by

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt, \quad \mathrm{Re}(x) > 0.$$

One of the most known formulas involving the gamma function is Stirling's formula which was known as

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x, \tag{1}$$

where the symbol $\sim$ means that the quotient of both sides converges to 1. Another important formula is Euler's reflection formula

$$\Gamma(1-x)\Gamma(x) = \frac{\pi}{\sin(\pi x)}, \quad 0 < x < 1. \tag{2}$$

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In this paper, we investigate some ratios of the Pochhammer k-symbol

$$R_{i,k}^n = \frac{(k-i)_{n,k}}{(k)_{n,k}}, \quad i = 1,2,...,k-1, \quad k \geqslant 2, \quad k \in \mathbb{Z}^+, \quad n \in \mathbb{N}. \tag{3}$$

Chen and Qi [6] presented the following inequalities for the Wallis ratio for every natural number n:

$$\frac{1}{\sqrt{\pi(n+\frac{4}{\pi}-1)}} \leqslant R_{1,2}^n = \frac{(2n-1)!!}{(2n)!!} \leqslant \frac{1}{\sqrt{\pi(n+\frac{1}{4})}},$$

where the constants $\frac{4}{\pi}-1$ and $\frac{1}{4}$ are the best possible. This inequality is a consequence of the complete monotonicity on $(0,\infty)$ of the function

$$\ln \frac{x\Gamma(x)}{\sqrt{x+\frac{1}{4}}\Gamma(x+\frac{1}{2})}.$$

Here Γ is the gamma function given by

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt, \quad x > 0.$$

Mortici et al. [14] gave

$$\frac{\frac{1}{3}}{\sqrt[3]{n^2}} \leqslant R_{1,3}^n = \frac{2.5...(3n-1)}{3.6...(3n)} \leqslant \frac{\frac{\sqrt{3}}{2\pi}\Gamma(\frac{2}{3})}{\sqrt[3]{n^2}},$$

and

$$\frac{\frac{2}{3}}{\sqrt[3]{n}} \leqslant R_{2,3}^n = \frac{1.4...(3n-2)}{3.6...(3n)} \leqslant \frac{\frac{1}{\Gamma(\frac{2}{3})}}{\sqrt[3]{n}}.$$

Recently, Cao and Wang [3] presented some sharp inequalities for the ratio of gamma functions $R_{1,2}^n, R_{1,3}^n, R_{2,3}^n$ and proposed some conjectures. Bai et al. [1] proved the following inequalities hold for every integer $n \geqslant 1$:

$$R_{i,k}^n \geqslant \frac{\nu_{i,k}}{\sqrt[k]{n^i+\beta n^{i-1}}} \exp\left\{ \frac{i(k-i)(3i^2-3ik-4i+2k)}{24k^3}\frac{1}{n^2} \right\}, \tag{4}$$

and

$$R_{i,k}^n \geqslant \frac{\nu_{i,k}}{\sqrt[k]{n^i+\beta n^{i-1}}} \exp\left\{ \frac{i(k-i)(3i^2-3ik-4i+2k)}{24k^3}\frac{1}{n^2} \right. \\ \left. + \frac{i(k-i)(i^4-2i^3k+i^2k^2-3i^2k+3ik^2-2i^2+6ik-2k^2)}{24k^4}\frac{1}{n^3} \right\}, \tag{5}$$

where $\nu_{i,k} = \frac{1}{\pi}\Gamma(\frac{i}{k})\sin(\frac{i}{k}\pi)$, $\beta = \frac{i(k-i)}{2k}$ and $i = 1,2,...,k-1$.

Motivated by the above work, in this paper we will continue our previous works [4, 2, 5, 15], and apply *multiple-correction method* to construct some new sharper double inequality of the ratio of gamma function $R_{i,k}^n$.

2. Some lemma

The following lemma gives a method for measuring the rate of convergence, it was proved by Mortici in [11] and used in [11, 12, 13, 10].

LEMMA 1. *If the sequence* $(x_n)_{n\in\mathbb{N}}$ *is convergent to zero and there exists the limit*

$$\lim_{n\to+\infty} n^s(x_n - x_{n+1}) = l \in [-\infty, +\infty] \tag{6}$$

with $s > 1$, *then*

$$\lim_{n\to+\infty} n^{s-1}x_n = \frac{l}{s-1}. \tag{7}$$

Throughout the paper, the notation $\Psi(k;x)$ means a polynomial of degree k in x with all of its non-zero coefficients positive, which may be different at each occurrence. While, $\Phi(k;x)$ denotes a polynomial of degree k in x with the leading coefficient equals one, which may be different at different subsection.

3. Further improvement and main theorem

According to the argument of Theorem 2.1 in [11], we can introduce a sequence $\{u_n\}_{n\geqslant 1}$ by the relation

$$R_{i,k}^n = \frac{\nu_{i,k}}{\sqrt[k]{n^i + an^{i-1}}}\exp(u_n), \tag{8}$$

and to say that an approximation $R_{i,k}^n \sim \frac{\nu_{i,k}}{\sqrt[k]{n^i+\alpha n^{i-1}}}$ is better if the speed of convergence of u_n is higher.

Step 1: *The first-correction.* From (8), we have

$$u_n - u_{n+1} = \ln\frac{R_{i,k}^n}{R_{i,k}^{n+1}} + \frac{1}{k}\ln\frac{n^i + an^{i-1}}{(n+1)^i + a(n+1)^{i-1}}. \tag{9}$$

Developing (9) into power series expansion in $1/n$, we have

$$u_n - u_{n+1} = \frac{i^2+2ak-ik}{2k^2}\frac{1}{n^2} + \frac{i^3-3i^2k-3a(1+a)k^2+2ik^2}{3k^3}\frac{1}{n^3} + O\Big(\frac{1}{n^4}\Big), \tag{10}$$

By Lemma 1, we know that the fastest possible sequence $\{u_n\}_{n\geqslant 1}$ is obtained as the first item on the right of (10) vanishes.

(i) If $a \neq \frac{-i^2+ik}{2k}$, then the rate of convergence of the sequence $\{u_n\}_{n\geqslant 1}$ is n^{-2}.

(ii) If $a = \frac{-i^2+ik}{2k}$, from (10) we have

$$u_n - u_{n+1} = \frac{i(-3i^3+2k^2-3ik(2+k)+i^2(4+6k))}{12k^3}\frac{1}{n^3} + O\Big(\frac{1}{n^4}\Big),$$

and the rate of convergence of the sequence $\{u_n\}_{n\geqslant 1}$ ia at least n^{-3}.

Step 2: *The second-correction.* We let

$$u_n - u_{n+1} = \ln\frac{R_{i,k}^n}{R_{i,k}^{n+1}} + \frac{1}{k}\ln\frac{n^i + an^{i-1} + bn^{i-2}}{(n+1)^i + a(n+1)^{i-1} + b(n+1)^{i-2}}. \tag{11}$$

Developing (11) into power series expansion in $1/n$, we have

$$u_n - u_{n+1} = \frac{-3i^4 + 24bk^2 + 2ik^2 - 3i^2k(2+k) + i^3(4+6k)}{12k^3} \cdot \frac{1}{n^3} + \left(\frac{-i^6 + 3i^5k - 24bk^3 - 2(1+6b)ik^3 + i^2k^2(8+12b+3k)}{8k^4} + \frac{i^4(2+3k-3k^2) + i^3k(-8-6k+k^2)}{8k^4} \right) \frac{1}{n^4} + O\left(\frac{1}{n^5}\right), \tag{12}$$

By Lemma 1, we know that the fastest possible sequence $\{u_n\}_{n\geqslant 1}$ is obtained as the first item on the right of (12) vanishes. So taking

$$b = \frac{-4i^3 + 3i^4 + 6i^2k - 6i^3k - 2ik^2 + 3i^2k^2}{24k^2},$$

we can get the rate of convergence of the sequence $\{u_n\}_{n\geqslant 1}$ ia at least n^{-4}.

Step 3: *The third-correction.* Similarly, We let

$$u_n - u_{n+1} = \ln \frac{R_{i,k}^n}{R_{i,k}^{n+1}} + \frac{1}{k} \ln \frac{n^i + an^{i-1} + bn^{i-2} + cn^{i-3}}{(n+1)^i + a(n+1)^{i-1} + b(n+1)^{i-2} + c(n+1)^{i-3}}. \tag{13}$$

Developing (13) into power series expansion in $1/n$, we have

$$u_n - u_{n+1} = \frac{i^6 + 48ck^3 + 2i^2k^2(2+k) - i^5(4+3k) - i^3k(8+8k+k^2) + i^4(4+10k+3k^2)}{16k^4} \cdot \frac{1}{n^4} + \left(\frac{45i^8 - 8640ck^4 - 48(1+60c)ik^4 - 60i^7(2+3k) - 20i^2k^3(36-144c+19k)}{1440k^5} + \frac{120i^3k^2(16+13k+k^2) + 10i^6(-8+24k+27k^2) + i^5(288+960k-180k^3)}{1440k^5} + \frac{5i^4k(-288-412k-48k^2+9k^3)}{1440k^5} \right) \frac{1}{n^5} + O\left(\frac{1}{n^6}\right), \tag{14}$$

By Lemma 1, we know that the fastest possible sequence $\{u_n\}_{n\geqslant 1}$ is obtained as the first item on the right of (14) vanishes. So taking

$$c = \frac{-4i^4 + 4i^5 - i^6 + 8i^3k - 10i^4k + 3i^5k - 4i^2k^2 + 8i^3k^2 - 3i^4k^2 - 2i^2k^3 + i^3k^3}{48k^3},$$

we can get the rate of convergence of the sequence $\{u_n\}_{n\geqslant 1}$ ia at least n^{-5}.

Now we present the following inequalities:

THEOREM 1. (Main) *For every positive integer n,*

$$\frac{v_{i,k}}{\sqrt[k]{n^i + an^{i-1} + bn^{i-2} + cn^{i-3}}} \exp\left\{ f(i,k)\frac{1}{n^4} \right\} \geqslant R_{i,k}^n \geqslant \frac{v_{i,k}}{\sqrt[k]{n^i + an^{i-1} + bn^{i-2} + cn^{i-3}}} \exp\left\{ f(i,k)\frac{1}{n^4} + g(i,k)\frac{1}{n^5} \right\}, \tag{15}$$

where

$$f(i,k) = \frac{i}{5760k^5}(288i^4 - 320i^5 + 120i^6 - 15i^7 - 720i^3k + 960i^4k - 420i^5k + 60i^6k + 480i^2k^2 - 980i^3k^2 + 540i^4k^2 - 90i^5k^2 + 360i^2k^3 - 300i^3k^3 + 60i^4k^3 - 48k^4 - 20ik^4 + 60i^2k^4 - 15i^3k^4),$$

$$g(i,k) = \frac{i}{2880k^6}(96i^5 - 40i^6 - 30i^7 + 20i^8 - 3i^9 - 576i^4k + 460i^5k - 75i^7k + 15i^8k + 960i^3k^2 - 1140i^4k^2 + 240i^5k^2 + 100i^6k^2 - 30i^7k^2 - 480i^2k^3 + 1080i^3k^3 - 420i^4k^3 - 50i^5k^3 + 30i^6k^3 - 48ik^4 - 380i^2k^4 + 270i^3k^4 - 15i^5k^4 + 48k^5 + 20ik^5 - 60i^2k^5 + 5i^3k^5 + 3i^4k^5).$$

Proof. We write

$$P_n = \ln R_{i,k}^n - \ln \nu_{i,k} + \frac{1}{k}\ln(n^i + an^{i-1} + bn^{i-2} + cn^{i-3}) - f(i,k)\frac{1}{n^4},$$

and

$$Q_n = \ln R_{i,k}^n - \ln \nu_{i,k} + \frac{1}{k}\ln(n^i + an^{i-1} + bn^{i-2} + cn^{i-3}) - f(i,k)\frac{1}{n^4} - g(i,k)\frac{1}{n^5},$$

As both P_n and Q_n converge to zero, it suffices to show that P_n is increasing and Q_n decreasing. Let $S(n) = P_{n+1} - P_n$ and $T(n) = Q_{n+1} - Q_n$, where

$$S(x) = \ln\frac{k-i+kx}{k+kx} + \frac{1}{k}\ln\frac{(x+1)^i + a(x+1)^{i-1} + b(x+1)^{i-2} + c(x+1)^{i-3}}{x^i + ax^{i-1} + bx^{i-2} + cx^{i-3}} - f(i,k)\left(\frac{1}{(x+1)^4} - \frac{1}{x^4}\right),$$

and

$$T(x) = S(x) - g(i,k)\left(\frac{1}{(x+1)^5} - \frac{1}{x^5}\right).$$

For $S(x)$ and $T(x)$, take the derivative with respect to x, and then replacing k with $i+m, m \in \mathbb{Z}+$, by Mathematica software we can obtain

$$S'(x) = -\frac{\Psi(10,x)}{\Psi(17,x)} < 0, \quad \text{and} \quad T'(x) = \frac{\Psi(12,x)}{\Psi(19,x)} > 0$$

Thus, $S(x)$ is strictly decreasing and $T(x)$ is strictly increasing on $[1,+\infty)$. Noting that $S(+\infty) = T(+\infty) = 0$, we have $S(x) > 0$ and $T(x) < 0$ on $[1,+\infty)$. □

REMARK 1. Clearly, the left sides of Theorem 3 and 4 of Mortici et al. [14] are included in (3.8), but the right sides can not be included in it. Theorem 1 provides some sharp inequalities for the ratio of Pochhammer k-symbol, which are superior to Theorem 3 of Bai et al. [1].

REMARK 2. It is worth to pointing out that Theorem 1 provides some sharp inequalities by multiple-correction method. Similarly, repeat the above approach step by step, we can get more sharp inequalities. But this maybe bring some computations increase, the details omitted here.

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