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LOCAL SPECTRAL PROPERTY OF RELATIVELY REGULAR OPERATORS

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Abstract. In this paper, we study some relatively regular operators T such that $T = TST$ for some $S \in \mathscr{L}(\mathscr{H})$. We give some spectral and local spectral properties between T and S. We also show that some relatively regular operators T have a nontrivial invariant subspace. Finally, we introduce and study the local spectral property of relatively regular operators modulo a nilpotent operator.

1. Introduction

Let $\mathscr{H}$ be a complex Hilbert space and let $\mathscr{L}(\mathscr{H})$ denote the algebra of all bounded linear operators on $\mathscr{H}$. As usual, we write $\sigma(T)$, $\sigma_e(T)$, $\sigma_p(T)$, and $\sigma_{su}(T)$ for the spectrum, the essential spectrum, the point spectrum, and surjective spectrum of T, respectively.

A subspace $\mathscr{M}$ of $\mathscr{H}$ is called an *invariant subspace* for an operator $T \in \mathscr{L}(\mathscr{H})$ if $T\mathscr{M} \subset \mathscr{M}$. We say that $\mathscr{M} \subset \mathscr{H}$ is a *hyperinvariant subspace* for $T \in \mathscr{L}(\mathscr{H})$ if $\mathscr{M}$ is an invariant subspace for every $S \in \mathscr{L}(\mathscr{H})$ commuting with T.

An operator X in $\mathscr{L}(\mathscr{H})$ is called a *quasiaffinity* if it has trivial kernel and dense range. An operator T in $\mathscr{L}(\mathscr{H})$ is said to be a *quasiaffine transform* of an operator S in $\mathscr{L}(\mathscr{H})$ if there is a quasiaffinity X in $\mathscr{L}(\mathscr{H})$ such that $XT = SX$, and this relation of S and T is denoted by $T \prec S$. If both $T \prec S$ and $S \prec T$, then we say that S and T are *quasisimilar*.

An operator $T \in \mathscr{L}(\mathscr{H})$ is said to have *the single-valued extension property*, abbreviated SVEP, if for every open subset G of $\mathbb{C}$ and any analytic function $f : G \to \mathscr{H}$ such that $(T-z)f(z) \equiv 0$ on G, it results $f(z) \equiv 0$ on G. For an operator $T \in \mathscr{L}(\mathscr{H})$ and $x \in \mathscr{H}$, *the resolvent set* $\rho_T(x)$ of T at x is defined to consist of z_0 in $\mathbb{C}$ such that there exists an analytic function $f(z)$ on a neighborhood of z_0, with values in $\mathscr{H}$, which verifies

$$(T-z)f(z) \equiv x. \tag{1}$$

We denote *the local spectrum* of T at x by $\sigma_T(x) = \mathbb{C} \setminus \rho_T(x)$, and by using local spectra, we define *the local spectral subspace* of T by $\mathscr{H}_T(F) = \{x \in \mathscr{H} : \sigma_T(x) \subset F\}$,

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where F is a subset of $\mathbb{C}$. An operator $T\in\mathscr{L}(\mathscr{H})$ is said to have *Dunford's property* (C) if $\mathscr{H}_T(F)$ is closed for each closed subset F of $\mathbb{C}$. An operator $T\in\mathscr{L}(\mathscr{H})$ is said to have *Bishop's property* (β) if for every open subset G of $\mathbb{C}$ and every sequence $f_n : G\to\mathscr{H}$ of $\mathscr{H}$-valued analytic functions such that $(T-z)f_n(z)$ converges uniformly to 0 in norm on compact subsets of G, then $f_n(z)$ converges uniformly to 0 in norm on compact subsets of G. It is known [10] that

$$\text{Bishop's property } (\beta) \Rightarrow \text{Dunford's property } (C) \Rightarrow \text{SVEP.}$$

An operator $T\in\mathscr{L}(\mathscr{H})$ is called *scalar* of order m if it possesses a spectral distribution of order m, i.e., if there is a continuous unital morphism of topological algebras

$$\Phi : C_0^m(\mathbb{C})\to\mathscr{L}(\mathscr{H})$$

such that $\Phi(z)=T$, where z stands for the identical function on $\mathbb{C}$ and $C_0^m(\mathbb{C})$ for the space of all compactly supported functions continuously differentiable of order m, $0\leqslant m\leqslant\infty$. An operator is said to be *subscalar* of order m if it is similar to the restriction of a scalar operator of order m to an invariant subspace.

We say that $T\in\mathscr{L}(\mathscr{H})$ is *relatively regular* if $TST=T$ for some $S\in\mathscr{L}(\mathscr{H})$, or equivalently, when both the range and kernel of T are closed complemented subspace. This concept is well known in ring theory and has been investigated by I. Kaplansky (see [6]). The relatively regular operators are very useful in solving linear equations. For example, if $Tx=y$ has a solution x_0 for a given $y\in\mathscr{H}$, then $TSy=TSTx_0=Tx_0=y$. Hence Sy is a solution. Several authors have studied about these subjects (see [1], [2], [3], and [11], etc.). In this paper, we emphasize on the local spectral theory (i.e., the single-valued extension property, the property (β), etc.) for relatively regular operators. Furthermore, we introduce and study the local spectral property of relatively regular operators modulo a nilpotent operator.

2. Preliminaries

Let z be the coordinate function in the complex plane $\mathbb{C}$ and $d\mu(z)$ the planar Lebegue measure. Consider a bounded (connected) open subset U of $\mathbb{C}$. We shall denote by $L^2(U,\mathscr{H})$ the Hilbert space of measurable functions $f:U\to\mathscr{H}$ such that

$$\|f\|_{2,U}=(\int_U\|f(z)\|^2d\mu(z))^{\frac{1}{2}}<\infty.$$

The space of functions $f\in L^2(U,\mathscr{H})$ which are analytic functions in U is denoted by

$$A^2(U,\mathscr{H})=L^2(U,\mathscr{H})\cap\mathscr{O}(U,\mathscr{H})$$

where $\mathscr{O}(U,\mathscr{H})$ denotes the Fr$\acute{e}$chet space of $\mathscr{H}$-valued analytic functions on U with respect to uniform topology. The space $A^2(U,\mathscr{H})$ is called *the Bergman space* for U, and it is a Hilbert space.

Now let us define a special Sobolev type space. Let U be again a bounded open subset of $\mathbb{C}$ and m be a fixed non-negative integer. The vector-valued Sobolev space

$W^m(U,\mathscr{H})$ with respect to $\bar{\partial}$ and of order m will be the space of those functions $f \in L^2(U,\mathscr{H})$ whose derivatives $\bar{\partial} f, \cdots, \bar{\partial}^m f$ in the sense of distributions still belong to $L^2(U,\mathscr{H})$. Endowed with the norm

$$\|f\|^2_{W^m} = \sum_{i=0}^{m} \|\bar{\partial}^i f\|^2_{2,U},$$

$W^m(U,\mathscr{H})$ becomes a Hilbert space contained continuously in $L^2(U,\mathscr{H})$. Note that the linear operator M of multiplication by z on $W^m(U,\mathscr{H})$ is continuous and it has a spectral distribution $\Phi_M : C_0^m(\mathbb{C}) \to \mathscr{L}(W^m(U,\mathscr{H}))$ of order m defined by the following relation:

$$\Phi_M(\varphi)f = \varphi f \text{ for } \varphi \in C_0^m(\mathbb{C}) \text{ and } f \in W^m(U,\mathscr{H}).$$

Therefore, M is a scalar operator of order m, $0 \leqslant m \leqslant \infty$.

An operator $T \in \mathscr{L}(\mathscr{H})$ is called *semi-regular* if T has closed range and $ker(T^n) \subseteq ran(T)$, for all $n \geqslant 0$. We define the *semi-regular spectrum* $\sigma_{se}(T)$ by

$$\sigma_{se}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not semi-regular}\}.$$

An operator $T \in \mathscr{L}(\mathscr{H})$ is called *upper semi-Fredholm* if T has closed range and dim $ker(T) < \infty$, and T is called *lower semi-Fredholm* if T has closed range and $\dim(\mathscr{H}/ran(T)) < \infty$. When T is either upper semi-Fredholm or lower semi-Fredholm, it is called *semi-Fredholm*. The *index of a semi-Fredholm operator* $T \in \mathscr{L}(\mathscr{H})$, denoted index$(T)$, is given by index$(T) = \dim ker(T) - \dim(\mathscr{H}/ran(T))$ and this value is an integer or $\pm\infty$. Also an operator $T \in \mathscr{L}(\mathscr{H})$ is said to be *Fredholm* if it is both upper and lower semi-Fredholm. We define the *upper semi-Fredholm spectrum* $\sigma_{uf}(T)$ and the *semi-Fredholm spectrum* $\sigma_{sf}(T)$ by

$$\sigma_{uf}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not upper semi-Fredholm}\}$$

and

$$\sigma_{sf}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not semi-Fredholm}\}.$$

An operator $T \in \mathscr{L}(\mathscr{H})$ is said to be *Weyl* if it is Fredholm of index zero. For an operator $T \in \mathscr{L}(H)$, if we can choose the smallest positive integer m such that $ker(T^m) = ker(T^{m+1})$, then m is called *the ascent* of T and T is said to have *finite ascent*. Moreover, if there is the smallest positive integer n satisfying $ran(T^n) = ran(T^{n+1})$, then n is called *the descent* of T and T is said to have *finite descent*. An operator $T \in \mathscr{L}(\mathscr{H})$ is called *upper semi-Browder* if it is Fredholm of finite ascent, and T is called *lower semi-Browder* if it is Fredholm of finite decent. When T is either upper semi-Browder or lower semi-Browder, it is called *semi-Browder*. Also an operator $T \in \mathscr{L}(\mathscr{H})$ is said to be *Browder* if it is both upper and lower semi-Browder. Also We define *the Weyl spectrum* $\sigma_w(T)$, *the Browder spectrum* $\sigma_b(T)$, *the upper semi-Browder spectrum* $\sigma_{ub}(T)$ by

$$\sigma_w(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Weyl}\},$$

$$\sigma_b(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Browder}\},$$

and

$$\sigma_{ub}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not upper semi-Browder}\}.$$

It is evident that

$$\sigma_{sf}(T) \subset \sigma_{uf}(T) \subset \sigma_{ub}(T) \subset \sigma_b(T).$$

We say that *Weyl's theorem holds* for T if

$$\sigma(T) \setminus \pi_{00}(T) = \sigma_w(T), \text{ or equivalently, } \sigma(T) \setminus \sigma_w(T) = \pi_{00}(T)$$

where $\pi_{00}(T) := \{\lambda \in \text{iso } \sigma(T) : 0 < \dim ker(T-\lambda) < \infty\}$ and iso $\sigma(T)$ denotes the set of all isolated points of $\sigma(T)$. We say that *Browder's theorem holds* for $T \in \mathcal{L}(\mathcal{H})$ if $\sigma_b(T) = \sigma_w(T)$.

3. Relatively regular operators

In this section, we study the local spectral property of a relatively regular operator with the single-valued extension property. The single-valued extension property implies the existence of a maximal analytic extension $f(z)$ of $R(\cdot\,;T)x$ to $\rho_T(x)$ from (1). First of all, we begin with the following theorem.

THEOREM 1. *Let (S,T) be solutions of operator equations satisfying $TST = T$ and $STS = S$. If there are positive constants c_1 and c_2 such that $c_1||Tx|| \leqslant ||Sx|| \leqslant c_2||Tx||$ for all $x \in \mathcal{H}$ or $ker\ T = ker\ S$, then T has the single-valued extension property if and only if S has the single-valued extension property.*

Proof. If there are positive constants c_1 and c_2 such that $c_1||Tx|| \leqslant ||Sx|| \leqslant c_2||Tx||$ for all $x \in \mathcal{H}$, then $ker\ T = ker\ S$. Hence it suffices to consider the case $ker\ T = ker\ S$. Suppose that T has the single-valued extension property. Let $\lambda_0 \in \mathbb{C}$ and let G be an open connected set in $\mathbb{C}$ containing λ_0. Let f be any $\mathcal{H}$-valued analytic function on G such that

$$(S-\lambda)f(\lambda) \equiv 0 \tag{2}$$

for any $\lambda \in G$. Then

$$(STS - \lambda ST)f(\lambda) = 0$$

for any $\lambda \in G$. Since $STS = S$, we have

$$(S - \lambda ST)f(\lambda) = S(I - \lambda T)f(\lambda) \equiv 0. \tag{3}$$

Then $(I-\lambda T)f(\lambda) \in ker\ S = ker\ T$ on G. We get that

$$(I-\lambda T)Tf(\lambda) = T(I-\lambda T)f(\lambda) \equiv 0.$$

(i) If $0 \in G$, choose an open connected set G_0 such that $G_0 \subset G \setminus \{0\}$. Then

$$(T-\mu)Tf\left(\frac{1}{\mu}\right) = (T-\mu)T(f\circ g)(\mu) \equiv 0$$

where $g(\mu) = \frac{1}{\mu}$ and $\mu = \frac{1}{\lambda}$ on G_0. Since T has the single-valued extension property,

$$T(f \circ g)(\mu) \equiv 0$$

on G_0. By Identity Theorem, we obtain that $Tf(\lambda) \equiv 0$ on G. Therefore

$$\lambda STf(\lambda) \equiv 0 \tag{4}$$

on G. By (3) and (4), $Sf(\lambda) = 0$. Then from (2), $\lambda f(\lambda) = 0$ on G, and hence $f(\lambda) \equiv 0$ on G. Thus S has the single-valued extension property.

(ii) If $0 \notin G$, then we don't need to choose G_0 as in (i). Hence,

$$(T-\mu)Tf\left(\frac{1}{\mu}\right) = (T-\mu)T(f \circ g)(\mu) \equiv 0$$

where $g(\mu) = \frac{1}{\mu}$ and $\mu = \frac{1}{\lambda}$ on G. The remaining part is the same proof with (i).

The converse implication is similar. □

We observe that there are several classes of operators satisfying Theorem 1. For example, if T is the bilateral shift, more generally, unitary, or invertible, then Theorem 1 holds. We give other examples.

EXAMPLE 1. Let T_α be a diagonal operator with diagonals $\{\sqrt{\frac{n+1}{n}}\}_{n=1}^{\infty}$ and S_β be a diagonal operator with diagonals $\{\sqrt{\frac{n}{n+1}}\}_{n=1}^{\infty}$. Then $T_\alpha S_\beta T_\alpha = T_\alpha$ and $S_\beta T_\alpha S_\beta = S_\beta$. Moreover, $ker\ T_\alpha = ker\ S_\beta$ and T_α and S_β have the single-valued extension property.

EXAMPLE 2. If T is the unilateral shift defined by $Te_n = e_{n+1}$ for every positive integer n where $\{e_n\}_{n=1}^{\infty}$ is an orthonormal basis for $\mathscr{H}$, then T is relatively regular with $S = T^*$. Hence T has the single-valued extension property, but S does not have it (see [4, Example 1.7]). From Theorem 1, $ker\ T \neq ker\ S$. Let's consider another example. Let $T = U \oplus U^*$ and $S = U^* \oplus U$ in $\mathscr{L}(\mathscr{H} \oplus \mathscr{H})$ where U is the unilateral shift. Then $TST = T$ and $STS = S$. Since U^* does not have the single-valued extension property, neither T nor S has the single-valued extension property. From Theorem 1, $ker\ T \neq ker\ S$.

COROLLARY 1. *Let $T \in \mathscr{L}(\mathscr{H})$ have closed range. Then T has the single-valued extension property if and only if $(T|_{ran\ T^*})^{-1}P$ has the single-valued extension property where P is an orthogonal projection of $\mathscr{H}$ onto $ran\ T$.*

Proof. Since T has closed range, so does T^* by the closed range theorem. Since $\mathscr{H} = ker\ T \oplus (ker\ T)^{\perp}$ and $(ker\ T)^{\perp} = ran\ T^*$, $T|_{ran\ T^*}$ is one-to-one and $(T|_{ran\ T^*})(ran\ T^*) = ran\ T$ is closed. Hence $T|_{ran\ T^*}$ is invertible and has an inverse $A : ran\ T \to ran\ T^*$. Let P be an orthogonal projection of $\mathscr{H}$ onto $ran\ T$. Set $S = AP$. Then $TST = TAPT = T$ and $STS = APTAP = AP = S$. Since $ker\ T = ker\ S$, from Theorem 1 T has the single-valued extension property if and only if AP has the single-valued extension property. □

COROLLARY 2. *Let $T \in \mathscr{L}(\mathscr{H})$ be relatively regular such that $TST = T$. If $ker\ T = ker\ STS$ or $TS = ST$, then T has the single-valued extension property if and only if STS has the single-valued extension property.*

Proof. (i) Assume $ker\ T = ker\ STS$. Set $S_0 = STS$. Then

$$TS_0T = T(STS)T = TST = T$$

and

$$S_0TS_0 = (STS)T(STS) = STS = S_0.$$

By Theorem 1, T has the single-valued extension property if and only if $S_0 = STS$ has the single-valued extension property.

(ii) If $TS = ST$, then $ker\ T = ker\ STS$. Indeed, if $Tx = 0$, then $STSx = S^2Tx = 0$. Hence $ker\ T \subseteq ker\ STS$. Conversely, if $STSx = 0$, then $TSx = TSTSx = 0$. Hence $Tx = TSTx = T^2Sx = 0$. Thus $ker\ STS \subseteq ker\ T$. The proof follows from (i). □

REMARK 1. The converses of Theorem 1 and Corollary 2 do not hold, in general.

EXAMPLE 3. Let $T = \begin{pmatrix} 0 & 0 \\ I & H \end{pmatrix}$ be in $\mathscr{L}(\mathscr{H} \oplus \mathscr{H})$ where H is hyponormal (i.e., $H^*H \geqslant HH^*$, or equivalently, $||H^*x|| \leqslant ||Hx||$ for $x \in \mathscr{H}$).

Set $S = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix} \in \mathscr{L}(\mathscr{H} \oplus \mathscr{H})$. Then $TST = T$ and $STS = S$ hold. Moreover, it is known from [7] that T and $STS = S$ have the single-valued extension property. But $TS \neq ST$ and $ker\ T \neq ker\ S (= ker\ STS)$.

The following corollary states the approximate eigenvalue problem in some sense.

COROLLARY 3. *Under the same hypotheses with Theorem 1, if T has the single-valued extension property, then $\sigma_{ap}(S^*) = \sigma(S) = \sigma_{su}(S) = \cup_{x \in \mathscr{H}} \sigma_S(x)$ and $\sigma_{ap}(S) = \sigma_{se}(S)$.*

Proof. From Theorem 1, S has the single-valued extension property. Hence $\sigma(S) = \sigma_{su}(S) = \cup_{x \in \mathscr{H}} \sigma_S(x)$. Since $\sigma_{su}(S) = \sigma_{ap}(S^*)$ and $\sigma_{ap}(S) = \sigma_{se}(S)$ from [1], we complete the proof. □

COROLLARY 4. *Assume that T is relatively regular such that $TST = T$. If T is normal (i.e., $T^*T = TT^*$), $TS = ST$, and $|T|^2$ has dense range, then S is normal.*

Proof. Since T is normal and $TS = ST$, the Fuglede-Putnam theorem implies $T^*S = ST^*$. Since T is relatively regular,

$$\begin{aligned} 0 &= TT^* - T^*T \\ &= TSTT^*S^*T^* - T^*S^*T^*TST \\ &= T^*T[SS^* - S^*S]T^*T \\ &= |T|^2[SS^* - S^*S]|T|^2. \end{aligned}$$

Hence S is normal on $\overline{ran\,|T|^2}$. $\square$

There is a trivial example for Corollary 4. In fact, if T is invertible normal and $S = T^{-1}$, then T is relatively regular, $TS = ST = I$, and S is normal. We next consider another example. Let $\mathbb{D}$ be the open unit disk in the complex plane. The space $H^2(\mathbb{D})$ consists of all the analytic functions on $\mathbb{D}$ having power series representations with square summable complex coefficients. If φ is an analytic function mapping $\mathbb{D}$ into itself, *the composition operator* C_φ is the operator on $H^2(\mathbb{D})$ defined by $C_\varphi f = f \circ \varphi$. For any $\alpha \in \mathbb{D}$, the function $K_\alpha(z) = \frac{1}{1-\overline{\alpha}z}$ is called reproducing kernel for $\alpha \in \mathbb{D}$ such that $f(\alpha) = \langle f, K_\alpha \rangle$ for any $f \in H^2(\mathbb{D})$.

EXAMPLE 4. Let φ and ψ be analytic maps of the unit disk $\mathbb{D}$ to itself. If C_φ is normal such that $C_\varphi C_\psi C_\varphi = C_\varphi$ for some C_ψ, then C_ψ is normal. Indeed, since $C_\varphi^* C_\psi^* C_\varphi^* = C_\varphi^*$, for a reproducing kernel $K_\alpha(z) = \frac{1}{1-\overline{\alpha}z}$, $\alpha \in \mathbb{D}$, we get that for any $\alpha \in \mathbb{D}$ and $f \in H^2(\mathbb{D})$, $\langle f, C_\varphi^* K_\alpha \rangle = \langle C_\varphi f, K_\alpha \rangle = \langle f \circ \varphi, K_\alpha \rangle = f(\varphi(\alpha)) = \langle f, K_{\varphi(\alpha)} \rangle$ and hence $C_\varphi^* K_\alpha = K_{\varphi(\alpha)}$ and

$$K_{\varphi(\psi(\varphi(\alpha)))}(z) = C_\varphi^* C_\psi^* C_\varphi^* K_\alpha(z) = C_\varphi^* K_\alpha(z) = K_{\varphi(\alpha)}(z).$$

Hence $\varphi(\psi(\varphi(\alpha))) = \varphi(\alpha)$ for every $\alpha \in \mathbb{D}$. Since C_φ is normal, it is well known that $\varphi(z) = \gamma z$ where $|\gamma| \leqslant 1$. Thus $\psi(\gamma\alpha) = \alpha$ for every $\alpha \in \mathbb{D}$. Set $\psi(z) = b_0 + b_1(z) + \cdots$. Then $\alpha = \psi(\gamma\alpha) = b_0 + b_1\gamma\alpha + \cdots$. Hence $b_1 = \frac{1}{\gamma}$ and $b_n = 0$ where $n \neq 1$. Thus $\psi(z) = \frac{1}{\gamma}z$. Since $\psi(\mathbb{D}) \subseteq \mathbb{D}$, $|\frac{1}{\gamma}| \leqslant 1$, i.e., $|\gamma| \geqslant 1$. Thus $|\gamma| = 1$. Therefore, $\psi(z) = \frac{1}{\gamma}z$ for $|\gamma| = 1$. Hence C_ψ is normal. Moreover, it is easy to check that $C_\psi C_\varphi = C_\varphi C_\psi$ and $|C_\varphi|^2$ is dense range in $H^2(\mathbb{D})$.

COROLLARY 5. *Let T_i be relatively regular such that $T_iS_iT_i = T_i$ and $T_iS_i = S_iT_i$ for $i = 1,2$. If T_1 and T_2 are hyponormal, then $S_1T_1S_1 \oplus S_2T_2S_2$ has the single-valued extension property.*

Proof. Since $T_1 \oplus T_2$ is hyponormal, it has the single-valued extension property. Since $(T_1 \oplus T_2)(S_1 \oplus S_2) = (S_1 \oplus S_2)(T_1 \oplus T_2)$, $S_1T_1S_1 \oplus S_2T_2S_2$ has the single-valued extension property from Corollary 2. $\square$

Recall that an operator T in $\mathcal{L}(\mathcal{H})$ *admits a moment sequence* if there exists nonzero vectors x and y in $\mathcal{H}$ and a (finite, regular) Borel measure μ supported on $\sigma(T)$ such that

$$\langle T^n x, y \rangle = \int_{\sigma(T)} \lambda^n d\mu, \;\; n \in \mathbb{N} \cup \{0\}.$$

(We use the term *measure* here in the usual sense of a nonnegative-valued set function.) An operator T in $\mathcal{L}(\mathcal{H})$ is *algebraic* if there is a non-zero polynomial p such that $p(T) = 0$.

THEOREM 2. *Assume that $T \in \mathcal{L}(\mathcal{H})$ is relatively regular such that $TST = T$. Then the following statements hold.*

(i) If $T \neq 0$ and $TS \neq I$, then T has a nontrivial invariant subspace $\overline{ran\, T}$, $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} : \overline{ran\, T} \oplus ker\, T^* \to \overline{ran\, T} \oplus ker\, T^*$, where $T_1 = T|_{\overline{ran\, T}}$, and admits a moment sequence.

(ii) ST and TS are subscalar, and have the property (β), Dunford property (C), and the single-valued extension property. Hence $\mathscr{H}_{\mathscr{S}\mathscr{T}}(F)$ and $\mathscr{H}_{\mathscr{T}\mathscr{S}}(F)$ are hyper-invariant under ST and TS, respectively, where F is closed.

Proof. (i) Since $(I-TS)T = 0$, $\{0\} \neq ran\, T \subseteq ker\,(I-TS) \neq \mathscr{H}$. Hence $\overline{ran\, T}$ is a nontrivial invariant subspace for T. Hence T can be written with respect to the decomposition $\mathscr{H} = \overline{ran\, T} \oplus ker\, T^*$ as $T = \begin{pmatrix} T|_{\overline{ran\, T}} & T_2 \\ 0 & T_3 \end{pmatrix}$. Choose $x \in \overline{ranT}$ and $y \in kerT^*$. If we define $\mu \equiv 0$ on $\sigma(T)$, then $\langle T^n x, y\rangle = 0 = \int_{\sigma(T)} \lambda^n d\mu$. Hence T admits a moment sequence.

(ii) Since $(ST)^2 = ST$ and $(TS)^2 = TS$, ST and TS are algebraic such that $p(ST) = p(TS) = 0$ where $p(z) = z^2 - z$. Hence ST and TS are subscalar from [8, Corollary 4.8]. Thus ST and TS have the property (β), Dunford property (C), and the single-valued extension property. Since $\mathscr{H}_{\mathscr{S}\mathscr{T}}(F)$ and $\mathscr{H}_{\mathscr{T}\mathscr{S}}(F)$ are closed, the proof follows from [4]. □

As some applications of Theorem 2, we get the following corollaries.

COROLLARY 6. *If $T \in \mathscr{L}(\mathscr{H})$ has closed range such that $T \neq 0$ and $TS \neq I$ where $S = (T|_{ran\, T^*})^{-1}P$ and P is an orthogonal projection of $\mathscr{H}$ onto $ran\, T$, then it has a nontrivial invariant subspace.*

Proof. Since $T \in \mathscr{L}(\mathscr{H})$ has closed range, it is relatively regular from the proof of Corollary 1. Hence the proof follows from Theorem 2. □

COROLLARY 7. *Assume that $T \in \mathscr{L}(\mathscr{H})$ is relatively regular such that $TST = T$ and $TS = ST$ where $T \neq 0$ and $TS \neq I$. If TST has the single-valued extension property and $T(\overline{ran\, T}) = \overline{ran\, T}$, then T is bijective on $\overline{ran\, T}$.*

Proof. From Corollary 2, we know that T has the single-valued extension property. Since $\overline{ran\, T}$ is a nontrivial invariant subspace for T from Theorem 2 and $T(\overline{ran\, T}) = \overline{ran\, T}$, the proof follows from [9]. □

COROLLARY 8. *If $T \in \mathscr{L}(\mathscr{H})$ is a relatively regular operator such that $T = TST$, then both Weyl's theorem and Browder's theorem hold for $f(ST)$ and $\sigma_\omega(f(ST)) = \sigma_b(f(ST)) = f(\sigma_\omega(ST)) = f(\sigma_b(ST))$ where f is any function analytic on a neighborhood of $\sigma(ST)$.*

Proof. Let f be any function analytic on a neighborhood of $\sigma(ST)$. Since ST is subscalar from Theorem 2, so is $f(ST)$ and thus Weyl's theorem holds for $f(ST)$ from [1]. Moreover, since $f(ST)$ has the single-valued extension property by Theorem 2, Browder's theorem holds for $f(ST)$ and the given equalities are satisfied from [1]. □

COROLLARY 9. *Assume that* $T \in \mathscr{L}(\mathscr{H})$ *is relatively regular such that* $TST = T$. *Then* $\sigma((ST)|_{\mathscr{H}_{ST}(F)}) \subset \sigma(ST) \cap F$ *and* $\sigma((TS)|_{\mathscr{H}_{TS}(F)}) \subset \sigma(TS) \cap F$.

Proof. Since TS and ST have the Dunford property (C) from Theorem 2, the proof follows from [4]. □

PROPOSITION 1. *Let* $T \in \mathscr{L}(\mathscr{H})$ *be a relatively regular operator such that* $TST = T$. *If* $\mathscr{M} \in \text{Lat } T \cap \text{Lat } S$, *then* $T|_{\mathscr{M}}$ *is a relatively regular operator.*

Proof. Let P be the orthogonal projection of $\mathscr{H}$ onto $\mathscr{M}$. Since $TST = T$, $P[TST - T]P = 0$. Since $PTP = TP$ and $PSP = SP$,

$$0 = P[TST - T]P = PTSTP - PTP = (PTP)(PSP)(PTP) - PTP.$$

Hence $PTP = TP$ is relatively regular. □

The following proposition provides some spectral relations.

PROPOSITION 2. *Assume that* $T \in \mathscr{L}(\mathscr{H})$ *is relatively regular such that* $TST = T$. *The following statements hold.*

(i) $ST - \lambda$ *is bounded below for all* $\lambda \neq 1$ *if and only if* T *is bounded below.*

(ii) $ST - \lambda$ *is one-to-one for all* $\lambda \neq 1$ *if and only if* T *is one-to-one.*

(iii) If T *is invertible, then* S *is invertible.*

Proof. (i) If $ST - \lambda$ is not bounded below for $\lambda \neq 1$, then $\lambda \in \sigma_{ap}(ST)$ for $\lambda \neq 1$ and there is a sequence $\{x_n\}$ of unit vectors in $\mathscr{H}$ such that

$$\lim_{n\to\infty} \|(ST - \lambda)x_n\| = 0.$$

Hence we get that,

$$\begin{aligned} 0 &= \lim_{n\to\infty} \|T(ST - \lambda)x_n\| \\ &= \lim_{n\to\infty} \|(TST - \lambda T)x_n\| \\ &= \lim_{n\to\infty} \|T(1 - \lambda)x_n\|. \end{aligned}$$

Hence $\lim_{n\to\infty} \|Tx_n\| = 0$ and $0 \in \sigma_{ap}(T)$, i.e., T is not bounded below.

If T is not bounded below, then $0 \in \sigma_{ap}(T)$ and there is a sequence $\{x_n\}$ of unit vectors in $\mathscr{H}$ such that

$$\lim_{n\to\infty} \|Tx_n\| = 0.$$

Hence we get that, for $\lambda \neq 1$,

$$\begin{aligned} 0 &= \lim_{n\to\infty} \|T(1 - \lambda)x_n\| \\ &= \lim_{n\to\infty} \|(TST - \lambda T)x_n\| \\ &= \lim_{n\to\infty} \|T(ST - \lambda)x_n\|. \end{aligned}$$

Hence $\lim_{n\to\infty}\|(ST-\lambda)x_n\|=0$ for $\lambda\neq 1$ and $\lambda\in\sigma_{ap}(ST)$ for $\lambda\neq 1$, i.e., $ST-\lambda$ is not bounded below.

(ii) The proof follows from the same argument as (i) with a constant sequence $\{x_n\}$.

(iii) If T is invertible, then there is an operator $T^{-1}\in\mathcal{L}(\mathcal{H})$. Since $T=TST$, we have $ST=TS=I$ by multiplying T^{-1} to both sides. Therefore S is invertible. □

The following proposition provides some spectral relations.

PROPOSITION 3. *Assume that $T\in\mathcal{L}(\mathcal{H})$ is relatively regular such that $TST=T$. The following statements hold.*

(i) $\sigma(ST)=\cup_{x\in\mathcal{H}}\sigma_{ST}(x)$ and $\sigma(TS)=\cup_{x\in\mathcal{H}}\sigma_{TS}(x)$. In particular, if T is invertible, $\cup_{x\in\mathcal{H}}\sigma_{ST}(x)=\cup_{x\in\mathcal{H}}\sigma_{TS}(x)$.

(ii) $\sigma_I(Tx)\subseteq\sigma_{ST}(x)$, $\sigma_{TS}(Tx)\subseteq\sigma_{ST}(x)$, and $\sigma_I(STx)\subseteq\sigma_{ST}(x)$.

(iii) $\cup_{x\in\mathcal{H}}\sigma_I(Tx)\subseteq\sigma(ST)$.

(iv) $T\mathcal{H}_{ST}(F)\subseteq\mathcal{H}_I(F)$, $ST\mathcal{H}_{ST}(F)\subseteq\mathcal{H}_T(F)$, and $T\mathcal{H}_{ST}(F)\subseteq\mathcal{H}_{TS}(F)$ for all closed set $F\subset\mathbb{C}$.

Proof. (i) Since ST and TS have the single-valued extension property from Theorem 2, $\sigma(ST)=\cup_{x\in\mathcal{H}}\sigma_{ST}(x)$ and $\sigma(TS)=\cup_{x\in\mathcal{H}}\sigma_{TS}(x)$ by [1]. In particular, if T is invertible, then TS and ST are invertible from Proposition 2. Hence $\sigma(ST)=\sigma(TS)=\cup_{x\in\mathcal{H}}\sigma_{ST}(x)=\cup_{x\in\mathcal{H}}\sigma_{TS}(x)$.

(ii) If $z\notin\sigma_{ST}(x)$, there exists an analytic function f defined on a neighborhood of z such that

$$(ST-z)f(z)\equiv x.$$

Multiplying both sides by T, we get that

$$\begin{aligned} Tx &\equiv (TST-zT)f(z)\\ &=(I-z)Tf(z). \end{aligned}\tag{5}$$

Hence $z\notin\sigma_I(Tx)$. Thus $\sigma_I(Tx)\subseteq\sigma_{ST}(x)$. Since

$$(TS-z)Tf(z)\equiv Tx\tag{6}$$

by (5), then $z\notin\sigma_{TS}(Tx)$. Hence $\sigma_{TS}(Tx)\subseteq\sigma_{ST}(x)$. If we multiply both sides of (5) with S, then

$$STx\equiv(I-z)STf(z).\tag{7}$$

Hence $z\notin\sigma_I(STx)$. Thus $\sigma_I(STx)\subseteq\sigma_{ST}(x)$.

(iii) From (ii), $\cup_{x\in\mathcal{H}}\sigma_I(Tx)\subseteq\cup_{x\in\mathcal{H}}\sigma_{ST}(x)$. Since ST is the single-valued extension property by Theorem 2, $\cup_{x\in\mathcal{H}}\sigma_{ST}(x)=\sigma(ST)$ from [10].

(iv) If $x\in\mathcal{H}_{ST}(F)$ for all closed F set of $\mathbb{C}$, then $\sigma_{ST}(x)\subset F$. By (i), $\sigma_I(Tx)\subset F$. Thus $Tx\in\mathcal{H}_I(F)$. Therefore $T\mathcal{H}_{ST}(F)\subseteq\mathcal{H}_I(F)$ for all closed set $F\subset\mathbb{C}$. Similarly, we get that $ST\mathcal{H}_{ST}(F)\subseteq\mathcal{H}_T(F)$ and $T\mathcal{H}_{ST}(F)\subseteq\mathcal{H}_{TS}(F)$ for all closed set $F\subset\mathbb{C}$. □

4. Relatively regular operators modulo a nilpotent operator

In this section, we introduce and study the relatively regular operators modulo a nilpotent operator. In particular, we focus on the local spectral property of such operators.

DEFINITION 1. If $TST - T = N$ where $N^k = 0$ and $NT = TN$, we say that T is *relatively regular modulo* N with order k.

We next give an example for Definition 1.

EXAMPLE 5. Let $T = \begin{pmatrix} U & U \\ 0 & U \end{pmatrix}$ be in $\mathcal{L}(\mathcal{H} \oplus \mathcal{H})$ where U is the unilateral shift. Then U is a relatively regular operators such that $UU^*U = U$. Moreover, T is relatively regular modulo $N = \begin{pmatrix} 0 & U \\ 0 & 0 \end{pmatrix}$ with order 2 where $S = U^* \oplus U^*$.

THEOREM 3. *Let $T \in \mathcal{L}(\mathcal{H})$ be relatively regular modulo N with order k for some $S \in \mathcal{L}(\mathcal{H})$. If $0 \notin \sigma_{ap}(T^k)$, then ST and TS have the property (β).*

Proof. Let G be any open set of $\mathbb{C}$, and let $\{f_n\}_{n=1}^{\infty}$ be a sequence of $\mathcal{H}$-valued analytic function on G such that

$$\lim_{n\to\infty}(ST-\lambda)f_n(\lambda) = 0$$

uniformly on compact subsets K of G. Then

$$\lim_{n\to\infty}(TST-\lambda T)f_n(\lambda) = \lim_{n\to\infty}(T+N-\lambda T)f_n(\lambda) = 0$$

uniformly on compact subsets K of G. Hence

$$\lim_{n\to\infty}[(I-\lambda)T+N]f_n(\lambda) = 0 \tag{8}$$

uniformly on compact subsets K of G. Since $N^k = 0$,

$$\lim_{n\to\infty}[(I-\lambda)N^{k-1}T+N^k]f_n(\lambda) = \lim_{n\to\infty}(I-\lambda)N^{k-1}Tf_n(\lambda) = 0$$

uniformly on compact subsets K of G. Since I has the property (β),

$$\lim_{n\to\infty}N^{k-1}Tf_n(\lambda) = 0 \tag{9}$$

uniformly on compact subsets K of G. From (8),

$$\lim_{n\to\infty}[(I-\lambda)T^2+NT]f_n(\lambda) = 0$$

uniformly on compact subsets K of G. Therefore from (9) we have

$$\lim_{n\to\infty}[(I-\lambda)T^2N^{k-2}+N^{k-1}T]f_n(\lambda) = \lim_{n\to\infty}(I-\lambda)T^2N^{k-2}f_n(\lambda) = 0$$

uniformly on compact subsets K of G. Since I has the property (β), we get that $\lim_{n\to\infty} T^2N^{k-2}f_n(\lambda) = 0$.

Assume that $\lim_{n\to\infty} T^jN^{k-j}f_n(\lambda) = 0$ for all $j = 1,2,\cdots,k-1$. Then from (8)

$$\lim_{n\to\infty}[(I-\lambda)T^{j+1}N^{k-(j+1)} + T^jN^{k-j}]f_n(\lambda) = 0$$

uniformly on compact subsets K of G. Hence

$$\lim_{n\to\infty}(I-\lambda)T^{j+1}N^{k-(j+1)}f_n(\lambda) = 0$$

uniformly on compact subsets K of G. Since I has the property (β),

$$\lim_{n\to\infty} T^{j+1}N^{k-(j+1)}f_n(\lambda) = 0$$

for all $j = 1,2,\cdots,k-1$. Hence by the induction, we get that

$$\lim_{n\to\infty} T^jN^{k-j}f_n(\lambda) = 0 \tag{10}$$

for all $j = 1,2,\cdots,k$, where $N^0 = I$. In particular, when $j = k$, $\lim_{n\to\infty} T^kf_n(\lambda) = 0$ uniformly on compact subsets K of G. Since $0 \notin \sigma_{ap}(T^k)$, we get that $\lim_{n\to\infty} f_n(\lambda) = 0$ uniformly on compact subsets K of G. Hence ST has the property (β).

Let G be any open set of $\mathbb{C}$, and let $\{f_n\}_{n=1}^{\infty}$ be a sequence of $\mathscr{H}$-valued analytic function on G such that

$$\lim_{n\to\infty}(TS-\lambda)f_n(\lambda) = 0 \tag{11}$$

uniformly on compact subsets K of G. Then

$$\lim_{n\to\infty}(TSTS-\lambda TS)f_n(\lambda) = \lim_{n\to\infty}[(T+N)-\lambda T]Sf_n(\lambda) = 0$$

uniformly on compact subsets K of G. Hence

$$\lim_{n\to\infty}[(I-\lambda)T+N]Sf_n(\lambda) = 0$$

uniformly on compact subsets K of G. Repeating the process from (8) to (10), we get that

$$\lim_{n\to\infty} T^jN^{k-j}Sf_n(\lambda) = 0$$

for all $j = 1,2,\cdots,k$. In particular, $\lim_{n\to\infty} T^kSf_n(\lambda) = 0$ uniformly on compact subsets K of G. Since $0 \notin \sigma_{ap}(T^k)$, $\lim_{n\to\infty} Sf_n(\lambda) = 0$ uniformly on compact subsets K of G. Thus $\lim_{n\to\infty} TSf_n(\lambda) = 0$ uniformly on compact subsets K of G. From (11), $\lim_{n\to\infty} \lambda f_n(\lambda) = 0$ uniformly on compact subsets K of G, and since a zero operator has the property (β), $\lim_{n\to\infty} f_n(\lambda) = 0$ uniformly on compact subsets K of G. Therefore TS has the property (β). □

REMARK 2. We observe that we can replace $0 \notin \sigma_{ap}(T^k)$ by $0 \notin \sigma_p(T^k)$ for the single-valued extension property of ST and TS in Theorem 3.

COROLLARY 10. *Under the same hypotheses with Theorem 3, then following statements hold.*

(i) ST and TS have the Dunford's property (C) and the single-valued extension property.

(ii) If ST and TS are quasisimilar, then $\sigma(ST) = \sigma(TS)$ and $\sigma_e(ST) = \sigma_e(TS)$.

(iii) If $\sigma(ST)$ has nonempty interior in the complex plane, then ST and TS have nontrivial invariant subspaces.

Proof. (i) Since ST and TS have the property (β) by Theorem 3, the proof follows from [1].

(ii) The proof follows from [1].

(iii) Since $\sigma(ST) \cup \{0\} = \sigma(TS) \cup \{0\}$, the proof follows from Theorem 3 and [10]. □

COROLLARY 11. *Let $T = \begin{pmatrix} T_1 & T_3 \\ 0 & T_2 \end{pmatrix}$ be in $\mathscr{L}(\mathscr{H} \oplus \mathscr{H})$ where T_1 and T_2 are commuting relatively regular operators such that $T_i = T_i S_i T_i$ for $i = 1, 2$. If*

$$T_1^2 S_1 T_3 + T_1 T_3 S_2 T_2 - T_1 T_3 = T_1 S_1 T_3 T_2 + T_3 S_2 T_2^2 - T_3 T_2 \tag{12}$$

and $0 \notin \sigma_{ap}(T^2)$, then TS and ST have the property (β) where $S = S_1 \oplus S_2$.

Proof. Set $S = S_1 \oplus S_2$. Then $TST - T = N$ where

$$N = \begin{pmatrix} 0 & T_1 S_1 T_3 + T_3 S_2 T_2 - T_3 \\ 0 & 0 \end{pmatrix}$$

is nilpotent of order 2. Since $TN = NT$ by the hypothesis, T is relatively regular modulo N with order 2. Since $0 \notin \sigma_{ap}(T^2)$, TS and ST have the property (β) from Theorem 3. □

EXAMPLE 6. Let $T = \begin{pmatrix} U & U \\ 0 & U \end{pmatrix}$ be in $\mathscr{L}(\mathscr{H} \oplus \mathscr{H})$ where U is the unilateral shift. Since $0 \notin \sigma_{ap}(T^2)$, $TS = \begin{pmatrix} UU^* & UU^* \\ 0 & UU^* \end{pmatrix}$ and $ST = \begin{pmatrix} I & I \\ 0 & I \end{pmatrix}$ have the property (β) from Corollary 11 where $S = U^* \oplus U^*$.

THEOREM 4. *Let $T \in \mathscr{L}(\mathscr{H})$ be relatively regular modulo N with order k for some $S \in \mathscr{L}(\mathscr{H})$. If $0 \notin \sigma_p(T^k)$, S has the single-valued extension property, and $N^{k-1}S = SN^{k-1}$, then T has the single-valued extension property.*

Proof. Assume S has the single-valued extension property. Let $\lambda_0 \in \mathbb{C}$ and let G be an open connected set in $\mathbb{C}$ containing λ_0. Let f be any $\mathscr{H}$-valued analytic function on G such that

$$(T - \lambda) f(\lambda) \equiv 0.$$

Then

$$(TST - \lambda TS) f(\lambda) = (T + N - \lambda TS) f(\lambda) \equiv 0. \tag{13}$$

Since $N^k = 0$, we obtain that

$$(N^{k-1}T + N^k - \lambda N^{k-1}TS)f(\lambda) = N^{k-1}T(I - \lambda S)f(\lambda) \equiv 0$$

on G. Since $TN = NT$, $T^kN^{k-1}(I - \lambda S)f(\lambda) \equiv 0$. Since $0 \notin \sigma_p(T^k)$, $N^{k-1}(I - \lambda S)f(\lambda) \equiv 0$. Since $N^{k-1}S = SN^{k-1}$, $(I - \lambda S)N^{k-1}f(\lambda) \equiv 0$.

(i) If $0 \in G$, choose an open connected set G_0 such that $G_0 \subset G \setminus \{0\}$. Then

$$(S - \mu)N^{k-1}(f \circ g)(\mu) \equiv 0$$

where $g(\mu) = \frac{1}{\mu}$ and $\mu = \frac{1}{\lambda}$ on G_0. Since S has the single-valued extension property,

$$N^{k-1}(f \circ g)(\mu) \equiv 0 \tag{14}$$

on G_0. By the Identity Theorem, we obtain that $N^{k-1}f(\lambda) \equiv 0$ on G. If we apply (13) to (14), then

$$(N^{k-2}T + N^{k-1} - \lambda N^{k-2}TS)f(\lambda) = N^{k-2}T(I - \lambda S)f(\lambda) \equiv 0.$$

By the similar method above,

$$N^{k-2}f(\lambda) \equiv 0$$

on G. By the induction, we get that $f(\lambda) \equiv 0$ on G. Hence T has the single-valued extension property.

(ii) If $0 \notin G$, then we don't need to choose G_0 as in (i). Hence,

$$(S - \mu)N^{k-1}(f \circ g)(\mu) \equiv 0$$

where $g(\mu) = \frac{1}{\mu}$ and $\mu = \frac{1}{\lambda}$ on G. The remaining part is the same proof with (i). □

COROLLARY 12. *Let* $T = \begin{pmatrix} T_1 & T_3 \\ 0 & T_2 \end{pmatrix}$ *be in* $\mathcal{L}(\mathcal{H} \oplus \mathcal{H})$ *where* T_1 *and* T_2 *are commuting relatively regular operators such that* $T_i = T_iS_iT_i$ *for* $i = 1, 2$, $T_1T_3 = T_3T_2$, *and the condition (12) is satisfied. If* $0 \notin \sigma_p(T^2)$, S_1 *and* S_2 *have the single-valued extension property, and* $S_1(T_1S_1T_3 + T_3S_2T_2 - T_3) = (T_1S_1T_3 + T_3S_2T_2 - T_3)S_2$, *then* T *has the single-valued extension property.*

Proof. From the proof of Corollary 11, T is relatively regular modulo N with order 2 where

$$N = \begin{pmatrix} 0 & T_1S_1T_3 + T_3S_2T_2 - T_3 \\ 0 & 0 \end{pmatrix}$$

and $S = S_1 \oplus S_2$. Since $0 \notin \sigma_p(T^2)$, S has the single-valued extension property, and $SN = NS$ by the hypotheses, T has the single-valued extension property from Theorem 4. □

EXAMPLE 7. Let $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_1 \end{pmatrix}$ in $\mathcal{L}(\mathcal{H} \oplus \mathcal{H})$ where T_1 and T_2 are commuting unitary operators. Then T_1 and T_2 are relatively regular operators. Moreover, the condition (12) is satisfied and $T_1^*T_2 = T_2T_1^*$ by using the Fuglede-Putnam theorem. Since $S = T_1^* \oplus T_1^*$ has the single-valued extension property and $0 \notin \sigma_p(T^2)$, T has the single-valued extension property from Corollary 12.

COROLLARY 13. *Under the same hypotheses with Corollary 11, if S_1 and S_2 have the single-valued extension property,*

$$S_1(T_1S_1T_3 + T_3S_2T_2 - T_3) = (T_1S_1T_3 + T_3S_2T_2 - T_3)S_2,$$

and $T_1T_3 = T_3T_2$, then $\sigma(T_1) \cup \sigma(T_2) = \sigma(T)$ and $\sigma_e(T_1) \cup \sigma_e(T_2) = \sigma_e(T)$.

Proof. By Corollary 12, it is known that T has the single-valued extension property. Since $T_1T_3 = T_3T_2$, T_1 and T_2 have the single-valued extension property from [7]. Hence the proof follows from [5]. □

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