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A LYAPUNOV–TYPE INEQUALITY FOR A FRACTIONAL DIFFERENTIAL EQUATION UNDER STURM–LIOUVILLE BOUNDARY CONDITIONS

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Abstract. Lyapunov-type inequality is established for a fractional differential equation under Sturm-Liouville boundary conditions. Our results cover many results in the literature.

1. Introduction

The well-known result of Lyapunov [8] states that if $u(t)$ is a nontrivial solution of the differential system

$$\begin{aligned} &u''(t)+r(t)u(t)=0, \quad t\in(a,b),\\ &u(a)=0=u(b), \end{aligned} \tag{1}$$

where $r(t)$ is a continuous and nonnegative function defined in $[a,b]$, then

$$\int_a^b r(t)dt > \frac{4}{b-a}, \tag{2}$$

and the constant 4 cannot be replaced by a larger number.

Since the appearance of Lyapunov's fundamental paper, there are many improvements and generalizations of (2) in some literatures. A thorough literature review of continuous and discrete Lyapunov-type inequalities and their applications can be found in the survey articles by Cheng [2], Brown and Hinton [1] and Tiryaki [9].

However, there are few papers to discuss the Lyapunov-type inequality related to ordinary fractional differential equations.

The study of Lyapunov-type inequalities for the differential equation depends on a fractional differential operator was initiated by Rui A. C. Ferreira [3]. He first obtained a Lyapunov-type inequality when the differential equation depends on the Riemann-Liouville fractional derivative, the main result is as follows.

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THEOREM 1. *If the following fractional boundary value problem (FBVP)*

$$({}_aD^{\alpha}u)(t)+q(t)u(t)=0, \quad a<t<b,\ 1<\alpha\leqslant 2, \tag{3}$$
$$u(a)=0=u(b), \tag{4}$$

has a nontrivial solution, where q is a real and continuous function, then

$$\int_a^b |q(s)|ds > \Gamma(\alpha)\left(\frac{4}{b-a}\right)^{\alpha-1}. \tag{5}$$

Meanwhile, a Lyapunov-type inequality when the differential equation depends on the Caputo fractional derivative was also obtained by Rui A. C. Ferreira [4].

THEOREM 2. *If a nontrivial continuous solution of the fractional boundary value problem (FBVP)*

$$({}_a^CD^{\alpha}u)(t)+q(t)u(t)=0, \quad a<t<b,\ 1<\alpha\leqslant 2, \tag{6}$$
$$u(a)=0=u(b), \tag{7}$$

exists, where q is a real and continuous function, then

$$\int_a^b |q(s)|ds > \frac{\Gamma(\alpha)\alpha^{\alpha}}{[(\alpha-1)(b-a)]^{\alpha-1}}. \tag{8}$$

Recently, M. Jleli and B. Samet [5] investigated Lyapunov-type inequalities for fractional differential equation involving the Caputo fractional derivative under two types of mixed boundary conditions. The results are as follows.

THEOREM 3. *If a nontrivial continuous solution of the fractional boundary value problem*

$$({}_a^CD^{\alpha}u)(t)+q(t)u(t)=0, \quad a<t<b,\ 1<\alpha\leqslant 2, \tag{9}$$
$$u(a)=u'(b)=0, \tag{10}$$

exists, where q is a real and continuous function in [a,b], then

$$\int_a^b (b-s)^{\alpha-2}|q(s)|ds \geqslant \frac{\Gamma(\alpha)}{\max\{\alpha-1,2-\alpha\}(b-a)}. \tag{11}$$

THEOREM 4. *If a nontrivial continuous solution of the fractional boundary value problem*

$$({}_a^CD^{\alpha}u)(t)+q(t)u(t)=0, \quad a<t<b,\ 1<\alpha\leqslant 2, \tag{12}$$
$$u'(a)=u(b)=0, \tag{13}$$

exists, where q is a real and continuous function in [a,b], then

$$\int_a^b (b-s)^{\alpha-1}|q(s)|ds \geqslant \Gamma(\alpha). \tag{14}$$

Very recently, M. Jleli, L. Ragoub and B. Samet [6] considered a Caputo fractional differential equation under Robin boundary conditions. They give the following result.

THEOREM 5. *If a nontrivial continuous solution of the fractional boundary value problem*

$$({}_a^C D^\alpha u)(t) + q(t)u(t) = 0, \quad a < t < b,\ 1 < \alpha \leqslant 2, \tag{15}$$
$$u(a) - u'(a) = u(b) + u'(b) = 0, \tag{16}$$

exists, where q is a real and continuous function in [a,b], then

$$\int_a^b (b-s)^{\alpha-2}(b-s+\alpha-1)|q(s)|ds \geqslant \frac{(b-a+2)\Gamma(\alpha)}{\max\left\{\frac{2-\alpha}{\alpha-1}(b-a)-1, b-a+1\right\}}. \tag{17}$$

Motivated by the above cited works, the purpose of this paper is to obtain Lyapunov-type inequality for fractional differential equation involving the Caputo fractional derivative under Sturm-Liouville-type boundary conditions. More precisely, we consider the fractional differential equation

$$({}_a^C D^\alpha u)(t) + q(t)u(t) = 0, \quad a < t < b,\ 1 < \alpha \leqslant 2, \tag{18}$$

under the boundary conditions

$$\mu u(a) - \nu u'(a) = 0,\ \gamma u(b) + \delta u'(b) = 0, \tag{19}$$

where $\mu \geqslant 0,\ \nu \geqslant 0,\ \gamma \geqslant 0,\ \delta \geqslant 0$ and $\Delta = \mu\gamma(b-a) + \mu\delta + \gamma\nu > 0$.

2. Preliminaries

In this section, we recall the concepts of the Riemann-Liouville fractional integral and the Caputo fractional derivative of order $\alpha \geqslant 0$. For more details, we refer to [7].

DEFINITION 1. Let $\alpha \geqslant 0$ and f be a real function defined on $[a,b]$. The Riemann-Liouville fractional integral of order α is defined by $({}_aI^0 f) \equiv f$ and

$$({}_aI^\alpha f)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)ds, \quad \alpha > 0,\ t \in [a,b].$$

DEFINITION 2. The Caputo fractional derivative of order $\alpha \geqslant 0$ is defined by $({}_a^C D^0 f) \equiv f$ and $({}_a^C D^\alpha f)(t) = ({}_aI^{m-\alpha} D^m f)(t)$ for $\alpha > 0$, where m is the smallest integer greater or equal to α.

LEMMA 1. *$u \in C[a,b]$ is a solution of the boundary value problem* (18)–(19) *if and only if u satisfies the integral equation*

$$u(t) = \int_a^b G(t,s)q(s)u(s)ds,$$

where $G(t,s)$ *is given by*

$$G(t,s)=\frac{(b-s)^{\alpha-2}[\gamma(b-s)+\delta(\alpha-1)]}{\Delta\Gamma(\alpha)}H(t,s),$$

$$H(t,s)=\begin{cases} g_1(t,s), & a\leqslant s\leqslant t\leqslant b,\\ g_2(t,s), & a\leqslant t\leqslant s\leqslant b,\end{cases}$$

$$g_1(t,s)=\nu+\mu(t-a)-\frac{\Delta(t-s)^{\alpha-1}}{\gamma(b-s)^{\alpha-1}+\delta(\alpha-1)(b-s)^{\alpha-2}},$$

$$g_2(t,s)=\nu+\mu(t-a).$$

Proof. It is a standard result within the fractional calculus theory involving the Caputo differential operator that $u\in C[a,b]$ is a solution of (18) if and only if

$$u(t)=c_0+c_1(t-a)-\frac{1}{\Gamma(\alpha)}\int_a^t(t-s)^{\alpha-1}q(s)u(s)ds,$$

where c_0 and c_1 are some real constants. On the other hand, we have

$$u'(t)=c_1-\frac{1}{\Gamma(\alpha)}\int_a^t(\alpha-1)(t-s)^{\alpha-2}q(s)u(s)ds.$$

By the boundary condition $\mu u(a)-\nu u'(a)=0,\ \gamma u(b)+\delta u'(b)=0$, we have

$$\mu c_0-\nu c_1=0,$$

$$\gamma c_0+[(b-a)\gamma+\delta]c_1=\frac{1}{\Gamma(\alpha)}\int_a^b\frac{\gamma(b-s)+\delta(\alpha-1)}{(b-s)^{2-\alpha}}q(s)u(s)ds,$$

obviously,

$$\Delta=\begin{vmatrix}\mu & -\nu\\ \gamma & (b-a)\gamma+\delta\end{vmatrix}=\mu\gamma(b-a)+\mu\delta+\gamma\nu,$$

thus,

$$c_0=\frac{\nu}{\Delta\Gamma(\alpha)}\int_a^b\frac{\gamma(b-s)+\delta(\alpha-1)}{(b-s)^{2-\alpha}}q(s)u(s)ds,$$

$$c_1=\frac{\mu}{\Delta\Gamma(\alpha)}\int_a^b\frac{\gamma(b-s)+\delta(\alpha-1)}{(b-s)^{2-\alpha}}q(s)u(s)ds.$$

Then, we get

$$\begin{aligned}u(t)&=c_0+c_1(t-a)-\frac{1}{\Gamma(\alpha)}\int_a^t(t-s)^{\alpha-1}q(s)u(s)ds\\ &=\frac{1}{\Delta\Gamma(\alpha)}\int_a^b\frac{[\nu+\mu(t-a)][\gamma(b-s)+\delta(\alpha-1)]}{(b-s)^{2-\alpha}}q(s)u(s)ds\\ &\quad-\frac{1}{\Gamma(\alpha)}\int_a^t(t-s)^{\alpha-1}q(s)u(s)ds\\ &=\int_a^b G(t,s)q(s)u(s)ds.\end{aligned}$$

which concludes the proof. □

LEMMA 2. *For all* $(t,s) \in [a,b] \times (a,b)$, *we have*

$$|H(t,s)| \leqslant \max\left\{\mu \frac{2-\alpha}{\alpha-1}(b-a)-\nu, \nu+\mu(b-a)\right\}.$$

Proof. It is easy to see that, for $a \leqslant t \leqslant s \leqslant b$, we have

$$0 \leqslant g_2(t,s) \leqslant \nu+\mu(b-a). \tag{20}$$

Now, we divide our proof into two parts.

Part I. If $\mu = 0$, for $a \leqslant s \leqslant t \leqslant b$, we have

$$\frac{\partial g_1}{\partial t}(t,s) = -\frac{\Delta(\alpha-1)(t-s)^{\alpha-2}}{\gamma(b-s)^{\alpha-1}+\delta(\alpha-1)(b-s)^{\alpha-2}} \leqslant 0,$$

and

$$\begin{aligned} g_1(b,s) &= \nu - \frac{\Delta(b-s)^{\alpha-1}}{\gamma(b-s)^{\alpha-1}+\delta(\alpha-1)(b-s)^{\alpha-2}} \\ &= \frac{\delta\nu(\alpha-1)}{\gamma(b-s)+\delta(\alpha-1)} \\ &> 0, \end{aligned}$$

so we have

$$0 < g_1(b,s) \leqslant g_1(t,s) \leqslant g_1(s,s) = \nu \leqslant \nu+0(b-a),$$

therefore

$$0 \leqslant g_1(t,s) \leqslant \nu+0(b-a).$$

Part II. If $\mu > 0$, for $a \leqslant s \leqslant t \leqslant b$, we have

$$\frac{\partial g_1}{\partial t}(t,s) = \mu - \frac{\Delta(\alpha-1)(t-s)^{\alpha-2}}{\gamma(b-s)^{\alpha-1}+\delta(\alpha-1)(b-s)^{\alpha-2}}.$$

Hence

$$\lim_{t\to s^+}\frac{\partial g_1}{\partial t}(t,s) = -\infty, \quad \alpha < 2, \text{ while } \frac{\partial g_1}{\partial t}(s,s) < 0, \quad \alpha = 2. \tag{21}$$

Now, for fixed s in (a,b), we want to study the variation of the function $t \mapsto g_1(t,s)$ for t in $[s,b]$. First, we have

$$\left.\frac{\partial g_1}{\partial t}(t,s)\right|_{t=b} = \mu - \frac{\Delta(\alpha-1)}{\gamma(b-s)+\delta(\alpha-1)}. \tag{22}$$

Let

$$a^* = b - \frac{(\alpha-1)[\mu(b-a)+\nu]}{\mu}.$$

A calculation shows that

$$s \geqslant a^* \Leftrightarrow \mu(b-s) \leqslant (\alpha-1)[\mu(b-a)+\nu] \Leftrightarrow \frac{\partial g_1}{\partial t}(t,s)\Big|_{t=b} \leqslant 0.$$

We distinguish two eventual cases according to the value of a^*.

Case 1. If $a^* \leqslant a$, in this case, $s > a \geqslant a^*$, by above calculation, we have

$$\frac{\partial g_1}{\partial t}(t,s)\Big|_{t=b} \leqslant 0. \quad s \in (a,b). \tag{23}$$

Since $\frac{\partial^2 g_1}{\partial t^2}(t,s) \geqslant 0$, then $\frac{\partial g_1}{\partial t}(t,s)$ is an increasing function with respect to t, from (21) and (23), we deduce

$$\frac{\partial g_1}{\partial t}(t,s) \leqslant 0, \quad s < t. \tag{24}$$

This yields

$$\begin{aligned} g_1(b,s) &= \nu + \mu(b-a) - \frac{\Delta(b-s)}{\gamma(b-s)+\delta(\alpha-1)} \\ &\leqslant g_1(t,s) \leqslant g_1(s,s) \leqslant \nu + \mu(b-a). \end{aligned} \tag{25}$$

In this case, by $a^* \leqslant a$, we get $\mu(b-a)(\alpha-2)+\nu(\alpha-1) \geqslant 0$. So we have

$$\begin{aligned} g_1(b,s) &= \nu + \mu(b-a) - \frac{\Delta(b-s)}{\gamma(b-s)+\delta(\alpha-1)} \\ &= \frac{\mu\delta(s-b)+\delta(\alpha-1)[\nu+\mu(b-a)]}{\gamma(b-s)+\delta(\alpha-1)} \\ &\geqslant \frac{\mu\delta(a-b)+\delta(\alpha-1)[\nu+\mu(b-a)]}{\gamma(b-s)+\delta(\alpha-1)} \\ &= \frac{\mu\delta(b-a)(\alpha-2)+\delta\nu(\alpha-1)}{\gamma(b-s)+\delta(\alpha-1)} \\ &\geqslant 0. \end{aligned} \tag{26}$$

From (25) and (26), we deduce

$$0 \leqslant g_1(t,s) \leqslant \nu + \mu(b-a). \tag{27}$$

Case 2. If $a < a^* \leqslant b$, in this case, we have two possibilities.

(i) If $a^* \leqslant s < b$, in this case, we have

$$\frac{\partial g_1}{\partial t}(t,s)\Big|_{t=b} \leqslant 0. \tag{28}$$

Therefore, we conclude that

$$g_1(b,s) \leqslant g_1(t,s) \leqslant g_1(s,s) \leqslant \nu + \mu(b-a). \tag{29}$$

In this case, by $a^* \leqslant s$, we get $(\alpha-1)[\nu+\mu(b-a)] \geqslant \mu(b-s)$. So we have

$$\begin{aligned} g_1(b,s) &= \nu+\mu(b-a)-\frac{\Delta(b-s)}{\gamma(b-s)+\delta(\alpha-1)} \\ &= \frac{[\nu+\mu(b-a)][\gamma(b-s)+\delta(\alpha-1)]-\Delta(b-s)}{\gamma(b-s)+\delta(\alpha-1)} \\ &= \frac{\mu\delta(s-b)+\delta(\alpha-1)[\nu+\mu(b-a)]}{\gamma(b-s)+\delta(\alpha-1)} \\ &\geqslant 0. \end{aligned} \tag{30}$$

From (29) and (30), we deduce

$$0 \leqslant g_1(t,s) \leqslant \nu+\mu(b-a). \tag{31}$$

(ii) If $a<s<a^*$, in this case, we have

$$\left.\frac{\partial g_1}{\partial t}(t,s)\right|_{t=b} > 0. \tag{32}$$

Hence, there would exist $t^* \in (s,b)$ such that

$$\left.\frac{\partial g_1}{\partial t}(t,s)\right|_{t=t^*} = 0. \tag{33}$$

In this case, it is easy to verify that $g_1(s,s) \geqslant 0$, and by $s<a^*$, we get $\mu(s-b)+(\alpha-1)[\nu+\mu(b-a)]<0$, so we have

$$\begin{aligned} g_1(b,s) &= \nu+\mu(b-a)-\frac{\Delta(b-s)}{\gamma(b-s)+\delta(\alpha-1)} \\ &= \frac{\mu\delta(s-b)+\delta(\alpha-1)[\nu+\mu(b-a)]}{\gamma(b-s)+\delta(\alpha-1)} \\ &< 0. \end{aligned} \tag{34}$$

This yields

$$g_1(t^*,s) \leqslant g_1(b,s) \leqslant 0 \leqslant g_1(s,s) \leqslant \nu+\mu(b-a). \tag{35}$$

Then

$$|g_1(t,s)| \leqslant \max\{-g_1(t^*,s), \nu+\mu(b-a)\}. \tag{36}$$

Observe that $\left.\frac{\partial g_1}{\partial t}(t,s)\right|_{t=t^*} = 0$ is equivalent to

$$\mu = \frac{\Delta(\alpha-1)(t^*-s)^{\alpha-2}}{\gamma(b-s)^{\alpha-1}+\delta(\alpha-1)(b-s)^{\alpha-2}}.$$

Therefore, we get

$$\begin{aligned} g_1(t^*,s) &= \nu+\mu(t^*-a)-\frac{\Delta(t^*-s)^{\alpha-1}}{\gamma(b-s)^{\alpha-1}+\delta(\alpha-1)(b-s)^{\alpha-2}} \\ &= \nu+\mu(t^*-a)-\frac{\mu(t^*-s)}{\alpha-1} \\ &= \nu-\mu a+\frac{\mu(\alpha-2)}{\alpha-1}t^*+\frac{\mu s}{\alpha-1} \\ &\geqslant \nu-\mu a+\frac{\mu(\alpha-2)}{\alpha-1}b+\frac{\mu a}{\alpha-1} \\ &= \nu+\mu\frac{\alpha-2}{\alpha-1}(b-a). \end{aligned}$$

Finally, using the above inequality and (36), we obtain

$$|g_1(t,s)| \leqslant \max\left\{\mu\frac{2-\alpha}{\alpha-1}(b-a)-\nu, \nu+\mu(b-a)\right\}. \tag{37}$$

which makes end to the proof. □

3. Main results

Our main result is as follows.

THEOREM 6. *If a nontrivial continuous solution of the fractional boundary value problem* (18)–(19) *exists, then*

$$\int_a^b (b-s)^{\alpha-2}[\gamma(b-s)+\delta(\alpha-1)]|q(s)|ds \geqslant \frac{[\mu\gamma(b-a)+\mu\delta+\gamma\nu]\Gamma(\alpha)}{\max\left\{\mu\frac{2-\alpha}{\alpha-1}(b-a)-\nu, \nu+\mu(b-a)\right\}}. \tag{38}$$

Proof. Let $\mathbf{B}=C[a,b]$ be the Banach space endowed with norm $\|x\| = \sup_{t\in[a,b]} |x(t)|$. From Lemma 1, for all $t\in[a,b]$, we have

$$u(t) = \frac{1}{\Delta\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-2}[\gamma(b-s)+\delta(\alpha-1)]H(t,s)q(s)u(s)ds.$$

Now, an application of Lemma 2 yields

$$\begin{aligned} \|u\| \leqslant & \frac{\max\left\{\mu\frac{2-\alpha}{\alpha-1}(b-a)-\nu, \nu+\mu(b-a)\right\}}{[\mu\gamma(b-a)+\mu\delta+\gamma\nu]\Gamma(\alpha)}\|u\| \\ & \times\int_a^b (b-s)^{\alpha-2}[\gamma(b-s)+\delta(\alpha-1)]|q(s)|ds, \end{aligned}$$

from which inequality in (38) follows. □

REMARK 1. Let $\mu = \nu = \gamma = \delta = 1$ in Theorem 6, then we get inequality (17).

Let $\gamma = 0,\ \delta = 1$ in Theorem 6, then we obtain the following result.

COROLLARY 1. *If a nontrivial continuous solution of the fractional boundary value problem*

$$\begin{aligned}&({}_a^C D^\alpha u)(t) + q(t)u(t) = 0, \quad a < t < b,\ 1 < \alpha \leqslant 2,\\ &\mu u(a) - \nu u'(a) = 0,\ u'(b) = 0,\end{aligned}$$

exists, where $\mu > 0, \nu \geqslant 0$, *q is a real and continuous function in* $[a,b]$, *then*

$$\int_a^b (b-s)^{\alpha-2}|q(s)|ds \geqslant \frac{\frac{\mu}{\alpha-1}\Gamma(\alpha)}{\max\left\{\mu\frac{2-\alpha}{\alpha-1}(b-a) - \nu, \nu + \mu(b-a)\right\}}. \tag{39}$$

REMARK 2. Note that one obtains Lyapunov-type inequality (11) when $\mu = 1, \nu = 0$ in (39).

Let $\mu = 0,\ \nu = 1$ in Theorem 6, then we obtain the following result.

COROLLARY 2. *If a nontrivial continuous solution of the fractional boundary value problem*

$$\begin{aligned}&({}_a^C D^\alpha u)(t) + q(t)u(t) = 0, \quad a < t < b,\ 1 < \alpha \leqslant 2,\\ &u'(a) = 0,\ \gamma u(b) + \delta u'(b) = 0,\end{aligned}$$

exists, where $\gamma > 0,\ \delta \geqslant 0$, *q is a real and continuous function in* $[a,b]$, *then*

$$\int_a^b [\gamma(b-s) + \delta(\alpha-1)](b-s)^{\alpha-2}|q(s)|ds \geqslant \gamma\Gamma(\alpha). \tag{40}$$

REMARK 3. Note that one obtains Lyapunov-type inequality (14) when $\gamma = 1$, $\delta = 0$ in (40).

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