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POINTWISE CONVERGENCE AND CESÀRO SUMMABILITY OF DOUBLE ORTHOGONAL SERIES

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Abstract. Let $(X,\mathscr{F},\mu)$ be a positive measure space and $\{\phi_{j,k}(x) : j,k = 1,2,\dots\}$ be a double orthonormal system of real-valued functions on X. We extend four previous results of Borgen [2] and Tandori [4, 5] from single to double orthogonal series.

1. Known results for single orthogonal series

Let $(X,\mathscr{F},\mu)$ be a positive measure space and $\{\phi_j(x) : j = 1,2,\dots\}$ be an orthonormal system of real-valued functions on X, in abbreviation: ONS. We consider the orthogonal series

$$\sum_{j=1}^{\infty} c_j\phi_j(x), \tag{1.1}$$

where $\{c_j : j = 1,2,\dots\}$ is a sequence of real numbers (so-called coefficients) satisfying the condition

$$\sum_{j=1}^{\infty} c_j^2 < \infty. \tag{1.2}$$

By the Riesz–Fischer theorem, there exists a function $f(x) \in L^2 = L^2(X,\mathscr{F},\mu)$ such that (1.1) is the generalized Fourier series of $f(x)$ with respect to the system $\{\phi_j(x)\}$ and the partial sums

$$s_m(x) := \sum_{j=1}^{m} c_j\phi_j(x), \qquad m = 1,2,\dots$$

of the orthogonal series (1.1) converge to $f(x)$ in L^2-norm:

$$\lim_{m\to\infty} \int |s_m(x) - f(x)|^2\, d\mu(x) = 0. \tag{1.3}$$

Here and in the sequel, the integrals are taken over the entire space X.

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It is well known that condition (1.2) does not ensure the pointwise convergence of the partial sums $s_m(x)$ to $f(x)$ as $m \to \infty$. The Rademacher–Menshov theorem (see, e.g., [1, Theorem 2.3.2, p. 80]) reads as follows: If

$$\sum_{j=1}^{\infty} c_j^2 \big[\log(j+1)\big]^2 < \infty, \tag{1.4}$$

where the logarithm is to the base 2, then

$$\lim_{m\to\infty} s_m(x) = f(x) \quad \text{a.e.}, \tag{1.5}$$

where $f(x)$ is the sum of the orthogonal series (1.1) in the L^2-norm (see in (1.3)).

The Cesàro summability $(C,1)$ of the orthogonal series is defined by the convergence of the arithmetic means

$$\sigma_n(x) := \frac{1}{n}\sum_{m=1}^{n} s_m(x), \qquad n = 1,2,\dots$$

of the partial sums. It is well known that the a.e. $(C,1)$ summability is guaranteed by a weaker condition than (1.4). In fact, the Menshov–Kaczmarz theorem (see, e.g., [1, Theorem 2.8.1, p. 125]) reads as follows: If

$$\sum_{j=1}^{\infty} c_j^2 \big[\log\log(j+3)\big]^2 < \infty, \tag{1.6}$$

then

$$\lim_{n\to\infty} \sigma_n(x) = f(x) \quad \text{a.e.},$$

where $f(x)$ occurs in (1.3).

Among others, the following theorems were proved in [2] by Borges and in [4, 5] by Tandori.

THEOREM A. *If condition (1.2) is satisfied, then*

$$\lim_{m\to\infty} \big(s_{2^m}(x) - \sigma_{2^m}(x)\big) = 0 \quad \textit{a.e.}$$

THEOREM B. *If condition (1.2) is satisfied, then*

$$\sum_{n=1}^{\infty} (n+1)|\Delta\sigma_n(x)|^2 < \infty \quad \textit{a.e.},$$

where

$$\Delta\sigma_n(x) := \sigma_{n+1}(x) - \sigma_n(x), \qquad n = 1,2,\dots.$$

THEOREM C. *If condition (1.2) is satisfied, and* $\big\{\sigma_{2^m}(x) : m = 1,2,\dots\big\}$ *converges a.e., then* $\big\{\sigma_n(x) : n = 1,2,\dots\big\}$ *also converges a.e.*

Our goal in this paper is to extend Theorems A, B, C from single to double orthogonal series.

In the proof of the extension of Theorem C from single to double orthogonal series (see Corollary 4 below), we will make use of such an argument that is analogous to the one in the proof of Theorem C. For the reader's convenience, we present it.

Proof of Theorem C. Suppose

$$2^m < p \leqslant 2^{m+1}, \tag{1.7}$$

where $m, p \in \mathbb{N}$. By the familiar Cauchy inequality for sequences of real numbers, we may estimate as follows

$$\begin{aligned} \left|\sigma_p(x) - \sigma_{2^m}(x)\right| &= \sum_{k=2^m}^{p-1} \left|\Delta\sigma_k(x)\right| \\ &\leqslant \left(\sum_{k=2^m}^{p-1} (k+1)\left|\Delta\sigma_k(x)\right|^2\right)^{1/2} \left(\sum_{k=2^m}^{p-1} \frac{1}{k+1}\right)^{1/2}. \end{aligned} \tag{1.8}$$

For $p \leqslant 2^{m+1}$ (see in (1.7)), we clearly have that

$$\sum_{k=2^m}^{p-1} \frac{1}{k+1} \leqslant \sum_{k=2^m}^{2^{m+1}-1} \frac{1}{k+1} \leqslant \frac{2^m}{2^m+1} \leqslant 1. \tag{1.9}$$

It follows from (1.8) and (1.9) that

$$\max_{2^m < p \leqslant 2^{m+1}} \left|\sigma_p(x) - \sigma_{2^m}(x)\right| \leqslant \left(\sum_{k=2^m}^{2^{m+1}-1} (k+1)\left|\Delta\sigma_k(x)\right|^2\right)^{1/2} \to 0 \quad \text{a.e. as } m \to \infty, \tag{1.10}$$

due to Theorem B.

By assumption, the subsequence $\{\sigma_{2^m}(x) : m = 1, 2, \dots\}$ converges a.e. Now, the limit in (1.10) clearly shows that the whole sequence $\{\sigma_p(x) : p = 1, 2, \dots\}$ also converges a.e. The proof of Theorem C is complete. □

2. Known results for double orthogonal series

We consider the double ONS $\{\phi_{j,k}(x) : j, k = 1, 2, \dots\}$ of real-valued functions on a positive measure space $(X, \mathscr{F}, \mu)$. We investigate the pointwise convergence and summability of the double orthogonal series

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} c_{j,k}\phi_{j,k}(x), \tag{2.1}$$

where $\{c_{j,k} : j, k = 1, 2, \dots\}$ is a double sequence of real numbers satisfying the condition

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} c_{j,k}^2 < \infty. \tag{2.2}$$

By the Riesz–Fischer theorem, there exists a function $g(x) \in L^2$ such that (2.1) is the generalized double Fourier series of $g(x)$ with respect to the system $\{\phi_{j,k}(x)\}$ and the rectangular partial sums

$$s_{m,n}(x) := \sum_{j=1}^{m} \sum_{k=1}^{n} c_{j,k}\phi_{j,k}(x) \tag{2.3}$$

of the double orthogonal series (2.1) converge to $g(x)$ in L^2-norm:

$$\lim_{m,n\to\infty} \int |s_{m,n}(x) - g(x)|^2 \, d\mu(x) = 0. \tag{2.4}$$

It is also well known that condition (2.2) does not ensure the pointwise convergence of the rectangular partial sums $s_{m,n}(x)$ as $m,n \to \infty$. The extension of the Rademacher–Menshov theorem (see, e.g., [3, Theorem A]) reads as follows: If the condition

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} c_{j,k}^2 \big[\log(j+1)\big]^2 \big[\log(k+1)\big]^2 < \infty, \tag{2.5}$$

is satisfied, then

$$\lim_{m,n\to\infty} s_{m,n}(x) = g(x) \quad \text{a.e.,}$$

where $g(x)$ occurs in (2.4).

The a.e. Cesàro summability $(C,1,1)$ of the double orthogonal series (2.1) is defined by the a.e. convergence of the arithmetic means

$$\sigma_{M,N}(x) := \frac{1}{MN} \sum_{m=1}^{M} \sum_{n=1}^{N} s_{m,n}(x), \qquad M,N = 1,2,\ldots \tag{2.6}$$

of the rectangular partial sums, can be guaranteed under a weaker condition than (2.5). In fact, the extension of the Menshov–Kaczmarz theorem (proved in [3, Corollary 4]) reads as follows: If the condition

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} c_{j,k}^2 \big[\log\log(j+3)\big]^2 \big[\log\log(k+3)\big]^2 < \infty \tag{2.7}$$

is satisfied, then we have

$$\lim_{M,N\to\infty} \sigma_{M,N}(x) = g(x) \quad \text{a.e.}$$

3. New results

Our first new result reads as follows.

THEOREM 1. *Suppose the double orthogonal series* (2.1) *is such that condition* (2.2) *is satisfied, then*

$$\lim_{m,n\to\infty} \big(s_{2^m,2^n}(x) - \sigma_{2^m,2^n}(x)\big) = 0 \quad \textit{a.e.} \tag{3.1}$$

Making use of Theorem 1 and the extension of the Menshov–Kaczmarz theorem, we can get a short proof for the extension of [1, Theorem 2.3.4, p. 83].

COROLLARY 2. *Suppose the double orthogonal series (2.1) is such that condition (2.7) is satisfied, then we have*

$$\lim_{m,n\to\infty} s_{2^m,2^n}(x) = g(x) \quad a.e. \tag{3.2}$$

Our third new result reads as follows.

THEOREM 3. *Suppose the double orthogonal series (2.1) is such that condition (2.2) is satisfied, then*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(m+1)(n+1)\left|\Delta_{1,1}\sigma_{m,n}(x)\right|^2 < \infty \quad a.e., \tag{3.3}$$

where

$$\Delta_{1,1}\sigma_{m,n}(x) := \sigma_{m+1,n+1}(x) - \sigma_{m+1,n}(x) - \sigma_{m,n+1}(x) + \sigma_{m,n}(x). \tag{3.4}$$

The following Corollary 4 of Theorem 3 is also of special interest.

COROLLARY 4. *Suppose the double orthogonal series (2.1) is such that condition (2.2) is satisfied. If the double subsequences*

$$\left\{\sigma_{m,2^N}(x) : m = 1,2,\ldots; N = 0,1,\ldots\right\}$$

and

$$\left\{\sigma_{2^M,n}(x) : M = 0,1,\ldots; n = 1,2,\ldots\right\}$$

of the Cesàro means of (2.1) converge a.e. to the same limit $\ell(x)$, then the whole double sequence $\{\sigma_{m,n}(x) : m,n = 1,2,\ldots\}$ also converges to the same $\ell(x)$ a.e.

In the case of the above conditions, the subsequence $\{\sigma_{2^M,2^N} : M,N = 0,1,\ldots\}$ clearly converges to the same limit $\ell(x)$ a.e., since it is a subsequence of both $\{\sigma_{m,2^N}\}$ and $\{\sigma_{2^M,n}\}$.

4. Proof of the new results

Proof of Theorem 1. It is routine to check that from (2.3) and (2.6) it follows that

$$s_{2^m,2^n}(x) - \sigma_{2^m,2^n}(x) = \sum_{j=2}^{2^m}\sum_{k=2}^{2^n}\frac{(j-1)(k-1)}{2^m 2^n}c_{j,k}\phi_{j,k}(x), \quad m,n = 1,2,\ldots.$$

Making use of orthonormality of the system $\{\phi_{j,k}(x)\}$, integration gives

$$\int \left|s_{2^m,2^n}(x) - \sigma_{2^m,2^n}(x)\right|^2 d\mu(x) = \sum_{j=2}^{2^m}\sum_{k=2}^{2^n}\frac{(j-1)^2(k-1)^2}{2^{2m}2^{2n}}c_{j,k}^2, \quad m,n = 1,2,\ldots. \tag{4.1}$$

Clearly, it follows from (4.1) that

$$\begin{aligned}
&\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\int \left|s_{2^m,2^n}(x)-\sigma_{2^m,2^n}(x)\right|^2 d\mu(x)\\
&=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{j=2}^{2^m}\sum_{k=2}^{2^n}\frac{(j-1)^2(k-1)^2}{2^{2m}2^{2n}}c_{j,k}^2\\
&=\sum_{j=2}^{\infty}\sum_{k=2}^{\infty}(j-1)^2(k-1)^2c_{j,k}^2\sum_{m:2^m\geqslant j}\sum_{n:2^n\geqslant k}\frac{1}{2^{2m}2^{2n}}, \qquad (4.2)
\end{aligned}$$

where we interchanged the order of summations with respect to j and m as well as with respect to k and n.

It is easy to check that

$$\sum_{m:2^m\geqslant j}\frac{1}{2^{2m}}=\sum_{m=[\log j]}^{\infty}\frac{1}{4^m}\leqslant\frac{4}{3\cdot 4^{\log j}}=\frac{4}{3j^2},$$

and analogously, we have

$$\sum_{n:2^n\geqslant k}\frac{1}{2^{2n}}\leqslant\frac{4}{3k^2}.$$

Taking into account the last two inequalities, from (4.2) it follows that

$$\begin{aligned}
\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\int \left|s_{2^m,2^n}(x)-\sigma_{2^m,2^n}(x)\right|^2 d\mu(x) &\leqslant \frac{16}{9}\sum_{j=2}^{\infty}\sum_{k=2}^{\infty}(j-1)^2(k-1)^2c_{j,k}^2\frac{1}{j^2k^2}\\
&\leqslant \frac{16}{9}\sum_{j=2}^{\infty}\sum_{k=2}^{\infty}c_{j,k}^2<\infty, \qquad (4.3)
\end{aligned}$$

due to our assumption (2.2).

By the monotone convergence theorem of the Lebesgue integral, we conclude from (4.3) that the double series

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left|s_{2^m,2^n}(x)-\sigma_{2^m,2^n}(x)\right|^2<\infty \quad \text{a.e.}$$

Thus, our Theorem 1 has been proved. □

Proof of Corollary 2. We start with the elementary inequality

$$|s_{2^m,2^n}(x)-g(x)|^2\leqslant 2\left(|s_{2^m,2^n}(x)-\sigma_{2^m,2^n}(x)|^2+|\sigma_{2^m,2^n}(x)-g(x)|^2\right). \qquad (4.4)$$

By Theorem 1, the first term on the right-hand side of (4.4) converges to 0 as $m,n\to\infty$. As to the second term there, due to condition (2.7) and the extended Menshov–Kaczmarz theorem it is clear that the second term also tends to 0 a.e. as $m,n\to\infty$. The proof of (3.2) is complete. □

Proof of Theorem 3. Let $m,n \geqslant 1$. By (2.6) and (3.4), it is routine to check that

$$\Delta_{1,1}\sigma_{m,n}(x) = \sum_{j=2}^{m+1}\sum_{k=2}^{n+1} \frac{(j-1)(k-1)}{m(m+1)n(n+1)} c_{j,k}\phi_{j,k}(x).$$

Making use of orthonormality of the system $\{\phi_{j,k}(x)\}$, integration gives

$$(m+1)(n+1)\int |\Delta_{1,1}\sigma_{m,n}(x)|^2\, d\mu(x) = \frac{1}{m^2(m+1)n^2(n+1)} \sum_{j=2}^{m+1}\sum_{k=2}^{n+1}(j-1)^2(k-1)^2 c_{j,k}^2,$$

whence we get

$$\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}(m+1)(n+1)\int |\Delta_{1,1}\sigma_{m,n}(x)|^2\, d\mu(x)$$
$$= \sum_{m=2}^{\infty}\sum_{n=2}^{\infty} \frac{1}{m^2(m+1)n^2(n+1)} \sum_{j=2}^{m+1}\sum_{k=2}^{n+1}(j-1)^2(k-1)^2 c_{j,k}^2. \tag{4.5}$$

Next, we interchange the order of summations with respect to j and m as well as with respect to k and n. As a result, we obtain

$$\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}(m+1)(n+1)\int |\Delta_{1,1}\sigma_{m,n}(x)|^2\, d\mu(x)$$
$$= \sum_{j=2}^{\infty}\sum_{k=2}^{\infty}(j-1)^2(k-1)^2 c_{j,k}^2 \sum_{m=j-1}^{\infty}\sum_{n=k-1}^{\infty} \frac{1}{m^2(m+1)n^2(n+1)}$$
$$\leqslant \frac{1}{4}\sum_{j=2}^{\infty}\sum_{k=2}^{\infty} c_{j,k}^2 < \infty, \tag{4.6}$$

due to our assumption (2.2) and the fact that

$$\sum_{m=j-1}^{\infty} \frac{1}{m^2(m+1)} < \int_{j-1}^{\infty} \frac{dt}{t^3} = \frac{1}{2(j-1)^2}.$$

By the monotone convergence theorem of the Lebesgue integral, we conclude from (4.6) that the double series in (3.3) converges a.e. The proof of Theorem 3 is complete. □

Proof of Corollary 4. We will imitate the proof of Theorem C in Section 1. To start with, suppose

$$2^M < p \leqslant 2^{M+1} \text{ and } 2^N < q \leqslant 2^{N+1}, \qquad M,N = 1,2,\ldots. \tag{4.7}$$

We observe that the following sum is a telescopic one, and due to this fact we have

$$\sum_{m=2^M}^{p-1}\sum_{n=2^N}^{q-1} \Delta_{1,1}\sigma_{m,n}(x) = \sigma_{p,q}(x) - \sigma_{2^M,q}(x) - \sigma_{p,2^N}(x) + \sigma_{2^M,2^N}(x).$$

Keeping this in mind and using the familiar Cauchy inequality for sequences of numbers, we may estimate as follows

$$
\begin{aligned}
&\max_{2^M < p \leqslant 2^{M+1}} \max_{2^N < q \leqslant 2^{N+1}} \left|\sigma_{p,q}(x) - \sigma_{2^M,q}(x) - \sigma_{p,2^N}(x) + \sigma_{2^M,2^N}(x)\right| \\
&= \max_{2^M < p \leqslant 2^{M+1}} \max_{2^N < q \leqslant 2^{N+1}} \left|\sum_{m=2^M}^{p-1} \sum_{n=2^N}^{q-1} \Delta_{1,1}\sigma_{m,n}(x)\right| \\
&\leqslant \left(\sum_{m=2^M}^{2^{M+1}-1} \sum_{n=2^N}^{2^{N+1}-1} (m+1)(n+1)\left|\Delta_{1,1}\sigma_{m,n}(x)\right|^2\right)^{1/2} \left(\sum_{m=2^M}^{2^{M+1}-1} \sum_{n=2^N}^{2^{N+1}-1} \frac{1}{(m+1)(n+1)}\right)^{1/2} \\
&\leqslant \left(\sum_{m=2^M}^{2^{M+1}-1} \sum_{n=2^N}^{2^{N+1}-1} (m+1)(n+1)\left|\Delta_{1,1}\sigma_{m,n}(x)\right|^2\right)^{1/2} \to 0 \quad \text{a.e. as } M,N \to \infty, \qquad (4.8)
\end{aligned}
$$

due to Theorem 3.

Now, by the assumptions of Corollary 4 and the inequality (4.8), for $2^M < p \leqslant 2^{M+1}$ and $2^N < q \leqslant 2^{N+1}$ we may estimate as follows

$$
\begin{aligned}
\left|\sigma_{p,q}(x) - \ell(x)\right| &= \Big|\big(\sigma_{p,q}(x) - \sigma_{2^M,q}(x) - \sigma_{p,2^N}(x) + \sigma_{2^M,2^N}(x)\big) \\
&\quad + \big(\sigma_{2^M,q}(x) - \ell(x)\big) + \big(\sigma_{p,2^N}(x) - \ell(x)\big) - \big(\sigma_{2^M,2^N}(x) - \ell(x)\big)\Big| \\
&\leqslant \left|\sigma_{p,q}(x) - \sigma_{2^M,q}(x) - \sigma_{p,2^N}(x) + \sigma_{2^M,2^N}(x)\right| \\
&\quad + \left|\sigma_{2^M,q}(x) - \ell(x)\right| + \left|\sigma_{p,2^N}(x) - \ell(x)\right| + \left|\sigma_{2^M,2^N}(x) - \ell(x)\right|.
\end{aligned}
$$

Taking the maximum in the inequality just received with respect to p and q subject to (4.7), we obtain that

$$
\begin{aligned}
&\max_{2^M < p \leqslant 2^{M+1}} \max_{2^N < q \leqslant 2^{N+1}} \left|\sigma_{p,q}(x) - \ell(x)\right| \\
&\leqslant \max_{2^M < p \leqslant 2^{M+1}} \max_{2^N < q \leqslant 2^{N+1}} \left|\sigma_{p,q}(x) - \sigma_{2^M,q}(x) - \sigma_{p,2^N}(x) + \sigma_{2^M,2^N}(x)\right| \\
&\quad + \max_{2^N < q \leqslant 2^{N+1}} \left|\sigma_{2^M,q}(x) - \ell(x)\right| + \max_{2^M < p \leqslant 2^{M+1}} \left|\sigma_{p,2^N}(x) - \ell(x)\right| \\
&\quad + \left|\sigma_{2^M,2^N}(x) - \ell(x)\right|. \qquad (4.9)
\end{aligned}
$$

Letting $M,N \to \infty$ in (4.9), it follows from (4.8) and the assumptions in Corollary 4 that the finite limit

$$\lim_{p,q\to\infty} \sigma_{p,q}(x) = \ell(x) \quad \text{a.e.}$$

exists. The proof of Corollary 4 is complete. □

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