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CONVOLUTION INEQUALITIES IN WEIGHTED LORENTZ SPACES: CASE $0 < q < 1$

MARTIN KŘEPELA

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Abstract. Let g be a fixed nonnegative radially decreasing kernel g. In this paper, boundedness of the convolution operator $T_g f := f * g$ between the weighted Lorentz spaces $\Gamma^q(w)$ and $\Lambda^p(v)$ is characterized in the case $0 < q < 1$. The conditions are sufficient if the kernel g is just a general measurable function.

Furthermore, the largest rearrangement-invariant (quasi-)space Y is found such that the Young-type inequality

$$\|f * g\|_{\Gamma^q(w)} \leqslant C\|f\|_{\Lambda^p(v)}\|g\|_Y$$

holds for all $f \in \Lambda^p(v)$ and $g \in Y$.

1. Introduction

Denote by $\mathscr{M}$ the cone of all measurable functions on $\mathbb{R}^n$. If $f, g \in \mathscr{M}$, the *convolution* of f and g is given by

$$(f * g)(x) = \int_{\mathbb{R}^n} f(y)g(x-y)\,\mathrm{d}y$$

for any $x \in \mathbb{R}^n$ for which the integral is defined. If $g \in \mathscr{M}$ is fixed, it is possible to define the convolution operator T_g by

$$T_g f(x) := (f * g)(x)$$

for $f \in \mathscr{M}$ and $x \in \mathbb{R}^n$, provided that the right-hand side is well defined.

In [10], the author characterized boundedness of the operator T_g between weighted Lorentz spaces $\Lambda^p(v)$ and $\Gamma^q(w)$ (see the definitions below) in the cases $0 < p < \infty$, $1 \leqslant q < \infty$ and $p = \infty$, $0 < q \leqslant \infty$. In the present article, the case $0 < q < 1$, $0 < p < \infty$ is treated, completing the results for the whole range $p, q \in (0, \infty]$.

Let $f \in \mathscr{M}$. The symbol f^* stands for the *nonincreasing rearrangement* of f, and f^{**} is the *Hardy-Littlewood maximal function* given by $f^{**}(t) := \frac{1}{t}\int_0^t f^*(s)\,\mathrm{d}s$ for $t > 0$ (see [2] for details).

A *weight* is a nonnegative measurable function w defined on $(0, \infty)$ and such that $0 < W(t) < \infty$ for all $t \in (0, \infty)$, where $W(t) := \int_0^t w(s)\,\mathrm{d}s$. The notation $V(t)$ has an analogous meaning for a weight v.

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Let v be a weight and $p \in (0,\infty)$. The *weighted Lebesgue space* $L^p(v)$ is the set of all measurable functions h on $(0,\infty)$ such that

$$\|h\|_{L^p(v)} := \left(\int_0^\infty |h(t)|^p v(t)\,\mathrm{d}t \right)^{\frac{1}{p}} < \infty.$$

Naturally, an analogy for $p = \infty$ also exists. By L^1 one denotes the space $L^1(\mathbb{R}^n)$, and L^1_{loc} stands for the space of all locally integrable functions on $\mathbb{R}^n$.

The *weighted Lorentz spaces* $\Lambda^p(v)$ and $\Gamma^p(v)$ are defined by

$$\Lambda^p(v) := \left\{ f \in \mathscr{M};\ \|f\|_{\Lambda^p(v)} := \|f^*\|_{L^p(v)} < \infty \right\},$$
$$\Gamma^p(v) := \left\{ f \in \mathscr{M};\ \|f\|_{\Gamma^p(v)} := \|f^{**}\|_{L^p(v)} < \infty \right\}.$$

For definitions of *rearrangement-invariant (r.i.) spaces, quasi-spaces* and *lattices*, see e.g. [2, 10]. The $\Lambda^p(v)$ and $\Gamma^p(v)$ “spaces” are always at least r.i. lattices, questions of their linearity and (quasi-)normability are treated e.g. in [6, 16] and articles referred therein.

An r.i. lattice X is said to be *essentially larger* than an r.i. lattice Y if $Y \subset X$ and for every $k \in \mathbb{N}$ there exists a function $f_k \in X$ such that $k\|f_k\|_X \leqslant \|f_k\|_Y$. In other words, X is essentially larger than Y if $Y \subset X$ and X is not embedded in Y.

The notation $A \lesssim B$ means that for every $p,q \in (0,\infty)$ there exists a constant $C = C(p,q) \in [0,\infty)$ such that $A \leqslant CB$. The constant C hence depends only on the parameters p and q. If both $A \lesssim B$ and $B \lesssim A$, one writes $A \approx B$.

The problem of boundedness of convolution-type operators between various function spaces was studied in a great number of articles, see e.g. [1,3,10,11,12,8,15,14,18] and the references therein. The technique employed in [10], which is also relevant for this paper, was based on using the O’Neil inequality

$$(f*g)^{**}(t) \leqslant tf^{**}(t)g^{**}(t) + \int_t^\infty f^*(s)g^*(s)\,\mathrm{d}s, \qquad f,g \in \mathscr{M}, \quad t > 0,$$

proved in [15], and its reverse version

$$tf^{**}(t)g^{**}(t) + \int_t^\infty f^*(s)g^*(s)\,\mathrm{d}s \leqslant C(n)(f*g)^{**}(t), \qquad t > 0,$$

which holds for all nonnegative radially decreasing functions f,g on $\mathbb{R}^n$ with the constant $C(n)$ depending only on the dimension of $\mathbb{R}^n$. The reverse variant for functions from $\mathbb{R}^n$ was proved e.g. in [9]. The O’Neil inequalities were used in [10] to prove the following lemma.

LEMMA 1.1. *Let X be an r.i. lattice, w be a weight, $g \in \mathscr{M}$ and $q \in (0,\infty]$. For $f \in \mathscr{M}$ and $t > 0$ define*

$$R_g^1 f(t) := tf^{**}(t)g^{**}(t), \quad R_g^2 f(t) := \int_t^\infty f^*(s)g^*(s)\,\mathrm{d}s, \quad R_g f(t) := R_g^1 f(t) + R_g^2 f(t).$$

Then

(i) *If* $R_g : X \to L^q(w)$ *is bounded, then* $T_g : X \to \Gamma^q(w)$ *is bounded and*

$$\|T_g\|_{X\to\Gamma^q(w)} \lesssim \|R_g\|_{X\to L^q(w)} < \infty.$$

(ii) *Let* g *be nonnegative and radially decreasing. If* $T_g : X \to \Gamma^q(w)$ *is bounded, then* $R_g : X \to L^q(w)$ *is bounded and*

$$\|R_g\|_{X\to L^q(w)} \lesssim \|T_g\|_{X\to\Gamma^q(w)} < \infty.$$

(iii) *Suppose there exists an r.i. lattice* Y *such that* $\|R_g\|_{X\to L^q(w)} \approx \|g\|_Y$ *for all* $g \in \mathscr{M}$. *Then* Y *is the essentially largest r.i. lattice such that the inequality*

$$\|f*g\|_{\Gamma^q(w)} \lesssim \|f\|_X \|g\|_Y$$

holds for all $f \in X$ *and* $g \in Y$.

This result was proved as [10, Theorem 3.1] for both ordinary and periodic functions on $\mathbb{R}$ but it holds even for functions on $\mathbb{R}^n$. Further results in here will have the $\mathbb{R}^n$-form but they may be simply modified to cover periodic functions on $\mathbb{R}$. Besides this, the statement of [10, Theorem 3.1] contains the term "r.i. space" in place of the more general "r.i. lattice" used in Lemma 1.1(iii). However, both versions are correct as the space structure is not important to prove the result.

Thanks to Lemma 1.1, the problem of boundedness of T_g between $\Lambda^p(v)$ and $\Gamma^q(w)$ reduces to characterizing boundedness of R_g^1 and R_g^2 between $\Lambda^p(v)$ and $L^q(w)$. The problem for R_g^1 was completely solved for the whole range $p,q \in (0,\infty]$ (see [5,4]). Similar characterizations for R_g^2 were known as well [7], but only for $q \geqslant 1$, at the time of publishing of [10]. Although [7] contains conditions even for $0<q<1$, in this case they have a discrete form which could not be applied. The case $0<q<1$ was therefore missing in [10]. However, recent progress in the required characterizations of Hardy-type inequalities [13] allows for completing the missing cases. Hence, this paper together with [10] cover the $\Lambda^p(v) \to \Gamma^q(w)$ convolution-operator boundedness for the whole range $p,q \in (0,\infty]$.

Before stating and proving the main result, it is useful to state the following technical lemma based on partial integration. Results of this type are well known (cf. [17, p. 176]) and are frequently used whenever weighted Hardy inequalities on monotone functions are studied.

LEMMA 1.2. *Let* φ, $\psi \in \mathscr{M}_+$ *and* φ *be locally integrable. Let* $0<q<p<\infty$ *and* $r = \frac{pq}{p-q}$. *Then*

$$\int_0^\infty \left(\int_t^\infty \varphi(x)\,\mathrm{d}x\right)^{\frac{r}{p}} \varphi(t) \sup_{s\in(0,t)} \psi(s)\,\mathrm{d}t$$
$$\approx \left(\int_0^\infty \varphi(t)\,\mathrm{d}t\right)^{\frac{r}{q}} \limsup_{s\to0+} \psi(s) + \int_0^\infty \left(\int_t^\infty \varphi(x)\,\mathrm{d}x\right)^{\frac{r}{q}} \frac{\mathrm{d}}{\mathrm{d}t}\left(\sup_{s\in(0,t)} \psi(s)\right)\mathrm{d}t.$$

Proof. Integration by parts yields

$$\int_0^\infty \left(\int_t^\infty \varphi(x)\,\mathrm{d}x\right)^{\frac{r}{p}} \varphi(t) \sup_{s\in(0,t)} \psi(s)\,\mathrm{d}t + \frac{q}{r}\lim_{s\to\infty}\left(\int_s^\infty \varphi(t)\,\mathrm{d}t\right)^{\frac{r}{q}} \sup_{x\in(0,s)} \psi(x)$$
$$= \frac{q}{r}\lim_{s\to 0+}\left(\int_s^\infty \varphi(t)\,\mathrm{d}t\right)^{\frac{r}{q}} \sup_{x\in(0,s)} \psi(x) + \frac{q}{r}\int_0^\infty \left(\int_t^\infty \varphi(x)\,\mathrm{d}x\right)^{\frac{r}{q}} \frac{\mathrm{d}}{\mathrm{d}t}\left(\sup_{s\in(0,t)} \psi(s)\right)\mathrm{d}t.$$

By monotonicity, one has

$$\lim_{s\to 0+}\left(\int_s^\infty \varphi(t)\,\mathrm{d}t\right)^{\frac{r}{q}} \sup_{x\in(0,s)} \psi(x) = \left(\int_0^\infty \varphi(t)\,\mathrm{d}t\right)^{\frac{r}{q}} \limsup_{s\to 0+} \psi(s).$$

Furthermore, for every $s>0$ it holds

$$\begin{aligned}\frac{q}{r}\left(\int_s^\infty \varphi(t)\,\mathrm{d}t\right)^{\frac{r}{q}} \sup_{x\in(0,s)} \psi(x) &= \int_s^\infty \left(\int_t^\infty \varphi(y)\,\mathrm{d}y\right)^{\frac{r}{p}} \varphi(t)\,\mathrm{d}t \sup_{x\in(0,s)} \psi(x)\\ &\leqslant \int_s^\infty \left(\int_t^\infty \varphi(y)\,\mathrm{d}y\right)^{\frac{r}{p}} \varphi(t) \sup_{x\in(0,t)} \psi(x)\,\mathrm{d}t,\end{aligned}$$

hence

$$\frac{q}{r}\lim_{s\to\infty}\left(\int_s^\infty \varphi(t)\,\mathrm{d}t\right)^{\frac{r}{q}} \sup_{x\in(0,s)} \psi(x) \leqslant \int_s^\infty \left(\int_t^\infty \varphi(y)\,\mathrm{d}y\right)^{\frac{r}{p}} \varphi(t) \sup_{x\in(0,t)} \psi(x)\,\mathrm{d}t.$$

Combining all these observations gives the result. □

2. Results

In all what follows, the convention $0.\infty := 0$ is strictly enforced. For example, any expression of the form $CV^{-\frac{1}{p}}(\infty)$ is equal to zero whenever $V(\infty)=\infty$, even if $C=\infty$.

The theorem below is formulated for convolution of functions from $\mathbb{R}^n$. It might be easily modified to the case of periodic functions on $\mathbb{R}$ in spirit of [10].

THEOREM 2.1. *Let v, w be weights.*
(i) *Let $0<q<p<1$ and $r=\frac{pq}{p-q}$. For any $g\in\mathscr{M}$ define*

$$A_1(g) := \left(\int_0^\infty W^{\frac{r}{p}}(t)w(t) \sup_{s\in(t,\infty)} (g^{**}(s))^r s^r V^{-\frac{r}{p}}(s)\,\mathrm{d}t\right)^{\frac{1}{r}},$$

$$A_2(g) := \left(\int_0^\infty \left(\int_t^\infty (g^{**}(x))^q w(x)\,\mathrm{d}x\right)^{\frac{r}{q}} \frac{\mathrm{d}}{\mathrm{d}t}\left(\sup_{s\in(0,t)} s^r V^{-\frac{r}{p}}(s)\right)\mathrm{d}t\right)^{\frac{1}{r}}$$

$$A_3(g) := \left(\int_0^\infty (g^{**}(t))^q w(t)\,\mathrm{d}t\right)^{\frac{1}{q}} \limsup_{s\to 0+} sV^{-\frac{1}{p}}(s).$$

and

$$\|g\|_Y := A_1(g)+A_2(g)+A_3(g).$$

Then $(Y,\|g\|_Y)$ *is the essentially largest r.i. lattice such that the inequality*

$$\|f*g\|_{\Gamma^q(w)} \lesssim \|f\|_{\Lambda^p(v)}\|g\|_Y \tag{1}$$

holds for all $f\in\Lambda^p(v)$ *and* $g\in Y$. *Moreover, if* g *is nonnegative and radially decreasing, then*

$$\sup_{f\in\Lambda^p(v)} \frac{\|f*g\|_{\Gamma^q(w)}}{\|f\|_{\Lambda^p(v)}} \approx \|g\|_Y. \tag{2}$$

(ii) *Let* $0<q<1<p<\infty$ *and* $r=\frac{pq}{p-q}$. *For any* $g\in\mathscr{M}$ *define*

$$A_4(g) := \left(\int_0^\infty W^{\frac{r}{p}}(t)w(t)\left(\int_t^\infty (g^{**}(s))^{p'}s^{p'}V^{-p'}(s)v(s)\,\mathrm{d}s\right)^{\frac{r}{p'}}\mathrm{d}t\right)^{\frac{1}{r}},$$

$$A_5(g) := \left(\int_0^\infty \left(\int_t^\infty (g^{**}(x))^q w(x)\,\mathrm{d}x\right)^{\frac{r}{q}}\left(\int_0^t V^{-p'}(s)v(s)s^{p'}\,\mathrm{d}s\right)^{\frac{r}{q'}}V^{-p'}(t)v(t)t^{p'}\,\mathrm{d}t\right)^{\frac{1}{r}},$$

$$A_6(g) := \left(\int_0^\infty (g^{**}(t))^q w(t)\,\mathrm{d}t\right)^{\frac{1}{q}}\lim_{s\to 0+}\left(\int_0^s V^{-p'}(x)v(x)x^{p'}\,\mathrm{d}x\right)^{\frac{1}{p'}},$$

and

$$\|g\|_Y := A_4(g)+A_5(g)+A_6(g)+\|g\|_1 W^{\frac{1}{q}}(\infty)V^{-\frac{1}{p}}(\infty).$$

Then $(Y,\|g\|_Y)$ *is the essentially largest r.i. lattice such that* (1) *holds for all* $f\in\Lambda^p(v)$ *and* $g\in Y$. *Moreover, if* g *is nonnegative and radially decreasing, then* (2) *is satisfied.*

Proof. For the definitions of the operators R_g^1 and R_g^2 see Lemma 1.2.

(i) Fix $g\in\mathscr{M}$. By [4, Theorem 3.1] one gets $\|R_g^1\|_{\Lambda^p(v)\to L^q(w)}\approx B_1+B_2$, where

$$B_1 := \left(\int_0^\infty \left(\int_0^t \left(\int_0^s g^*(x)\,\mathrm{d}x\right)^q w(s)\,\mathrm{d}s\right)^{\frac{r}{p}}\left(\int_0^t g^*(y)\,\mathrm{d}y\right)^q w(t)V^{-\frac{r}{p}}(t)\,\mathrm{d}t\right)^{\frac{1}{r}},$$

$$B_2 := \left(\int_0^\infty \sup_{s\in(0,t)} s^r V^{-\frac{r}{p}}(s)\left(\int_t^\infty (g^{**}(x))^q w(x)\,\mathrm{d}x\right)^{\frac{r}{p}}(g^{**}(t))^q w(t)\,\mathrm{d}t\right)^{\frac{1}{r}}.$$

Next, [13, Theorem 13(i)] gives $\|R_g^2\|_{\Lambda^p(v)\to L^q(w)}\approx B_3+B_4$, where

$$B_3 := \left(\int_0^\infty W^{\frac{r}{p}}(t)w(t)\sup_{s\in(t,\infty)}\left(\int_t^s g^*(x)\,\mathrm{d}x\right)^r V^{-\frac{r}{p}}(s)\,\mathrm{d}t\right)^{\frac{1}{r}},$$

$$B_4 := \left(\int_0^\infty \left(\int_0^t w(x) \left(\int_x^t g^*(y)\,\mathrm{d}y \right)^q \mathrm{d}x \right)^{\frac{r}{p}} w(t) \sup_{s\in(t,\infty)} \left(\int_t^s g^*(x)\,\mathrm{d}x \right)^q V^{-\frac{r}{p}}(s)\,\mathrm{d}t \right)^{\frac{1}{r}}.$$

In view of Lemma 1.1, it suffices to prove that

$$B_1 + B_2 + B_3 + B_4 \approx A_1(g) + A_2(g) + A_3(g). \tag{3}$$

Lemma 1.2 implies that $B_2 \approx A_2(g) + A_3(g)$. Next, it is easy to see that $B_1 + B_3 + B_4 \lesssim A_1(g)$. Hence, the "$\lesssim$" inequality in (3) is verified.

The following part is aimed at proving the opposite estimate. Observe that $A_1(g) \approx B_3 + B_5$, where

$$B_5 := \left(\int_0^\infty W^{\frac{r}{p}}(t) w(t) \left(\int_0^t g^*(x)\,\mathrm{d}x \right)^r V^{-\frac{r}{p}}(t)\,\mathrm{d}t \right)^{\frac{1}{r}}.$$

Assume that $W(\infty) = \infty$. There exists a (not necessarily unique) sequence $\{t_k\}_{k\in\mathbb{Z}}$ such that for all $k \in \mathbb{Z}$ it holds

$$2^k = \int_0^{t_k} w(x)\,\mathrm{d}x = \int_{t_k}^{t_{k+1}} w(x)\,\mathrm{d}x. \tag{4}$$

One gets

$$\begin{aligned}
B_5^r &= \int_0^\infty W^{\frac{r}{p}}(t) w(t) \left(\int_0^t g^*(x)\,\mathrm{d}x \right)^r V^{-\frac{r}{p}}(t)\,\mathrm{d}t \\
&= \sum_{k\in\mathbb{Z}} \int_{t_k}^{t_{k+1}} W^{\frac{r}{p}}(t) w(t) \left(\int_0^t g^*(x)\,\mathrm{d}x \right)^r V^{-\frac{r}{p}}(t)\,\mathrm{d}t \\
&\leqslant \sum_{k\in\mathbb{Z}} \int_{t_k}^{t_{k+1}} W^{\frac{r}{p}}(t) w(t)\,\mathrm{d}t \sup_{t\in[t_k,t_{k+1}]} \left(\int_0^t g^*(x)\,\mathrm{d}x \right)^r V^{-\frac{r}{p}}(t) \\
&\lesssim \sum_{k\in\mathbb{Z}} 2^{\frac{kr}{q}} \sup_{t\in[t_k,t_{k+1}]} \left(\int_0^t g^*(x)\,\mathrm{d}x \right)^r V^{-\frac{r}{p}}(t) \qquad (5) \\
&\lesssim \sum_{k\in\mathbb{Z}} 2^{\frac{kr}{q}} \left(\int_0^{t_{k-1}} g^*(x)\,\mathrm{d}x \right)^r V^{-\frac{r}{p}}(t_k) + \sum_{k\in\mathbb{Z}} 2^{\frac{kr}{q}} \sup_{t\in[t_k,t_{k+1}]} \left(\int_{t_{k-1}}^t g^*(x)\,\mathrm{d}x \right)^r V^{-\frac{r}{p}}(t) \\
&=: B_6^r + B_7^r.
\end{aligned}$$

Inequality (5) follows from (4). The estimate then continues as follows.

$$\begin{aligned}
B_6^r &= \sum_{k\in\mathbb{Z}} 2^{\frac{kr}{q}} \left(\int_0^{t_{k-1}} g^*(x)\,\mathrm{d}x \right)^r V^{-\frac{r}{p}}(t_k) \\
&\lesssim \sum_{k\in\mathbb{Z}} \int_{t_{k-1}}^{t_k} \left(\int_{t_{k-1}}^t w(s)\,\mathrm{d}s \right)^{\frac{r}{p}} w(t)\,\mathrm{d}t \left(\int_0^{t_{k-1}} g^*(x)\,\mathrm{d}x \right)^r V^{-\frac{r}{p}}(t_k) \qquad (6) \\
&= \sum_{k\in\mathbb{Z}} \int_{t_{k-1}}^{t_k} \left(\int_{t_{k-1}}^t w(s)\,\mathrm{d}s \right)^{\frac{r}{p}} w(t)\,\mathrm{d}t \left(\int_0^{t_{k-1}} g^*(x)\,\mathrm{d}x \right)^{\frac{rq}{p}} \left(\int_0^{t_{k-1}} g^*(y)\,\mathrm{d}y \right)^q V^{-\frac{r}{p}}(t_k)
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \sum_{k\in\mathbb{Z}} \int_{t_{k-1}}^{t_k} \left(\int_{t_{k-1}}^{t} \left(\int_0^s g^*(x)\,\mathrm{d}x \right)^q w(s)\,\mathrm{d}s \right)^{\frac{r}{p}} \left(\int_0^t g^*(y)\,\mathrm{d}y \right)^q w(t)\,\mathrm{d}t\, V^{-\frac{r}{p}}(t_k) \\
&\leqslant \sum_{k\in\mathbb{Z}} \int_{t_{k-1}}^{t_k} \left(\int_0^{t} \left(\int_0^s g^*(x)\,\mathrm{d}x \right)^q w(s)\,\mathrm{d}s \right)^{\frac{r}{p}} \left(\int_0^t g^*(y)\,\mathrm{d}y \right)^q w(t) V^{-\frac{r}{p}}(t)\,\mathrm{d}t \\
&= B_1^r.
\end{aligned}
$$

In the step (6) one uses (4). For each $k\in\mathbb{Z}$ there exists $z_k\in[t_k,t_{k+1}]$ such that

$$
\left(\int_{t_{k-1}}^{z_k} g^*(x)\,\mathrm{d}x \right)^r V^{-\frac{r}{p}}(z_k) = \sup_{t\in[t_k,t_{k+1}]} \left(\int_{t_{k-1}}^{t} g^*(x)\,\mathrm{d}x \right)^r V^{-\frac{r}{p}}(t), \tag{7}
$$

since the argument of the supremum is a continuous function. The term B_7 is then estimated by

$$
\begin{aligned}
B_7^r &= \sum_{k\in\mathbb{Z}} 2^{\frac{kr}{q}} \sup_{t\in[t_k,t_{k+1}]} \left(\int_{t_{k-1}}^{t} g^*(x)\,\mathrm{d}x \right)^r V^{-\frac{r}{p}}(t) \\
&= \sum_{k\in\mathbb{Z}} 2^{\frac{kr}{q}} \left(\int_{t_{k-1}}^{z_k} g^*(x)\,\mathrm{d}x \right)^r V^{-\frac{r}{p}}(z_k) \qquad (8) \\
&\lesssim \sum_{k\in\mathbb{Z}} \int_{t_{k-2}}^{t_{k-1}} W^{\frac{r}{p}}(t) w(t)\,\mathrm{d}t \left(\int_{t_{k-1}}^{z_k} g^*(x)\,\mathrm{d}x \right)^r V^{-\frac{r}{p}}(z_k) \qquad (9) \\
&\leqslant \sum_{k\in\mathbb{Z}} \int_{t_{k-2}}^{t_{k-1}} W^{\frac{r}{p}}(t) w(t) \sup_{s\in(t,\infty)} \left(\int_t^s g^*(x)\,\mathrm{d}x \right)^r V^{-\frac{r}{p}}(t)\,\mathrm{d}t \\
&= B_3^r.
\end{aligned}
$$

Relation (7) implies (8), and (9) follows from (4). The obtained estimates yield the equivalence

$$
A_1(g) \approx B_3 + B_5 \lesssim B_3 + B_6 + B_7 \lesssim B_1 + B_3,
$$

which together with the known relation $B_2 \approx A_2(g) + A_3(g)$ gives the "$\gtrsim$" inequality in (3). Hence, (3) is proved. If $W(\infty) < \infty$, the proof is carried out analogously with appropriate minor modifications. Part (i) is now complete.

(ii) Fix $g\in\mathscr{M}$. By [5, Theorem 4.1(ii)] it holds $\|R_g^1\|_{\Lambda^p(v)\to L^q(w)} \approx B_1 + B_8$, where

$$
B_8 := \left(\int_0^\infty \left(\int_0^t V^{-p'}(s) v(s) s^{p'}\,\mathrm{d}s \right)^{\frac{r}{p'}} \left(\int_t^\infty (g^{**}(x))^q w(x)\,\mathrm{d}x \right)^{\frac{r}{p}} (g^{**}(t))^q w(t)\,\mathrm{d}t \right)^{\frac{1}{r}}.
$$

Furthermore, from [13, Theorem 13(ii)] it follows that $\|R_g^2\|_{\Lambda^p(v)\to L^q(w)} \approx B_4 + B_9 + B_{10}$, where

$$
B_9 := \left(\int_0^\infty W^{\frac{r}{p}}(t) w(t) \left(\int_t^\infty \left(\int_t^s g^*(x)\,\mathrm{d}x \right)^{p'} V^{-p'}(s) v(s)\,\mathrm{d}s \right)^{\frac{r}{p'}} \mathrm{d}t \right)^{\frac{1}{r}},
$$

$$B_{10} := \left(\int_0^\infty \left(\int_t^\infty g^*(x)\,\mathrm{d}x\right)^q w(t)\,\mathrm{d}t\right)^{\frac{1}{q}} V^{-\frac{1}{p}}(\infty).$$

By Lemma 1.1, the proof will be complete once the equivalence

$$B_1 + B_4 + B_8 + B_9 + B_{10} \approx A_4(g) + A_5(g) + A_6(g) + \|g\|_1 W^{\frac{1}{q}}(\infty) V^{-\frac{1}{p}}(\infty) \tag{10}$$

is established. Lemma 1.2 gives $A_5(g) + A_6(g) \approx B_8$. Next, it holds

$$\begin{aligned}
B_1^r + B_4^r &\lesssim \int_0^\infty W^{\frac{r}{p}}(t) w(t) \sup_{s\in(t,\infty)} \left(\int_0^s g^*(x)\,\mathrm{d}x\right)^r V^{-\frac{r}{p}}(s)\,\mathrm{d}t \\
&\lesssim \int_0^\infty W^{\frac{r}{p}}(t) w(t) \sup_{s\in(t,\infty)} \left(\int_0^s g^*(x)\,\mathrm{d}x\right)^r \left(\int_s^\infty V^{-p'}(y) v(y) y^{p'}\,\mathrm{d}y\right)^{\frac{r}{p'}} \mathrm{d}t \\
&\quad + \int_0^\infty W^{\frac{r}{p}}(t) w(t)\,\mathrm{d}t \left(\int_0^\infty g^*(x)\,\mathrm{d}x\right)^r V^{-\frac{r}{p}}(\infty) \\
&\lesssim A_4^r(g) + \|g\|_1^r W^{\frac{r}{q}}(\infty) V^{-\frac{r}{p}}(\infty).
\end{aligned}$$

Obviously, the inequalities $B_9 \leqslant A_4(g)$ and $B_{10} \leqslant \|g\|_1 W^{\frac{1}{q}}(\infty) V^{-\frac{1}{p}}(\infty)$ are also valid. This proves the "$\lesssim$" inequality in (10).

To prove the converse part of (10), the same approach as in (i) is used. Suppose that $W(\infty) = \infty$ and let $\{t_k\}_{k\in\mathbb{Z}}$ be a sequence of points such that (4) hold in each of them. Then it holds

$$\begin{aligned}
A_4^r(g) &:= \int_0^\infty W^{\frac{r}{p}}(t) w(t) \left(\int_t^\infty \left(\int_0^s g^*(x)\,\mathrm{d}x\right)^{p'} V^{-p'}(s) v(s)\,\mathrm{d}s\right)^{\frac{r}{p'}} \mathrm{d}t \\
&\leqslant \sum_{k\in\mathbb{Z}} \int_{t_k}^{t_{k+1}} W^{\frac{r}{p}}(t) w(t)\,\mathrm{d}t \left(\int_{t_k}^\infty \left(\int_0^s g^*(x)\,\mathrm{d}x\right)^{p'} V^{-p'}(s) v(s)\,\mathrm{d}s\right)^{\frac{r}{p'}} \\
&\lesssim \sum_{k\in\mathbb{Z}} 2^{\frac{kr}{q}} \left(\int_{t_k}^\infty \left(\int_0^s g^*(x)\,\mathrm{d}x\right)^{p'} V^{-p'}(s) v(s)\,\mathrm{d}s\right)^{\frac{r}{p'}} \\
&\lesssim \sum_{k\in\mathbb{Z}} 2^{\frac{kr}{q}} \left(\int_0^{t_{k-1}} g^*(x)\,\mathrm{d}x\right)^r \left(\int_{t_k}^\infty V^{-p'}(s) v(s)\,\mathrm{d}s\right)^{\frac{r}{p'}} \\
&\quad + \sum_{k\in\mathbb{Z}} 2^{\frac{kr}{q}} \left(\int_{t_k}^\infty \left(\int_{t_{k-1}}^s g^*(x)\,\mathrm{d}x\right)^{p'} V^{-p'}(s) v(s)\,\mathrm{d}s\right)^{\frac{r}{p'}} \\
&\lesssim B_6^r + \sum_{k\in\mathbb{Z}} 2^{\frac{kr}{q}} \left(\int_{t_{k-1}}^\infty \left(\int_{t_{k-1}}^s g^*(x)\,\mathrm{d}x\right)^{p'} V^{-p'}(s) v(s)\,\mathrm{d}s\right)^{\frac{r}{p'}}
\end{aligned} \tag{11}$$

$$\lesssim B_6^r + \sum_{k\in\mathbb{Z}} \int_{t_{k-2}}^{t_{k-1}} W^{\frac{r}{p}}(t)w(t)\,\mathrm{d}t \left(\int_{t_{k-1}}^{\infty} \left(\int_{t_{k-1}}^{s} g^*(x)\,\mathrm{d}x \right)^{p'} V^{-p'}(s)v(s)\,\mathrm{d}s \right)^{\frac{r}{p'}} \tag{12}$$
$$\lesssim B_6^r + B_9^r.$$

Both the steps (11) and (12) are based on (4). Moreover, in part (i) it was proved that $B_6 \lesssim B_1$ and this estimate holds even this case, i.e. for $p>1$. Hence, one obtains $A_4(g) \lesssim B_6 + B_9 \lesssim B_1 + B_9$. Next, it holds

$$\begin{aligned}
&\|g\|_1 W^{\frac{1}{q}}(\infty)V^{-\frac{1}{p}}(\infty)\\
&\approx \left(\int_0^{\infty} \left(\int_0^{t} g^*(x)\,\mathrm{d}x \right)^{q} w(t)\,\mathrm{d}t \right)^{\frac{1}{q}} V^{-\frac{1}{p}}(\infty) + B_{10}\\
&\approx \left(\int_0^{\infty} \left(\int_0^{t} \left(\int_0^{s} g^*(x)\,\mathrm{d}x \right)^{q} w(s)\,\mathrm{d}s \right)^{\frac{r}{p}} \left(\int_0^{t} g^*(x)\,\mathrm{d}x \right)^{q} w(t)\,\mathrm{d}t \right)^{\frac{1}{r}} V^{-\frac{1}{p}}(\infty) + B_{10}\\
&\leqslant B_1 + B_{10}.
\end{aligned}$$

The relation $A_5(g)+A_6(g) \approx B_8$ was mentioned earlier. The obtained estimates of $A_4(g)$, $A_5(g)$, $A_6(g)$ and $\|g\|_1 W^{\frac{1}{q}}(\infty)V^{-\frac{1}{p}}(\infty)$ together yield the "$\gtrsim$" inequality in (10). Hence, (10) is proved and so is the whole theorem. □

REMARK 2.2. (i) In both cases of Theorem 2.1, the functional $\|\cdot\|_Y$ is equivalent to an r.i. quasi-norm. Indeed, each of the expressions $A_i(g)$, $i=1,\dots,6$ itself is an r.i. quasi-norm. Some of the properties of the r.i. quasi-spaces generated by such quasi-norms are described in [10].

(ii) The "space" $\Lambda^p(v)$ may admit functions which are not locally integrable. Namely, it holds (see e.g. [10, Remark 3.4]) that $\Lambda^p(v) \subset L^1_{\mathrm{loc}}$ if and only if

(a) $\limsup_{s\to 0+} sV^{-\frac{1}{p}}(s) < \infty$ in the case $0<p\leqslant 1$,

(b) there exists $\varepsilon>0$ such that $\int_0^{\varepsilon} V^{-p'}(s)v(s)s^{p'}\,\mathrm{d}s < \infty$ in the case $1<p<\infty$.

If $\Lambda^p(v)$ contains any $f \notin L^1_{\mathrm{loc}}$, then the operator T_g cannot be bounded between $\Lambda^p(v)$ and $\Gamma^q(w)$ unless $g=0$ a.e. This is reflected by the presence of the conditions $A_3(g)$ and $A_6(g)$ in the respective expressions $\|g\|_Y$ for $0<p\leqslant 1$ and $1<p$. If (a) is not satisfied, then $A_3(g)=\infty$ unless $g=0$ a.e. An analogy holds for (b) and $A_6(g)$. Moreover, the term $\lim_{s\to 0+} \left(\int_0^s V^{-p'}(x)v(x)x^{p'}\,\mathrm{d}x \right)^{\frac{1}{p'}}$ can attain only the values 0 or ∞ and thus so does $A_6(g)$. Hence, the term $A_6(g)$ is not present if $\Lambda^p(v) \subset L^1_{\mathrm{loc}}$.

(iii) If $V(\infty)<\infty$, the constant function $f\equiv 1$ belongs to $\Lambda^p(v)$. This f and any $g \in \mathscr{M}$ satisfy $T_g f \equiv \|g\|_1$. Hence, for T_g to be bounded between $\Lambda^p(v)$ and $\Gamma^q(w)$ it is necessary that $g \in L^1$ and $W(\infty)<\infty$. This corresponds to the fact that

$$\|g\|_1 W^{\frac{1}{q}}(\infty)V^{-\frac{1}{p}}(\infty) \lesssim \|g\|_Y$$

in both cases (i) and (ii) of Theorem 2.1. This inequality is obvious in case (ii). In (i), it follows from the estimate

$$\|g\|_1 W^{\frac{1}{q}}(\infty)V^{-\frac{1}{p}}(\infty) \approx \left(\int_0^\infty W^{\frac{r}{p}}(t)w(t)\,\mathrm{d}t \left(\int_0^\infty g^*(x)\,\mathrm{d}x\right)^r V^{-\frac{r}{p}}(\infty)\right)^{\frac{1}{r}} \leqslant A_1(g).$$

(iv) In view of the previous remark, the expressions of $\|g\|_Y$ in [10, Theorem 3.2] should be slightly corrected. Namely, in cases (iii) and (iv) thereof, the expression $\left(\int_0^m x^q (g^{**}(x))^q w(x)\,\mathrm{d}x\right)^{\frac{1}{q}} V^{-\frac{1}{p}}(m)$ should be replaced by $\|g\|_1 W^{\frac{1}{q}}(m) V^{-\frac{1}{p}}(m)$. This mistake in [10] seems to be caused by using [7, Theorem 5.1], which assumes $V(\infty) = \infty$, in the proof. Using [7, Theorem 2.1] instead would lead to the correct appearance of the term $\left(\int_0^m \left(\int_x^m g^*(y)\,\mathrm{d}y\right)^q w(x)\,\mathrm{d}x\right)^{\frac{1}{q}} V^{-\frac{1}{p}}(m)$ in the affected formulas. This term is not covered by other parts of $\|g\|_Y$ in cases (iii) and (iv) of [10, Theorem 3.2], unlike the cases (i) and (ii) thereof, which are correct.

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Martin Křepela
Karlstad University
Faculty of Health, Science and Technology
Department of Mathematics and Computer Science
651 88 Karlstad, Sweden
and
Charles University, Faculty of Mathematics and Physics
Department of Mathematical Analysis
Sokolovská 83, 186 75 Praha 8, Czech Republic
e-mail: martin.krepela@kau.se