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REMARKS ON SOME NORM INEQUALITIES FOR POSITIVE SEMIDEFINITE MATRICES AND QUESTIONS OF BOURIN

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Abstract. Let A and B be positive semidefinite matrices. It is shown that

$$\left\|A^sB^p+B^qA^t\right\|_2 \leqslant \left\|A^sB^p+A^tB^q\right\|_2$$

for all positive real numbers s,t,p,q for which

$$\left|\frac{s}{s+t}-\frac{1}{2}\right|+\left|\frac{p}{p+q}-\frac{1}{2}\right| \leqslant \frac{1}{2}.$$

This is a generalization of a recent inequality proved by Bhatia for the special case $s=q$, $t=p$ with

$$\frac{1}{4} \leqslant \frac{p}{p+q} \leqslant \frac{3}{4},$$

and it is a special case of a conjecture posed by Hayajneh and Kittaneh, which claims that for positive semidefinite matrices A_1,A_2,B_1,B_2 with $A_1B_1=B_1A_1$, $A_2B_2=B_2A_2$ and any unitarily invariant norm,

$$|||A_1B_2+A_2B_1||| \leqslant |||A_1B_2+B_1A_2|||.$$

For $i=1,\ldots,k$, let A_i and B_i be positive semidefinite matrices such that, for each i, A_i commutes with B_i. It is shown that for any unitarily invariant norm,

$$\left|\left|\left|\sum_{i=1}^{k} A_iB_i\right|\right|\right| \leqslant \left|\left|\left|\left(\sum_{i=1}^{k} A_i\right)^{\frac{1}{2}}\left(\sum_{i=1}^{k} B_i\right)\left(\sum_{i=1}^{k} A_i\right)^{\frac{1}{2}}\right|\right|\right|.$$

This is stronger than the inequality

$$\left|\left|\left|\sum_{i=1}^{k} A_iB_i\right|\right|\right| \leqslant \left|\left|\left|\left(\sum_{i=1}^{k} A_i\right)\left(\sum_{i=1}^{k} B_i\right)\right|\right|\right|,$$

which has been recently proved by Audenaert. Simple applications of these norm inequalities answer some questions of Bourin affirmatively.

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1. Introduction

In a recent paper [9], and in their investigations of the Lieb-Thirring trace inequalities, Hayajneh and Kittaneh proposed the following conjecture for commuting positive semidefinite matrices.

CONJECTURE 1.1. Let A_1,A_2,B_1,B_2 be positive semidefinite matrices with $A_1B_1 = B_1A_1$ and $A_2B_2 = B_2A_2$. Then for every unitarily invariant norm,

$$|||A_1B_2 + A_2B_1||| \leqslant |||A_1B_2 + B_1A_2|||. \tag{1}$$

An important special case of the inequality (1) is the inequality

$$\left|\left|\left|A^sB^p + B^qA^t\right|\right|\right| \leqslant \left|\left|\left|A^sB^p + A^tB^q\right|\right|\right|, \tag{2}$$

where A and B are positive semidefinite matrices and s,t,p,q are positive real numbers.

In this paper, we will investigate the Hilbert-Schmidt norm version of (2), i.e., the inequality

$$\left\|A^sB^p + B^qA^t\right\|_2 \leqslant \left\|A^sB^p + A^tB^q\right\|_2. \tag{3}$$

Section 2 is devoted to proving the inequality (3) under the condition that

$$\left|\frac{s}{s+t} - \frac{1}{2}\right| + \left|\frac{p}{p+q} - \frac{1}{2}\right| \leqslant \frac{1}{2}. \tag{4}$$

A special case of the inequality (3) when $s = q$, $t = p$ is the inequality

$$\|A^qB^p + B^qA^p\|_2 \leqslant \|A^qB^p + A^pB^q\|_2. \tag{5}$$

Recently, Bhatia [4] proved the inequality (5) under the condition that

$$\frac{1}{4} \leqslant \frac{p}{p+q} \leqslant \frac{3}{4}, \tag{6}$$

which is a significant improvement on a recent result of Hayajneh and Kittaneh [9], where they proved it for $q = 1,2$ or 3 and $p = 1$.

In our first main result (Theorem 1) in Section 2, we prove that the inequality (3) is true under the condition given in (4). This is a generalization of the result of Bhatia given in [4], where the particular inequality (5) is proved under the condition given in (6).

Another special case of (3) when $s = t = 1$ is the inequality

$$\|AB^p + B^qA\|_2 \leqslant \|AB^p + AB^q\|_2. \tag{7}$$

It has been pointed out to the authors by J. C. Bourin that the inequality (7) can also be concluded from Theorem 2.2 in [7]. In [9], Hayajneh and Kittaneh proved (7) using some number theory tools, and the proof goes in an algorithmic way.

REMARK 1. By taking $p=1$, $q=3$, $s=3$, and $t=2$ in the inequality (3), we have the following special case in view of the condition given in (4):

$$\left\|A^3B+B^3A^2\right\|_2 \leqslant \left\|A^3B+A^2B^3\right\|_2,$$

which does not follow from the inequalities (5) and (7). This demonstrates the power of the four parameters inequality (3) under the condition given in (4).

Bottazzi et al. [6] gave a counterexample to the following special case of the inequality (1):

$$|||A^qB^p+B^qA^p||| \leqslant |||A^qB^p+A^pB^q|||. \tag{8}$$

They answered it in the negative for just the spectral (or the operator) norm by exhibiting a pair of positive semidefinite matrices such that the claim does not hold. Inspite of the failure of the inequality (8) for the spectral norm, in view of the inequality (5), which is valid under the condition (6), it would be interesting to investigate the inequality (8) for other unitarily invariant norms like the Schatten norms.

In his investigation of matrix subadditivity inequalities, Bourin [8] asked the following question.

QUESTION 1.4. Given positive semidefinite matrices A,B and positive real numbers p,q, is it true that

$$\left|\left|\left|A^{p+q}+B^{p+q}\right|\right|\right| \leqslant |||(A^p+B^p)(A^q+B^q)|||? \tag{9}$$

Bourin also wondered whether the stronger inequality

$$\left|\left|\left|A^{p+q}+B^{p+q}\right|\right|\right| \leqslant \left|\left|\left|(A^p+B^p)^{\frac{1}{2}}(A^q+B^q)(A^p+B^p)^{\frac{1}{2}}\right|\right|\right| \tag{10}$$

holds true.

Recall that if X and Y are matrices such that XY is Hermitian, then for every unitarily invariant norm,

$$|||XY||| \leqslant |||\mathrm{Re}\, YX||| \leqslant |||YX|||. \tag{11}$$

(See [12].)

In [10], Hayajneh and Kittaneh conjectured that for all positive semidefinite matrices A_1,A_2,B_1,B_2 with $A_1B_1=B_1A_1$ and $A_2B_2=B_2A_2$, we have the inequality

$$|||A_1B_1+A_2B_2||| \leqslant |||(A_1+A_2)(B_1+B_2)|||, \tag{12}$$

which is more general than the inequality (9). They also wondered whether the stronger inequality

$$|||A_1B_1+A_2B_2||| \leqslant \left|\left|\left|(A_1+A_2)^{\frac{1}{2}}(B_1+B_2)(A_1+A_2)^{\frac{1}{2}}\right|\right|\right| \tag{13}$$

holds true. In fact, they proved (12) and (13) for the trace norm and the Hilbert-Schmidt norm.

In [2], Audenaert proved the following more general version of the inequality (12) for all unitarily invariant norms:

$$\left|\left|\left|\sum_{i=1}^{k} A_iB_i\right|\right|\right| \leqslant \left|\left|\left|\left(\sum_{i=1}^{k} A_i\right)\left(\sum_{i=1}^{k} B_i\right)\right|\right|\right|, \tag{14}$$

where A_i, B_i are positive semidefinite matrices such that, for each i, A_i commutes with B_i. See [11] and [13] for different proofs of the inequality (14). Section 3 is devoted to proving the stronger inequality

$$\left|\left|\left|\sum_{i=1}^{k} A_iB_i\right|\right|\right| \leqslant \left|\left|\left|\left(\sum_{i=1}^{k} A_i\right)^{\frac{1}{2}}\left(\sum_{i=1}^{k} B_i\right)\left(\sum_{i=1}^{k} A_i\right)^{\frac{1}{2}}\right|\right|\right|. \tag{15}$$

As a corollary of our second main result (Theorem 2), we obtain further refinements of the inequality (14), which answers the question of Bourin given in the inequality (10) affirmatively.

2. The first main result

We start with the following lemma, which will be used in proving our first main result. This lemma has been proved by Ando, Hiai and Okubo in [1], and it has played a crucial role in the proof of the main result in [4].

LEMMA 1. *Let A, B be positive semidefinite matrices, and let $\mu, \nu \in [0,1]$ such that*

$$\left|\mu - \frac{1}{2}\right| + \left|\nu - \frac{1}{2}\right| \leqslant \frac{1}{2}.$$

Then

$$\left|\operatorname{tr} A^{\mu}B^{\nu}A^{1-\mu}B^{1-\nu}\right| \leqslant \operatorname{tr} AB.$$

Our main result in this section can be stated as follows.

THEOREM 1. *Let A, B be positive semidefinite matrices, and let s, t, p, q be positive real numbers such that*

$$\left|\frac{s}{s+t} - \frac{1}{2}\right| + \left|\frac{p}{p+q} - \frac{1}{2}\right| \leqslant \frac{1}{2}.$$

Then

$$\left\|A^sB^p + B^qA^t\right\|_2 \leqslant \left\|A^sB^p + A^tB^q\right\|_2. \tag{16}$$

Proof. We have

$$\left\|A^sB^p+B^qA^t\right\|_2^2=\mathrm{tr}\left(A^{2s}B^{2p}+A^sB^qA^tB^p+A^tB^qA^sB^p+A^{2t}B^{2q}\right)$$

and

$$\left\|A^sB^p+A^tB^q\right\|_2^2=\mathrm{tr}\left(A^{2s}B^{2p}+2A^{s+t}B^{p+q}+A^{2t}B^{2q}\right).$$

Here we have used the fact that for all matrices X,Y, $\|X\|_2=(\mathrm{tr}\ X^*X)^{\frac{1}{2}}$ and the cyclicity of the trace, i.e., $\mathrm{tr}\ XY=\mathrm{tr}\ YX$.

Therefore, the inequality (16) is equivalent to the statement

$$\mathrm{Re}\ \mathrm{tr}\ A^sB^pA^tB^q\leqslant\mathrm{tr}\ A^{s+t}B^{p+q}. \tag{17}$$

Replacing A and B by $A^{\frac{1}{s+t}}$ and $B^{\frac{1}{p+q}}$, we see that this is equivalent to saying

$$\mathrm{Re}\ \mathrm{tr}\ A^{\mu}B^{\nu}A^{1-\mu}B^{1-\nu}\leqslant\mathrm{tr}\ AB \tag{18}$$

for $\mu,\nu\in[0,1]$.

By Lemma 1, the inequality (18) holds provided

$$\left|\mu-\frac{1}{2}\right|+\left|\nu-\frac{1}{2}\right|\leqslant\frac{1}{2}.$$

Replacing μ and ν by $\frac{s}{s+t}$ and $\frac{p}{p+q}$ in (18), we see that inequality (17) is valid if

$$\left|\frac{s}{s+t}-\frac{1}{2}\right|+\left|\frac{p}{p+q}-\frac{1}{2}\right|\leqslant\frac{1}{2}.$$

This completes the proof. □

3. The second main result

In this section, we denote the vectors of eigenvalues and singular values of a matrix A by $\lambda(A)$ and $\sigma(A)$, respectively, where they are obtained by arranging singular values and eigenvalues as well whenever they are real, in a non-increasing order. In general, for $x=(x_1,\dots,x_n)\in\mathbb{R}^n$, we will write $x^{\downarrow}$ for the vector obtained by rearranging the coordinates of x in a non-increasing order.

Let $x,y\in\mathbb{R}^n$. We say that x is weakly majorized by y, denoted $x\prec_w y$, if and only if for $k=1,\dots,n$, $\sum_{i=1}^k x_i^{\downarrow}\leqslant\sum_{i=1}^k y_i^{\downarrow}$.

The Fan dominance principle [3, p. 93] says that for any two matrices X,Y, we have $\sigma(X)\prec_w\sigma(Y)$ if and only if $|||X|||\leqslant|||Y|||$ for all unitarily invariant norms. Another fact that will be used says that $\lambda(XY)=\lambda(YX)$.

To prove our second main result, we need the following two lemmas. The first lemma is a part of Theorem 3.1 in [2], and the second lemma can be concluded from the proof of Lamma 2.1 in [2].

LEMMA 2. *For $i=1,\ldots,k$, let A_i,B_i be positive semidefinite matrices such that, for each i, A_i commutes with B_i. Then for all unitarily invariant norms,*

$$\left|\left|\left|\sum_{i=1}^{k} A_i B_i\right|\right|\right| \leqslant \left|\left|\left|\left(\sum_{i=1}^{k} A_i^{\frac{1}{2}} B_i^{\frac{1}{2}}\right)^2\right|\right|\right|.$$

LEMMA 3. *Let S be a general $n\times m$ complex matrix, and let L and M be two diagonal, positive semidefinite $m\times m$ matrices. Then*

$$\sigma\left(\left(S\left(LM\right)^{\frac{1}{2}} S^*\right)^2\right) \prec_w \lambda\left(SLS^*SMS^*\right).$$

Our main result in this section can be stated as follows.

THEOREM 2. *For $i=1,\ldots,k$, let A_i,B_i be positive semidefinite matrices such that, for each i, A_i commutes with B_i. Then for all unitarily invariant norms,*

$$\left|\left|\left|\left(\sum_{i=1}^{k} A_i^{\frac{1}{2}} B_i^{\frac{1}{2}}\right)^2\right|\right|\right| \leqslant \left|\left|\left|\left(\sum_{i=1}^{k} A_i\right)^{\frac{1}{2}}\left(\sum_{i=1}^{k} B_i\right)\left(\sum_{i=1}^{k} A_i\right)^{\frac{1}{2}}\right|\right|\right|.$$

Proof. Let A_i,B_i have spectral decompositions

$$A_i = U_i D_i U_i^*, B_i = U_i E_i U_i^*,$$

where U_i are unitary matrices, and D_i,E_i are positive semidefinite diagonal matrices.

Let

$$L=\begin{bmatrix} D_1 & 0 & \cdots & 0 & 0\\ 0 & D_2 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & D_{k-1} & 0\\ 0 & 0 & \cdots & 0 & D_k \end{bmatrix},\quad M=\begin{bmatrix} E_1 & 0 & \cdots & 0 & 0\\ 0 & E_2 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & E_{k-1} & 0\\ 0 & 0 & \cdots & 0 & E_k \end{bmatrix},$$

and

$$S=\left[U_1 |\ U_2| \cdots |U_{k-1}\ |U_k\right].$$

Then

$$\sum_{i=1}^{k} A_i = SLS^*,\quad \sum_{i=1}^{k} B_i = SMS^*,\quad \sum_{i=1}^{k} A_i^{\frac{1}{2}} B_i^{\frac{1}{2}} = S\left(LM\right)^{\frac{1}{2}} S^*.$$

We need to show that

$$\sigma\left(\left(S\left(LM\right)^{\frac{1}{2}} S^*\right)^2\right) \prec_w \sigma\left(\left(SLS^*\right)^{\frac{1}{2}} SMS^*\left(SLS^*\right)^{\frac{1}{2}}\right).$$

Now,

$$\begin{aligned}\sigma\left(\left(S(LM)^{\frac{1}{2}}S^*\right)^2\right) &\prec_w \lambda\left(SLS^*SMS^*\right) \qquad \text{(by Lemma 3)}\\ &= \lambda\left((SLS^*)^{\frac{1}{2}}SMS^*(SLS^*)^{\frac{1}{2}}\right)\\ &= \sigma\left((SLS^*)^{\frac{1}{2}}SMS^*(SLS^*)^{\frac{1}{2}}\right).\end{aligned}$$

Therefore,

$$\sigma\left(\left(\sum_{i=1}^k A_i^{\frac{1}{2}}B_i^{\frac{1}{2}}\right)^2\right) \prec_w \sigma\left(\left(\sum_{i=1}^k A_i\right)^{\frac{1}{2}}\left(\sum_{i=1}^k B_i\right)\left(\sum_{i=1}^k A_i\right)^{\frac{1}{2}}\right),$$

which is equivalent to

$$\left|\left|\left|\left(\sum_{i=1}^k A_i^{\frac{1}{2}}B_i^{\frac{1}{2}}\right)^2\right|\right|\right| \leqslant \left|\left|\left|\left(\sum_{i=1}^k A_i\right)^{\frac{1}{2}}\left(\sum_{i=1}^k B_i\right)\left(\sum_{i=1}^k A_i\right)^{\frac{1}{2}}\right|\right|\right|.$$

This completes the proof. □

Now, we are in a position to prove the inequality (15). Using Theorem 2, Lemma 2, and the inequality (11), we have the following refinements of the inequality (14), including the inequality (15).

COROLLARY 1. *For $i = 1,\ldots,k$, let A_i, B_i be positive semidefinite matrices such that, for each i, A_i commutes with B_i. Then for all unitarily invariant norms,*

$$\begin{aligned}\left|\left|\left|\sum_{i=1}^k A_iB_i\right|\right|\right| &\leqslant \left|\left|\left|\left(\sum_{i=1}^k A_i^{\frac{1}{2}}B_i^{\frac{1}{2}}\right)^2\right|\right|\right|\\ &\leqslant \left|\left|\left|\left(\sum_{i=1}^k A_i\right)^{\frac{1}{2}}\left(\sum_{i=1}^k B_i\right)\left(\sum_{i=1}^k A_i\right)^{\frac{1}{2}}\right|\right|\right|\\ &\leqslant \frac{1}{2}\left|\left|\left|\left(\sum_{i=1}^k A_i\right)\left(\sum_{i=1}^k B_i\right)+\left(\sum_{i=1}^k B_i\right)\left(\sum_{i=1}^k A_i\right)\right|\right|\right|\\ &\leqslant \left|\left|\left|\left(\sum_{i=1}^k A_i\right)\left(\sum_{i=1}^k B_i\right)\right|\right|\right|.\end{aligned}$$

Direct applications of the inequalities in Corollary 1 answer some questions of Bourin in the affirmative. In fact, letting $k = 2$, $A_1 = A^p$, $A_2 = B^p$, $B_1 = A^q$, and $B_2 = B^q$, we have the following chain of norm inequalities for positive semidefinite matrices:

$$
\begin{aligned}
\left|\left|\left|A^{p+q}+B^{p+q}\right|\right|\right| &\leqslant \left|\left|\left|\left(A^{\frac{p+q}{2}}+B^{\frac{p+q}{2}}\right)^2\right|\right|\right| \\
&\leqslant \left|\left|\left|(A^p+B^p)^{\frac{1}{2}}(A^q+B^q)(A^p+B^p)^{\frac{1}{2}}\right|\right|\right| \\
&\leqslant \frac{1}{2}\left|\left|\left|(A^p+B^p)(A^q+B^q)+(A^q+B^q)(A^p+B^p)\right|\right|\right| \\
&\leqslant \left|\left|\left|(A^p+B^p)(A^q+B^q)\right|\right|\right|.
\end{aligned} \tag{19}
$$

It should be mentioned here that the inequality (19) also follows from Theorem 2 in [5].

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