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AN INVOLUTION INEQUALITY FOR THE KULLBACK—LEIBLER DIVERGENCE

IOSIF PINELIS

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Abstract. Let $R_t := tP + (1-t)P\iota$, where $t \in (0,1)$, P is a probability measure, and $P\iota$ is the push-forward image of P under a measurable involution ι. An inequality involving the Kullback–Leibler divergence of R_t from P is given. It is shown that the role of the involution is essential.

Let $\mathscr{P}$ denote the set of all probability measures on a measurable space (X,Σ). Take any P and Q in $\mathscr{P}$ such that P absolutely continuous with respect to Q, with the Radon–Nikodym derivative $\frac{dP}{dQ}$ of P with respect to Q. Then the Kullback–Leibler (KL) divergence of Q from P can be defined by the formula

$$D(P||Q) := D_{\text{KL}}(P||Q) := \int_X dP \ln \frac{dP}{dQ}.$$

The KL divergence is well defined, in that the displayed integral exists and its value is in the interval $[0,\infty]$ [2]. In the original paper by Kullback and Leibler, the KL divergence was referred to as “the mean information for discrimination”, whereas the term “divergence” was used there for the symmetrized version $D(P||Q) + D(Q||P)$ of $D(P||Q)$; see pages 80 and 81 concerning formulas (2.4) and (2.6) in [2].

Suppose that $\iota : X \to X$ is an involution of X, that is, ι is a bijection and the inverse transformation ι^{-1} equals ι. For example, if X is an additive group, then the transformation $X \ni x \mapsto -x$ is an involution.

Suppose further that the involution ι is Σ-measurable. For any $P \in \mathscr{P}$, let $P\iota = P\iota^{-1}$ denote the push-forward (or, in this case, equivalently, the “push-back”) probability measure, defined by the condition that $(P\iota)(A) := P(\iota(A)) = P(\iota^{-1}(A))$ for all $A \in \Sigma$. In view of the example mentioned in the previous paragraph, $P\iota$ may be considered a generalized reflection of P.

THEOREM 1. *Take any $P \in \mathscr{P}$. For $t \in (0,1)$, consider the mixture*

$$R_t := tP + (1-t)P\iota \tag{1}$$

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of the probability measures P and $P\iota$, so that P is absolutely continuous with respect to R_t. Then

$$\frac{D(P||R_{1-t})}{\ln(1-t)} \geqslant \frac{D(P||R_t)}{\ln t} \tag{2}$$

for all $t \in (0,1/2)$.

Proof. In this proof, let t be an arbitrary number in the interval $(0,1)$, unless further specified. Let $\mu := (P+P\iota)/2$, so that $\mu\iota = \mu$. Let $p := \frac{dP}{d\mu}$. Then $\frac{dP\iota}{d\mu} = p\iota := p\circ\iota$ and hence $\frac{dR_t}{d\mu} = r_t := t\,p + (1-t)\,p\iota$. It follows that

$$D(P||R_t) = \int_X d\mu\; p\ln\frac{p}{r_t} = \int_{[p>0]} d\mu\; p\ln\frac{p}{r_t} = J_t + \alpha\ln\frac{1}{t},$$

where

$$\begin{aligned} J_t &:= \int_{[p>0,\,p\iota>0]} d\mu\; p\ln\frac{p}{r_t} = \int_{[p>0,\,p\iota>0]} d(\mu\iota)\; p\iota\ln\frac{p\iota}{r_t\iota} = \int_{[p>0,\,p\iota>0]} d\mu\; p\iota\ln\frac{p\iota}{r_{1-t}}, \\ \alpha &:= \int_{[p>0,\,p\iota=0]} d\mu\; p = \int_{[p\iota=0]} d\mu\; p = P([p\iota=0]) \in [0,1]; \end{aligned}$$

here and in what follows, $[p>0] := \{x\in X\colon\, p(x)>0\}$, and the sets $[p>0,\,p\iota>0]$, $[p>0,\,p\iota=0]$, $[p\iota=0]$ are defined similarly. Hence,

$$\begin{aligned} \frac{D(P||R_t)}{\ln(1/t)} - \alpha &= \frac{1}{2\ln(1/t)}\int_{[p>0,\,p\iota>0]} d\mu\left(p\ln\frac{p}{r_t} + p\iota\ln\frac{p\iota}{r_{1-t}}\right) \\ &= \frac{1}{2\ln(1/t)}\int_{[p>0,\,p\iota>0]} d\mu\left(p\ln\frac{p}{t\,p+(1-t)\,p\iota} + p\iota\ln\frac{p\iota}{t\,p\iota+(1-t)\,p}\right) \\ &= \frac{1}{2}\int_{[p>0,\,p\iota>0]} \mu(dx)\; p(x)\,F(t,\rho(x)), \end{aligned}$$

where $\rho(x) := \frac{p\iota(x)}{p(x)}$ and

$$F(t,u) := \frac{1}{\ln(1/t)}\left(\ln\frac{1}{t+(1-t)u} + u\ln\frac{1}{t+(1-t)/u}\right).$$

So, it suffices to show that

$$\Delta(t,u) := F(t,u) - F(1-t,u) > 0 \tag{3}$$

for $t\in(0,1/2)$ and $u\in(0,1)\cup(1,\infty)$ (clearly, $\Delta(t,1)=0$).

It is indeed not very hard to show that (3) is true. Note that $\Delta(t,1/u) = \Delta(t,u)/u$ for $u>0$. So, without loss of generality $u>1$. That is, it suffices to show that

$$h(u) := \Delta(t,u)\ln(1-t)\ln t$$

is positive for $t \in (0,1/2)$ and $u > 1$. Let

$$h_2(u) := h''(u)\frac{(1-t+tu)^2(t+(1-t)u)^2u}{1+u}$$
$$= -(u^2+1)(1-t)^2t^2\ln\left(\tfrac{1}{t}-1\right) - u\left((1-2t^2)(1-t)^2\ln(1-t) + t^2(1-4t+2t^2)\ln t\right).$$

So, $h_2(u)$ is a quadratic polynomial in u, with the coefficient of u^2 equal to $-(1-t)^2t^2\ln\left(\frac{1}{t}-1\right) < 0$ for $t \in (0,1/2)$. So, h_2 is concave and $h_2(\infty-) = -\infty$.

Next, $h_2(1) = g(t) := t^2\ln t - (1-t)^2\ln(1-t)$, $g(0+) = g(1/2) = 0$ and $g''(t) = -2\ln\left(\frac{1}{t}-1\right) < 0$, so that g is concave and hence $h_2(1) = g(t) > 0$ for $t \in (0,1/2)$. Because h_2 is concave and $h_2(\infty-) = -\infty$, it follows that $h_2(u)$ and hence $h''(u)$ switch their sign exactly once, from $+$ to $-$, as u increases from 1 to ∞. So, for some real $c = c(t) > 1$, the function h is convex on $[1,c]$ and concave on $[c,\infty)$. At that, $h(1) = h'(1) = 0$ and $h(\infty-) = \infty$. So, $h > 0$ on $(1,\infty)$. That is, indeed $\Delta(t,u) > 0$ for $t \in (0,1/2)$ and $u > 1$. $\square$

REMARK 1. Following the lines of the above proof of Theorem 1, one can see that inequality (2) turns into the equality if and only if $p\iota = p$ μ-almost everywhere on the set $A^+ := [p > 0,\, p\iota > 0]$, that is, if and only if $P(A^+ \cap A) = (P\iota)(A^+ \cap A)$ for all $A \in \Sigma$.

REMARK 2. The role of the involution ι in Theorem 1 is essential. Theorem 1 will fail to hold in general if, in the definition (1) of R_t, the involution $P\iota$ of P is replaced by an arbitrary $Q \in \mathscr{P}$. E.g., take $X = \{-1,0,1\}$, $t = \frac{1}{10}$, $P = \frac{1}{100}(57\delta_{-1} + 42\delta_0 + \delta_1)$, $Q = \frac{1}{100}(41\delta_{-1} + 12\delta_0 + 47\delta_1)$, where δ_x denotes the Dirac probability measure at x. Then inequality (2) will be false if $P\iota$ is replaced by Q in the definition (1) of R_t.

Theorem 1 is a generalization of an inequality conjectured on the MathOverflow site [1] and proved there by the author of the present note.

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Iosif Pinelis
Department of Mathematical Sciences
Michigan Technological University
Houghton, Michigan 49931, USA
e-mail: ipinelis@mtu.edu