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NEW INEQUALITIES FOR PRODUCTS OF CROSS–SECTION MEASURES

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Abstract. The main purpose of this paper is to sharpen some upper and lower bounds on products of cross-section measures of centered convex bodies. The bounds are given in terms of relative inner and outer radii of isoperimetrices of normed spaces, and improve previously known results in the symmetric case. Thus, our results mainly refer to the geometry of finite dimensional real Banach spaces.

1. Introduction

We recall that a *convex body* K in $\mathbb{R}^d, d \geqslant 2$, is a compact, convex set with nonempty interior, and that K is said to be *centered* if it is symmetric with respect to the origin o of $\mathbb{R}^d$. As usual, S^{d-1} denotes the standard Euclidean unit sphere in $\mathbb{R}^d$. We write λ_i for the i*-dimensional Lebesgue measure* (*volume*) in $\mathbb{R}^d$, where $1 \leqslant i \leqslant d$, and instead of λ_d we simply write λ. We denote by $u^{\perp}$ the $(d-1)$-dimensional subspace orthogonal to $u \in S^{d-1}$, and by l_u the 1-subspace parallel to u.

For a convex body $K \subset \mathbb{R}^d$ we denote by $\lambda_{d-1}(K,u^{\perp})$ and $\lambda_1(K,u)$ the $(d-1)$-*dimensional* and 1*-dimensional inner cross-section measures of* K, i.e., the maximal measure of a hyperplane section of K normal to $u \in S^{d-1}$, and the *maximal chord length* of K in the direction u, respectively. Furthermore, $\lambda_1(K|l_u)$ denotes the *width* of K at u, and $\lambda_{d-1}(K|u^{\perp})$ is called the $(d-1)$*-dimensional outer cross-section measure* or *brightness* of K at $u \in S^{d-1}$, where $K|u^{\perp}$ is the orthogonal projection of K onto $u^{\perp}$. These notions given above can be found in the monograph [3]. In [11] and [16] the following results for cross-section measures were derived.

For a convex body K in $\mathbb{R}^d$, $d \geqslant 2$, and every direction $u \in S^{d-1}$ we have

$$\lambda(K) \leqslant \lambda_{d-1}(K|u^{\perp})\lambda_1(K,u) \leqslant d\lambda(K), \tag{1}$$

and both sides are sharp.

On the other hand, for each $u \in S^{d-1}$ a convex body K in $\mathbb{R}^d$, $d \geqslant 2$, satisfies

$$\lambda(K) \leqslant \lambda_{d-1}(K,u^{\perp})\lambda_1(K|l_u) \leqslant d\lambda(K), \tag{2}$$

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again with sharpness on both sides.

Our main purpose is to establish strengthenings of (1) and (2) for centered convex bodies. The inner radius and outer radius of isoperimetrices of normed spaces (for Holmes-Thompson and Busemann measures) will be used to obtain these inequalities. Thus, our main results will be related to finite dimensional real Banach spaces.

For a convex body K in $\mathbb{R}^d$, the *polar body* K° of K is defined by

$$K^\circ = \{y \in \mathbb{R}^d : \langle x, y\rangle \leqslant 1, x \in K\}.$$

We identify $\mathbb{R}^d$ and its *dual space* $\mathbb{R}^{d*}$ by using the standard basis. In that case, λ_i and λ_i^* coincide in $\mathbb{R}^d$. The symbol ε_i stands for the volume of the standard Euclidean unit ball in $\mathbb{R}^i$.

For a convex body K in $\mathbb{R}^d$ and $u \in S^{d-1}$, the *support function* of K is defined by

$$h_K(u) = \sup\{\langle u, y\rangle : y \in K\},$$

and with o as an interior point of K its *radial function* $\rho_K(u)$ is defined by

$$\rho_K(u) = \max\{\alpha \geqslant 0 : \alpha u \in K\}.$$

It is well known that

$$\rho_{K^\circ}(u) = \frac{1}{h_K(u)}, \ u \in S^{d-1}. \tag{3}$$

If K is a centered convex body, then $2\rho_K(u) = \lambda_1(K \cap l_u)$, and $2h_K(u) = \lambda_1(K|l_u)$ for any $u \in S^{d-1}$.

The *projection body* ΠK of a convex body K in $\mathbb{R}^d$ is defined by $h_{\Pi K}(u) = \lambda_{d-1}(K|u^\perp)$ for each $u \in S^{d-1}$ (see [3, Chapter 4]). Note that any projection body is a *zonoid* (i.e., a limit of vector sums of segments). In particular, if K is a polytope, then its projection body is a *zonotope* centered at the origin (see [15] and [4] for many properties and applications of this interesting class of convex bodies). We also refer to [1], [6], [7], and [12] for affine isoperimetric inequalilties related to projection bodies. The *intersection body* IK of a convex body $K \subset \mathbb{R}^d$ is defined by $\rho_{IK}(u) = \lambda_{d-1}(K \cap u^\perp)$ for each $u \in S^{d-1}$ (cf. [5] and [3, Chapter 8]). If K is a centered convex body, then IK is also a centered convex body (see [2]).

We write $(\mathbb{R}^d, ||\cdot||) = \mathbb{M}^d$ for a d*-dimensional real Banach space*, i.e., a *Minkowski space* with *unit ball* B which is a centered convex body; see [17]. The *unit sphere of* $\mathbb{M}^d$ is the boundary ∂B of the unit ball.

2. Isoperimetrices and inner/outer radii in Minkowski spaces

A Minkowski space $\mathbb{M}^d$ possesses a Haar measure μ, and this measure is unique up to multiplying the Lebesgue measure by a constant, i.e., $\mu = \sigma_B \lambda$.

The following notions are well known; see [17, Chapter 5]. The d*-dimensional Holmes-Thompson volume* of a convex body K in $\mathbb{M}^d$ is defined by

$$\mu_B^{HT}(K) = \frac{\lambda(K)\lambda(B^\circ)}{\varepsilon_d}, \ \text{i.e.,} \ \sigma_B = \frac{\lambda(B^\circ)}{\varepsilon_d},$$

and the *d-dimensional Busemann volume* of K is defined by

$$\mu_B^{Bus}(K) = \frac{\varepsilon_d}{\lambda(B)}\lambda(K), \text{ i.e. }, \ \sigma_B = \frac{\varepsilon_d}{\lambda(B)} \ (\text{and } \mu_B^{Bus}(B) = \varepsilon_d).$$

In order to define the Minkowski surface area of a convex body, one has to define σ_B similarly in $\mathbb{M}^{d-1}$. That is, for the Holmes-Thompson measure we have $\sigma_B(u) = \lambda_{d-1}((B\cap u^\perp)^\circ)/\varepsilon_{d-1}$, and for the Busemann measure $\sigma_B(u) = \varepsilon_{d-1}/\lambda_{d-1}(B\cap u^\perp)$ (see [17, pp. 150–151]). The *Minkowski surface area* of K can be also defined in terms of mixed volumes (see [14] for notation and more about mixed volumes) by

$$\mu_B(\partial K) = dV(K[d-1], I_B),$$

where I_B is that convex body whose support function is $\sigma_B(u)$. For the Holmes-Thompson measure, I_B is given by $I_B^{HT} = \Pi(B^\circ)/\varepsilon_{d-1}$, (cf. [17, p. 150 and p. 157] for detailed explanation) and therefore it is a centered zonoid. For the Busemann measure we have $I_B^{Bus} = \varepsilon_{d-1}(IB)^\circ$ (see again [17, pp. 150–151]). Among all homothetic images of I_B a unique one is specified, which is called the *isoperimetrix* $\hat{I}_B$ and is determined by $\mu_B(\partial\hat{I}_B) = d\mu_B(\hat{I}_B)$. The *isoperimetrix for the Holmes-Thompson measure* is defined by

$$\hat{I}_B^{HT} = \frac{\varepsilon_d}{\lambda(B^\circ)}I_B^{HT} = \frac{\varepsilon_d}{\varepsilon_{d-1}}\frac{1}{\lambda(B^\circ)}\Pi B^\circ, \tag{4}$$

and the *isoperimetrix for the Busemann measure* by

$$\hat{I}_B^{Bus} = \frac{\lambda(B)}{\varepsilon_d}I_B^{Bus} = \frac{\varepsilon_{d-1}}{\varepsilon_d}\lambda(B)(IB)^\circ; \tag{5}$$

see [17, Chapter 5].

DEFINITION 1. If K is a convex body in $\mathbb{M}^d$, the *inner radius* of K is defined by

$$r(K,\hat{I}_B) := \max\{\alpha > 0 : \exists x \in \mathbb{M}^d \text{ with } \alpha\hat{I}_B \subseteq K + x\},$$

and the *outer radius* of K is defined by

$$R(K,\hat{I}_B) := \min\{\alpha > 0 : \exists x \in \mathbb{M}^d \text{ with } \alpha\hat{I}_B \supseteq K + x\}.$$

One should notice that $r(K,\hat{I}_B)$ and $R(K,\hat{I}_B)$ can be also defined in terms of the support functions of the involved sets. In particular, if K is a centered convex body, then $r(K,\hat{I}_B)$ is the maximum value of α such that $\alpha \leqslant h_K(u)/h_{\hat{I}_B}(u)$ for all $u \in S^{d-1}$. Similarly, $R(K,\hat{I}_B)$ is the minimum value of α such that $\alpha \geqslant h_K(u)/h_{\hat{I}_B}(u)$ for all $u \in S^{d-1}$ (see [13] and [18]).

3. New inequalities for cross-section measures

We present now the announced stronger inequalities than (1) and (2), and some consequences thereof.

THEOREM 2. *If B is a centered convex body in $\mathbb{R}^d$, and $u \in S^{d-1}$ is a unit vector, then*

$$\frac{1}{R(B^\circ,\hat{I}^{HT}_{B^\circ})}\frac{2\varepsilon_{d-1}}{\varepsilon_d} \leqslant \frac{\lambda_{d-1}(B|u^\perp)\lambda_1(B\cap l_u)}{\lambda(B)} \leqslant \frac{1}{r(B^\circ,\hat{I}^{HT}_{B^\circ})}\frac{2\varepsilon_{d-1}}{\varepsilon_d}.$$

Proof. Let B be a centered convex body in $\mathbb{R}^d$. Then

$$r(B,\hat{I}^{HT}_B)\hat{I}^{HT}_B \subseteq B \subseteq R(B,\hat{I}^{HT}_B)\hat{I}^{HT}_B.$$

This is equivalent to

$$r(B,\hat{I}^{HT}_B)\frac{\varepsilon_d}{\lambda(B^\circ)}\frac{\Pi B^\circ}{\varepsilon_{d-1}} \subseteq B \subseteq R(B,\hat{I}^{HT}_B)\frac{\varepsilon_d}{\lambda(B^\circ)}\frac{\Pi B^\circ}{\varepsilon_{d-1}}.$$

Hence

$$r(B,\hat{I}^{HT}_B)\frac{\varepsilon_d}{\lambda(B^\circ)\varepsilon_{d-1}}h_{\Pi B^\circ}(u) \leqslant h_B(u) \leqslant R(B,\hat{I}^{HT}_B)\frac{\varepsilon_d}{\lambda(B^\circ)\varepsilon_{d-1}}h_{\Pi B^\circ}(u).$$

If we set B to be B° and use the identity (3), we obtain

$$r(B^\circ,\hat{I}^{HT}_{B^\circ})\frac{\varepsilon_d}{\lambda(B)\varepsilon_{d-1}}h_{\Pi B}(u)\rho_B(u) \leqslant 1 \leqslant R(B^\circ,\hat{I}^{HT}_{B^\circ})\frac{\varepsilon_d}{\lambda(B)\varepsilon_{d-1}}h_{\Pi B}(u)\rho_B(u).$$

Therefore,

$$r(B^\circ,\hat{I}^{HT}_{B^\circ})\frac{\lambda_{d-1}(B|u^\perp)\lambda_1(B\cap l_u)}{\lambda(B)} \leqslant \frac{2\varepsilon_{d-1}}{\varepsilon_d} \leqslant R(B^\circ,\hat{I}^{HT}_{B^\circ})\frac{\lambda_{d-1}(B|u^\perp)\lambda_1(B\cap l_u)}{\lambda(B)},$$

yielding

$$\frac{1}{R(B^\circ,\hat{I}^{HT}_{B^\circ})}\frac{2\varepsilon_{d-1}}{\varepsilon_d} \leqslant \frac{\lambda_{d-1}(B|u^\perp)\lambda_1(B\cap l_u)}{\lambda(B)} \leqslant \frac{1}{r(B^\circ,\hat{I}^{HT}_{B^\circ})}\frac{2\varepsilon_{d-1}}{\varepsilon_d}. \qquad \square$$

REMARK 3. We recall (see [9]) that

$$\frac{1}{R(B^\circ,\hat{I}^{HT}_{B^\circ})}\frac{2\varepsilon_{d-1}}{\varepsilon_d} \geqslant 1,$$

and

$$\frac{1}{r(B^\circ,\hat{I}^{HT}_{B^\circ})}\frac{2\varepsilon_{d-1}}{\varepsilon_d} \leqslant d.$$

THEOREM 4. *If B is a centered convex body in $\mathbb{R}^d$ and $u \in S^{d-1}$ is a unit vector, then*

$$r(B,\hat{I}_B^{Bus})\frac{2\varepsilon_{d-1}}{\varepsilon_d} \leqslant \frac{\lambda_{d-1}(B\cap u^{\perp})\lambda_1(B|l_u)}{\lambda(B)} \leqslant R(B,\hat{I}_B^{Bus})\frac{2\varepsilon_{d-1}}{\varepsilon_d}.$$

Proof. Let B be a centered convex body in $\mathbb{R}^d$. Then

$$r(B,\hat{I}_B^{Bus})\hat{I}_B^{Bus} \subseteq B \subseteq R(B,\hat{I}_B^{Bus})\hat{I}_B^{Bus}.$$

This can also be written as

$$r(B,\hat{I}_B^{Bus})\frac{\lambda(B)}{\varepsilon_d}\varepsilon_{d-1}(IB)^{\circ} \subseteq B \subseteq R(B,\hat{I}_B^{Bus})\frac{\lambda(B)}{\varepsilon_d}\varepsilon_{d-1}(IB)^{\circ}.$$

Hence

$$r(B,\hat{I}_B^{Bus})\frac{\lambda(B)}{\varepsilon_d}\varepsilon_{d-1}h_{(IB)^{\circ}}(u) \leqslant h_B(u) \leqslant R(B,\hat{I}_B^{Bus})\frac{\lambda(B)}{\varepsilon_d}\varepsilon_{d-1}h_{(IB)^{\circ}}(u).$$

From the relation between the support function and the radial function we obtain

$$r(B,\hat{I}_B^{Bus})\frac{\lambda(B)}{\varepsilon_d}\varepsilon_{d-1} \leqslant h_B(u)\rho_{IB}(u) \leqslant R(B,\hat{I}_B^{Bus})\frac{\lambda(B)}{\varepsilon_d}\varepsilon_{d-1},$$

and therefore

$$r(B,\hat{I}_B^{Bus})\frac{2\varepsilon_{d-1}}{\varepsilon_d} \leqslant \frac{\lambda_{d-1}(B\cap u^{\perp})\lambda_1(B|l_u)}{\lambda(B)} \leqslant R(B,\hat{I}_B^{Bus})\frac{2\varepsilon_{d-1}}{\varepsilon_d}. \quad \square$$

REMARK 5. We recall (see [10]) that

$$r(B,\hat{I}_B^{Bus})\frac{2\varepsilon_{d-1}}{\varepsilon_d} \geqslant 1.$$

Also,

$$R(B,\hat{I}_B^{Bus})\frac{2\varepsilon_{d-1}}{\varepsilon_d} \leqslant d.$$

From Theorem 2 and Theorem 4, we deduce the following statement.

COROLLARY 6. *If B is a centered convex body in $\mathbb{R}^d$ and $u \in S^{d-1}$ is a unit vector, then*

$$r(B^{\circ},\hat{I}_{B^{\circ}}^{HT})r(B,\hat{I}_B^{Bus}) \leqslant \frac{\lambda_{d-1}(B\cap u^{\perp})\lambda_1(B|l_u)}{\lambda_{d-1}(B|u^{\perp})\lambda_1(B\cap l_u)} \leqslant R(B^{\circ},\hat{I}_{B^{\circ}}^{HT})R(B,\hat{I}_B^{Bus}).$$

REMARK 7. From Remarks 3 and 5 we get the inequalities $r(B^{\circ},\hat{I}_{B^{\circ}}^{HT})r(B,\hat{I}_B^{Bus}) \geqslant 1/d$ and $R(B^{\circ},\hat{I}_{B^{\circ}}^{HT})R(B,\hat{I}_B^{Bus}) \leqslant d$.

THEOREM 8. *If B is a centered convex body in $\mathbb{R}^d$, then*

$$\frac{1}{d} \leqslant R(B,\hat{I}_B^{Bus})r(B^\circ,\hat{I}_{B^\circ}^{HT}) \leqslant 1,$$

$$\frac{1}{d} \leqslant R(B,\hat{I}_B^{HT})r(B^\circ,\hat{I}_{B^\circ}^{Bus}) \leqslant 1.$$

Furthermore, for both inequalities the upper bound on the right cannot be reduced.

Proof. Let B be a centered convex body in $\mathbb{R}^d$. Since $R(B,\hat{I}_B^{HT}) \geqslant r(B,\hat{I}_B^{HT})$ and $R(B,\hat{I}_B^{Bus}) \geqslant r(B,\hat{I}_B^{Bus})$, we have $R(B,\hat{I}_B^{Bus})r(B^\circ,\hat{I}_{B^\circ}^{HT}) \geqslant r(B,\hat{I}_B^{Bus})r(B^\circ,\hat{I}_{B^\circ}^{HT})$, and $R(B,\hat{I}_B^{HT})r(B^\circ,\hat{I}_{B^\circ}^{Bus}) \geqslant r(B,\hat{I}_B^{HT})r(B^\circ,\hat{I}_{B^\circ}^{Bus})$. Thus, the lower bound for both inequalities follows from Remark 7.

It is well known that $IB \subseteq \Pi B$, and that for $d \geqslant 3$ equality holds iff B is a centered ellipsoid (see [8]). Therefore $(\Pi B)^\circ \subseteq (IB)^\circ$, also equivalent to

$$\frac{\lambda(B)}{\varepsilon_d}\left(\frac{\Pi B}{\varepsilon_{d-1}}\right)^\circ \subseteq \frac{\lambda(B)}{\varepsilon_d}\varepsilon_{d-1}(IB)^\circ.$$

Thus $\frac{\lambda(B)}{\varepsilon_d}(I_{B^\circ}^{HT})^\circ \subseteq \frac{\lambda(B)}{\varepsilon_d}I_B^{Bus}$, from which it follows that

$$(\hat{I}_{B^\circ}^{HT})^\circ \subseteq \hat{I}_B^{Bus}, \tag{6}$$

and for $d \geqslant 3$ equality holds iff B is a centered ellipsoid.

As we know,

$$r(B,\hat{I}_B^{HT})\hat{I}_B^{HT} \subseteq B \subseteq R(B,\hat{I}_B^{HT})\hat{I}_B^{HT}.$$

Therefore,

$$\frac{1}{R(B,\hat{I}_B^{HT})}(\hat{I}_B^{HT})^\circ \subseteq B^\circ \subseteq \frac{1}{r(B,\hat{I}_B^{HT})}(\hat{I}_B^{HT})^\circ.$$

Setting B° to be B and using (6), we obtain

$$B \subseteq \frac{1}{r(B^\circ,\hat{I}_{B^\circ}^{HT})}(\hat{I}_{B^\circ}^{HT})^\circ \subseteq \frac{1}{r(B^\circ,\hat{I}_{B^\circ}^{HT})}\hat{I}_B^{Bus}.$$

From the definition of the outer radius for the Busemann isoperimetrix, we get

$$R(B,\hat{I}_B^{Bus}) \leqslant \frac{1}{r(B^\circ,\hat{I}_{B^\circ}^{HT})},$$

which establishes the first inequality.

To obtain the second inequality, we write

$$r(B^\circ,\hat{I}_{B^\circ}^{Bus})\hat{I}_{B^\circ}^{Bus} \subseteq B^\circ \subseteq R(B^\circ,\hat{I}_{B^\circ}^{Bus})\hat{I}_{B^\circ}^{Bus}.$$

Hence

$$\frac{1}{R(B^\circ,\hat{I}_{B^\circ}^{Bus})}(\hat{I}_{B^\circ}^{Bus})^\circ \subseteq B \subseteq \frac{1}{r(B^\circ,\hat{I}_{B^\circ}^{Bus})}(\hat{I}_{B^\circ}^{Bus})^\circ.$$

Using (6) when B is replaced by B° and applying polarity we also get $(\hat{I}_{B^\circ}^{Bus})^\circ \subseteq \hat{I}_B^{HT}$. Therefore,

$$B \subseteq \frac{1}{r(B^\circ, \hat{I}_{B^\circ}^{Bus})} \hat{I}_B^{HT}.$$

Again, from the definition of the outer radius for the Holmes-Thompson case, we get

$$R(B, \hat{I}_B^{HT}) \leqslant \frac{1}{r(B^\circ, \hat{I}_{B^\circ}^{Bus})}.$$

The following example shows that the upper bound is attained for both inequalities. Let $B = \alpha E$, where α is a positive real number and E is the standard unit ball in $\mathbb{R}^d$. Then one can easily show that $\Pi B = \alpha^{d-1}\varepsilon_{d-1}E$, $\hat{I}_B^{HT} = \alpha E$, and $\hat{I}_{B^\circ}^{HT} = (1/\alpha)E$. Therefore, $R(B, \hat{I}_B^{HT}) = r(B, \hat{I}_B^{HT}) = R(B^\circ, \hat{I}_{B^\circ}^{HT}) = r(B^\circ, \hat{I}_{B^\circ}^{HT}) = 1$.

Similarly, $IB = \alpha^{d-1}\varepsilon_{d-1}E$, $\hat{I}_B^{Bus} = \alpha E$, and $\hat{I}_{B^\circ}^{Bus} = (1/\alpha)E$. We also get $R(B, \hat{I}_B^{Bus}) = r(B, \hat{I}_B^{Bus}) = R(B^\circ, \hat{I}_{B^\circ}^{Bus}) = r(B^\circ, \hat{I}_{B^\circ}^{Bus}) = 1$. $\square$

4. New inequalities for width and diameter of the isoperimetrix in Minkowski spaces

The following results refer to extremal values of the Minkowskian width function of convex bodies involving inner and outer radii. Let B be the unit ball of a Minkowski space. We denote by $w_B(K)$ and $D_B(K)$ the *Minkowski thickness* (or *Minkowski minimal width*) and the *Minkowski diameter* of a convex body K, respectively. They are defined as $w_B(K) = \min_{u \in S^{d-1}} \frac{2w(K,u)}{w(B,u)}$ and $D_B(K) = \max_{u \in S^{d-1}} \frac{2w(K,u)}{w(B,u)}$, where $w(K,u) = h_K(u) + h_K(-u)$ is the Euclidean width of K in the direction u. In particular, if K is a centered convex body, then $w(K,u) = 2h_K(u)$. Therefore, since $\hat{I}_B^{HT}$ is centered, using (4) we can expand $w_B(\hat{I}_B^{HT})$ as follows:

$$w_B(\hat{I}_B^{HT}) = \min_{u \in S^{d-1}} \frac{2w(\hat{I}_B^{HT}, u)}{w(B,u)} = \min_{u \in S^{d-1}} \frac{2h_{\hat{I}_B^{HT}}(u)}{h_B(u)} = \min_{u \in S^{d-1}} \frac{2\varepsilon_d}{\varepsilon_{d-1}} \frac{h_{\Pi B^\circ}(u)\rho_{B^\circ}(u)}{\lambda(B^\circ)}$$

$$= \min_{u \in S^{d-1}} \frac{\varepsilon_d}{\varepsilon_{d-1}} \frac{\lambda_{d-1}(B^\circ|u^\perp)\lambda_1(B^\circ \cap l_u)}{\lambda(B^\circ)}.$$

Similarly,

$$D_B(\hat{I}_B^{HT}) = \max_{u \in S^{d-1}} \frac{\varepsilon_d}{\varepsilon_{d-1}} \frac{\lambda_{d-1}(B^\circ|u^\perp)\lambda_1(B^\circ \cap l_u)}{\lambda(B^\circ)}.$$

It has been proven that $\varepsilon_d/\varepsilon_{d-1} \leqslant w_B(\hat{I}_B^{HT})$ and $D_B(\hat{I}_B^{HT}) \leqslant d\varepsilon_d/\varepsilon_{d-1}$ (see [10]). We improve these bounds as follows (see Remark 3).

PROPOSITION 9. *If B is the unit ball of a d-dimensional Minkowski space $\mathbb{M}^d$, then*

$$\frac{2}{R(B, \hat{I}_B^{HT})} \leqslant w_B(\hat{I}_B^{HT}),$$

$$D_B(\hat{I}_B^{HT}) \leqslant \frac{2}{r(B,\hat{I}_B^{HT})}.$$

Proof. From Theorem 2, we obtain that for any $u \in S^{d-1}$

$$\frac{2}{R(B,\hat{I}_B^{HT})} \leqslant \frac{\varepsilon_d}{\varepsilon_{d-1}} \frac{\lambda_{d-1}(B^\circ|u^\perp)\lambda_1(B^\circ \cap l_u)}{\lambda(B^\circ)} \leqslant \frac{2}{r(B,\hat{I}_B^{HT})}.$$

The results follow by taking min and max with respect to $u \in S^{d-1}$ in the left-hand inequality and the right-hand inequality, respectively. $\square$

Using (5) we can also expand $w_B(\hat{I}_B^{Bus})$ as follows:

$$\begin{aligned} w_B(\hat{I}_B^{Bus}) &= \min_{u\in S^{d-1}} \frac{2w(\hat{I}_B^{Bus},u)}{w(B,u)} = \min_{u\in S^{d-1}} \frac{2h_{\hat{I}_B^{Bus}}(u)}{h_B(u)} \\ &= \frac{2\varepsilon_{d-1}}{\varepsilon_d} \min_{u\in S^{d-1}} \frac{\lambda(B)}{\rho_{IB}(u)h_B(u)} = \frac{4\varepsilon_{d-1}}{\varepsilon_d} \min_{u\in S^{d-1}} \frac{\lambda(B)}{\lambda_{d-1}(B\cap u^\perp)\lambda_1(B|l_u)}. \end{aligned}$$

Similarly,

$$D_B(\hat{I}_B^{Bus}) = \frac{4\varepsilon_{d-1}}{\varepsilon_d} \max_{u\in S^{d-1}} \frac{\lambda(B)}{\lambda_{d-1}(B\cap u^\perp)\lambda_1(B|l_u)}.$$

It has also been proven that $4\varepsilon_{d-1}/(d\varepsilon_d) \leqslant w_B(\hat{I}_B^{Bus})$, and $D_B(\hat{I}_B^{Bus}) \leqslant 4\varepsilon_{d-1}/\varepsilon_d$ (see again [10]). We establish the following improved bounds for $w_B(\hat{I}_B^{Bus})$ and $D_B(\hat{I}_B^{Bus})$ (see Remark 5).

PROPOSITION 10. *If B is the unit ball of a d-dimensional Minkowski space $\mathbb{M}^d$, then*

$$\frac{2}{R(B,\hat{I}_B^{Bus})} \leqslant w_B(\hat{I}_B^{Bus}),$$

$$D_B(\hat{I}_B^{Bus}) \leqslant \frac{2}{r(B,\hat{I}_B^{Bus})}.$$

Proof. From Theorem 4, we get for any $u \in S^{d-1}$

$$\frac{1}{R(B,\hat{I}_B^{Bus})}\frac{\varepsilon_d}{2\varepsilon_{d-1}} \leqslant \frac{\lambda(B)}{\lambda_{d-1}(B\cap u^\perp)\lambda_1(B|l_u)} \leqslant \frac{1}{r(B,\hat{I}_B^{Bus})}\frac{\varepsilon_d}{2\varepsilon_{d-1}}.$$

The results follow by taking min and max with respect to $u \in S^{d-1}$ in the left-hand inequality and the right-hand inequality, respectively. $\square$

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