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ON A GENERALIZATION OF A THEOREM OF LEVIN AND STEČKIN AND INEQUALITIES OF THE HERMITE–HADAMARD TYPE

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Abstract. We give new necessary and sufficient conditions for higher order convex ordering. These results generalize the Levin-Stečkin theorem (1960) on convex ordering. The obtained results can be useful in the study of the Hermite-Hadamard type inequalities and in particular inequalities between the quadrature operators.

1. Introduction and preliminaries

In this paper we give useful criteria for the verification of higher order convex orders. These criteria can be used to prove the Hermite-Hadamard type inequalities for higher order convex functions.

Let $f\colon [a,b] \to \mathbb{R}$ be a convex function ($a,b \in \mathbb{R}$, $a<b$). The following double inequality

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)\,dx \leqslant \frac{f(a)+f(b)}{2} \tag{1.1}$$

is known as the Hermite-Hadamard inequality (see [9] for many generalizations and applications of (1.1)).

In many papers, the Hermite-Hadamard type inequalities are studied based on the convex stochastic ordering properties (see, for example, [10, 20, 18, 21, 23, 22, 15]). In the paper [20], to get a simple proof of some known Hermite-Hadamard type inequalities as well as to obtaining new Hermite-Hadamard type inequalities, is used the Ohlin lemma on sufficient conditions for convex stochastic ordering. Recently, the Ohlin lemma is also used to study the inequalities of the Hermite-Hadamard type in [18, 21, 23, 22, 15]. In the papers [23, 22, 15], furthermore, to examine the Hermite-Hadamard type inequalities is used the Levin-Stečkin theorem [13] (see also [14]), which gives necessary and sufficient conditions for the stochastic convex ordering.

Let us recall some basic notions and results on the stochastic convex order (see, for example, [8]). As usual, F_X denotes the distribution function of a random variable X

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and μ_X is the distribution corresponding to X. For real valued random variables X,Y with a finite expectation, we say that X is dominated by Y in *convex ordering* sense if

$$\mathbb{E}f(X) \leqslant \mathbb{E}f(Y)$$

for all convex functions $f\colon \mathbb{R} \to \mathbb{R}$ (for which the expectations exist). In that case we write $X \leqslant_{cx} Y$, or $\mu_X \leqslant_{cx} \mu_Y$.

In the following Ohlin's lemma [16] are given sufficient conditions for convex stochastic ordering.

LEMMA 1.1. ([16]) *Let X,Y be two random variables such that $\mathbb{E}X = \mathbb{E}Y$. If the distribution functions F_X, F_Y cross exactly one time, i.e., for some x_0 holds*

$$F_X(x) \leqslant F_Y(x) \text{ if } x < x_0 \quad \text{and} \quad F_X(x) \geqslant F_Y(x) \text{ if } x > x_0, \tag{1.2}$$

then

$$\mathbb{E}f(X) \leqslant \mathbb{E}f(Y) \tag{1.3}$$

for all convex functions $f\colon \mathbb{R} \to \mathbb{R}$.

REMARK 1.1. The inequality (1.1) may be easily proved with the use of the Ohlin lemma (see[20]). Indeed, let X, Y, Z be three random variables with the distributions $\mu_X = \delta_{(a+b)/2}$, μ_Y which is equally distributed in $[a,b]$ and $\mu_Z = \frac{1}{2}(\delta_a + \delta_b)$, respectively. It is easy to see that the pairs (X,Y) and (Y,Z) satisfy the assumptions of the Ohlin lemma, then using (1.3), we obtain (1.1).

As we can see, the Ohlin lemma is a strong tool, however, it is worth noticing that in the case of some inequalities, the distribution functions cross more than once. Therefore a simple application of the Ohlin lemma is impossible and some additional idea is needed.

In the papers [22, 15], the authors used the Levin-Stečkin theorem [13] (see also [14], Theorem 4.2.7), concerning necessary and sufficient conditions for convex ordering of functions with bounded variation, which are distribution functions of signed measures.

THEOREM 1.1. ([13]) *Let $a,b \in \mathbb{R}$, $a < b$ and let $F_1, F_2\colon [a,b] \to \mathbb{R}$ be functions with bounded variation such that $F_1(a) = F_2(a)$. Then, in order that*

$$\int_a^b f(x)dF_1(x) \leqslant \int_a^b f(x)dF_2(x)$$

for all continuous convex functions $f\colon [a,b] \to \mathbb{R}$, it is necessary and sufficient that F_1 and F_2 verify the following three conditions:

$$F_1(b) = F_2(b), \tag{1.4}$$

$$\int_a^b F_1(x)dx = \int_a^b F_2(x)dx, \tag{1.5}$$

$$\int_a^x F_1(t)dt \leqslant \int_a^x F_2(t)dt \quad \text{for all} \quad x \in (a,b). \tag{1.6}$$

Szostok [22] used Theorem 1.1 to make an observation, which is more general than Ohlin's lemma and concerns the situation when the functions F_1 and F_2 have more crossing points than one. First we need the following definitions.

Define the number of sign changes of a function $\varphi\colon \mathbb{R} \to \mathbb{R}$ by

$$S^-(\varphi) = \sup\{S^-[\varphi(x_1), \varphi(x_2), \ldots, \varphi(x_k)] : x_1 < x_2 < \ldots x_k \in \mathbb{R}, k \in \mathbb{N}\},$$

where $S^-[y_1, y_2, \ldots, y_k]$ denotes the number of sign changes in the sequence $y_1, y_2, \ldots, y_k$ (zero terms are being discarded). Two real functions φ_1, φ_2 are said to have n crossing points (or cross each other n-times) if $S^-(\varphi_1 - \varphi_2) = n$. Let $a = x_0 < x_1 < \ldots < x_n < x_{n+1} = b$. We say that the functions φ_1, φ_2 crosses n-times at the points $x_1, x_2, \ldots, , x_n$ (or that $x_1, x_2, \ldots, , x_n$ are the points of sign changes of $\varphi_1 - \varphi_2$) if $S^-(\varphi_1 - \varphi_2) = n$ and there exist $a < \xi_1 < x_1 < \ldots < \xi_n < x_n < \xi_{n+1} < b$ such that $S^-[\xi_1, \xi_2, \ldots, \xi_{n+1}] = n$.

In [22] is given some useful modification of the Levin-Stečkin theorem [13], which can be rewritten in the following form.

LEMMA 1.2. ([22]) *Let $a, b \in \mathbb{R}$, $a < b$ and let $F_1, F_2\colon (a,b) \to \mathbb{R}$ be functions with bounded variation such that $F(a) = F(b) = 0$, $\int_a^b F(x)dx = 0$, where $F = F_2 - F_1$. Let $a < x_1 < \ldots < x_m < b$ be the points of sign changes of the function F. Assume that $F(t) \geqslant 0$ for $t \in (a, x_1)$.*

- *If m is even then the inequality*

$$\int_a^b f(x)dF_1(x) \leqslant \int_a^b f(x)dF_2(x) \tag{1.7}$$

  is not satisfied by all continuous convex functions $f\colon [a,b] \to \mathbb{R}$.

- *If m is odd, define A_i ($i = 0, 1, \ldots, m$, $x_0 = a$, $x_{m+1} = b$)*

$$A_i = \int_{x_i}^{xi+1} |F(x)|dx.$$

  Then the inequality (1.7) is satisfied for all continuous convex functions $f\colon [a,b] \to \mathbb{R}$ if, and only if, the following inequalities hold true:

$$\begin{gathered} A_0 \geqslant A_1, \\ A_0 + A_2 \geqslant A_1 + A_3, \\ \vdots \\ A_0 + A_2 + \ldots + A_{m-3} \geqslant A_1 + A_3 + \ldots + A_{m-2}. \end{gathered} \tag{1.8}$$

REMARK 1.2. Let

$$H(x) = \int_a^x F(t)dt.$$

Then the inequalities (1.8) are equivalent to the following inequalities

$$H(x_2) \geqslant 0,\ H(x_4) \geqslant 0,\ H(x_6) \geqslant 0, \ldots, H(x_{m-1}) \geqslant 0.$$

Now we are going to study Hermite-Hadamard type inequalities for higher-order convex functions. Many results on higher order generalizations of the Hermite-Hadamard type inequality one can found, among others, in [1, 2, 3, 4, 5, 9, 20, 21]. In recent papers [20, 21] the theorem of M. Denuit, C. Lefèvre and M. Shaked [8] on sufficient conditions for s-convex ordering was used to prove Hermite-Hadamard type inequalities for higher-order convex functions.

Let us review some notations. The convexity of n-th order (or n-convexity) was defined in terms of divided differences by Popoviciu [17], however, we will not state it here. Instead we list some properties of n-th order convexity which are equivalent to Popoviciu's definition (see [12]).

PROPOSITION 1.1. *A function $f\colon (a,b) \to \mathbb{R}$ is n-convex on (a,b) $(n \geqslant 1)$ if, and only if, its derivative $f^{(n-1)}$ exists and is convex on (a,b) (with the convention $f^{(0)}(x) = f(x)$).*

PROPOSITION 1.2. *Assume that $f\colon [a,b] \to \mathbb{R}$ is $(n+1)$-times differentiable on (a,b) and continuous on $[a,b]$ ($n \geqslant 1$). Then f is n-convex if, and only if, $f^{(n+1)}(x) \geqslant 0$, $x \in (a,b)$.*

For real valued random variables X,Y and any integer $s \geqslant 2$ we say that X is dominated by Y in *s-convex ordering* sense if $\mathbb{E} f(X) \leqslant \mathbb{E} f(Y)$ for all $(s-1)$-convex functions $f\colon \mathbb{R} \to \mathbb{R}$, for which the expectations exist ([8]). In that case we write $X \leqslant_{s-cx} Y$, or $\mu_X \leqslant_{s-cx} \mu_Y$, or $F_X \leqslant_{s-cx} F_Y$. Then the order $\leqslant_{2-cx}$ is just the usual convex order $\leqslant_{cx}$.

A very useful criterion for the verification of the s-convex order is given by Denuit, Lefèvre and Shaked in [8].

PROPOSITION 1.3. ([8]) *Let X and Y be two random variables such that $\mathbb{E}(X^j - Y^j) = 0$, $j = 1,2,\ldots,s-1$ ($s \geqslant 2$). If $S^-(F_X - F_Y) = s-1$ and the last sign of $F_X - F_Y$ is positive, then $X \leqslant_{s-cx} Y$.*

Proposition 1.3 can be rewritten in the following form.

PROPOSITION 1.4. ([8]) *Let X and Y be two random variables such that*

$$\mathbb{E}(X^j - Y^j) = 0, \quad j = 1,2,\ldots,s \ \ (s \geqslant 1).$$

If the distribution functions F_X and F_Y cross exactly s-times at points $x_1 < x_2 < \ldots < x_s$ and

$$(-1)^{s+1}(F_Y(x) - F_X(x)) \geqslant 0 \quad \textit{for all} \ \ x \leqslant x_1,$$

then

$$\mathbb{E} f(X) \leqslant \mathbb{E} f(Y) \tag{1.9}$$

for all s-convex functions $f\colon \mathbb{R} \to \mathbb{R}$.

Proposition 1.4 is a counterpart of the Ohlin lemma concerning convex ordering. This proposition gives sufficient conditions for s-convex ordering, and is very useful for the verification of higher order convex orders, however, it is worth noticing that in the case of some inequalities, the distribution functions cross more than s-times. Therefore a simple application of this proposition is impossible and some additional idea is needed.

In this paper we give a theorem on necessary and sufficient conditions for higher order convex stochastic ordering, which is a counterpart of the Levin-Stečkin theorem [13] concerning convex stochastic ordering. Based on this theorem, we give useful criteria for the verification of higher order convex stochastic ordering, which can be useful in the study of Hermite-Hadamard type inequalities for higher order convex functions, and in particular inequalities between the quadrature operators. Moreover, our criteria can be easier to checking of higher order convex orders, than those given in [8, 11].

2. Main results

Let $F_1, F_2\colon [a,b] \to \mathbb{R}$ be two functions with bounded variation and μ_1, μ_2 be the signed measures corresponding to F_1, F_2, respectively. We say that F_1 is dominated by F_2 in $(n+1)$-*convex ordering* sense $(n \geqslant 1)$ if

$$\int_a^b f(x)dF_1(x) \leqslant \int_a^b f(x)dF_2(x)$$

for all continuous n-convex functions $f\colon [a,b] \to \mathbb{R}$. In that case we write $F_1 \leqslant_{(n+1)-cx} F_2$, or $\mu_1 \leqslant_{(n+1)-cx} \mu_2$.

In the following theorem we give necessary and sufficient conditions for $(n+1)$-convex ordering of two functions with bounded variation.

THEOREM 2.1. *Let $a,b \in \mathbb{R}$, $a<b$, $n \in \mathbb{N}$ and let $F_1, F_2\colon [a,b] \to \mathbb{R}$ be two functions with bounded variation such that $F_1(a) = F_2(a)$. Then, in order that*

$$\int_a^b f(x)dF_1(x) \leqslant \int_a^b f(x)dF_2(x)$$

for all continuous n-convex functions $f\colon [a,b] \to \mathbb{R}$, it is necessary and sufficient that F_1 and F_2 verify the following conditions:

$$F_1(b) = F_2(b), \tag{2.1}$$

$$\int_a^b F_1(x)dx = \int_a^b F_2(x)dx, \tag{2.2}$$

$$\int_a^b \int_a^{x_{k-1}} \dots \int_a^{x_1} F_1(t)dt\,dx_1 \dots dx_{k-1} = \int_a^b \int_a^{x_{k-1}} \dots \int_a^{x_1} F_2(t)dt\,dx_1 \dots dx_{k-1} \quad \textit{for } k = 2,\dots,n, \tag{2.3}$$

$$
\begin{aligned}
&(-1)^{n+1}\int_a^x\int_a^{x_{n-1}}\cdots\int_a^{x_1}F_1(t)dtdx_1\ldots dx_{n-1}\\
&\qquad\leqslant(-1)^{n+1}\int_a^x\int_a^{x_{n-1}}\cdots\int_a^{x_1}F_2(t)dtdx_1\ldots dx_{n-1}\quad\textit{for all }x\in(a,b).
\end{aligned}\tag{2.4}
$$

First we prove the following lemma.

LEMMA 2.1. *Let* $F\colon[a,b]\to\mathbb{R}$ *be a function with bounded variation. Let* $f\colon[a,b]\to\mathbb{R}$ *be an* n*-convex function of the class* C^{n+1} *on* (a,b)*. Then*

$$
\int_a^b f(x)dF(x)=\Big[F(x)f(x)\Big]_{x=a}^{x=b}-\int_a^b F(x)f'(x)dx,\tag{2.5}
$$

$$
\int_a^b f(x)dF(x)=\Big[F(x)f(x)\Big]_{x=a}^{x=b}-\Big[\int_a^x F(t)dtf'(x)\Big]_{x=a}^{x=b}+\int_a^b\int_a^x F(t)dtf''(x)dx,\tag{2.6}
$$

$$
\begin{aligned}
\int_a^b f(x)dF(x)=&\Big[F(x)f(x)\Big]_{x=a}^{x=b}-\Big[\int_a^x F(t)dtf'(x)\Big]_{x=a}^{x=b}+\Big[\int_a^x\int_a^{x_1}F(t)dtdx_1f''(x)\Big]_{x=a}^{x=b}\\
&+\ldots+\Big[(-1)^k\int_a^x\int_a^{x_{k-1}}\cdots\int_a^{x_1}F(t)dtdx_1\ldots dx_{k-1}f^{(k)}(x)\Big]_{x=a}^{x=b}\\
&+(-1)^{k+1}\int_a^b\int_a^x\int_a^{x_{k-1}}\cdots\int_a^{x_1}F(t)dtdx_1\ldots dx_{k-1}f^{(k+1)}(x)dx\\
&\textit{for }\ k=2,\ldots,n.
\end{aligned}\tag{2.7}
$$

Proof. The proof is by induction. Integrating by parts and using the equalities $F(x)=(\int_a^x F(t)dt)'$ and $\int_a^x F(t)dt=(\int_a^x\int_a^{x_1}F(t)dtdx_1)'$, we obtain immediately (2.5), (2.6) and (2.7) for $k=2$.

Put

$$
I_k(x)=\int_a^x\int_a^{x_k}\cdots\int_a^{x_1}F(t)dtdx_1\ldots dx_k\quad\text{for }\ x\in(a,b),\quad k=1,2,\ldots,n.
$$

Then we have

$$
I_{k-1}(x)=(I_k(x))'\quad\text{for }\ x\in(a,b),\ k=1,2,\ldots,n.\tag{2.8}
$$

Assume that (2.7) holds for some $k=2,\ldots,n-1$. Integrating by parts and using (2.8), we obtain that the last summand in (2.7) can be rewritten in the form

$$
\begin{aligned}
&(-1)^{k+1}\int_a^b\int_a^x\int_a^{x_{k-1}}\cdots\int_a^{x_1}F(t)dtdx_1\ldots dx_{k-1}f^{(k+1)}(x)dx\\
&=(-1)^{k+1}\int_a^b I_{k-1}(x)f^{(k+1)}(x)dx\\
&=\Big[I_k(x)f^{(k+1)}(x)\Big]_{x=a}^{x=b}+(-1)^{k+2}\int_a^b I_k(x)f^{(k+2)}(x)dx,
\end{aligned}
$$

which implies that (2.7) holds for $k+1$. Thus (2.7) holds for all $k = 2,\dots,n$. The lemma is proved. □

The proof of Theorem 2.1 is an immediate consequence of the following lemma.

LEMMA 2.2. *Let* $F\colon [a,b] \to \mathbb{R}$ *be a function with bounded variation such that* $F(a) = 0$. *Then in order that*

$$\int_a^b f(x)dF(x) \geqslant 0 \tag{2.9}$$

for all continuous n*-convex functions* $f\colon [a,b] \to \mathbb{R}$, *it is necessary and sufficient that* F *satisfies the following conditions:*

$$F(b) = 0, \tag{2.10}$$

$$\int_a^b F(x)dx = 0, \tag{2.11}$$

$$\int_a^b \int_a^{x_{k-1}} \dots \int_a^{x_1} F(t)dt dx_1 \dots dx_{k-1} = 0 \quad \textit{for } k = 2,\dots,n, \tag{2.12}$$

$$(-1)^{n+1} \int_a^x \int_a^{x_{n-1}} \dots \int_a^{x_1} F(t)dt dx_1 \dots dx_{n-1} \geqslant 0 \quad \textit{for all } x \in (a,b). \tag{2.13}$$

Proof. Via an approximation argument we may restrict to the case when f is of the class $C^{n+1}((a,b))$.

We now prove the sufficiency. By Lemma 2.1, using (2.6) and (2.7) with $k = n$, and taking into account (2.10)–(2.12) we get

$$\int_a^b f(x)dF(x) = (-1)^{n+1} \int_a^b \int_a^x \int_a^{x_{n-1}} \dots \int_a^{x_1} F(t)dt dx_1 \dots dx_{n-1} f^{(n+1)}(x)dx. \tag{2.14}$$

Then, by (2.13) and Proposition 1.2, we obtain (2.9).

We now prove the necessity. The necessity of (2.10) follows by checking our statement for $f = 1$ and $f = -1$.

The necessity of (2.11) follows by checking our statement for $f(x) = x$ and $f(x) = -x$ and by using (2.10), (2.5).

The necessity of (2.12) we prove by induction on k. The necessity of (2.12) for $k = 2$ follows by checking our statement for $f(x) = x^2$ and $f(x) = -x^2$, using (2.6) and taking into account (2.10), (2.11). Assume, that the equality

$$\int_a^b \int_a^{x_{l-1}} \dots \int_a^{x_1} F(t)dt dx_1 \dots dx_{l-1} = 0 \tag{2.15}$$

holds for some $k = 2,\dots,n-1$ and all $l = 2,\dots,k$. Then we check our statement for $f(x) = x^{k+1}$ and $f(x) = -x^{k+1}$. Using (2.7) and taking into account (2.10), (2.11) and (2.15) for $l = 2,\dots,k$, we obtain (2.15) for $l = k+1$. Consequently, we obtain that (2.12) is satisfied for all $k = 2,\dots,n$.

By (2.7) with $k=n$ and taking into account (2.10)–(2.12), we obtain that (2.14) holds. Then, for the necessity of (2.13), notice that

$$(-1)^{n+1}\int_a^x\int_a^{x_{n-1}}\ldots\int_a^{x_1}F(t)dtdx_1\ldots dx_{n-1}<0$$

for some $x\in(a,b)$, yields an interval I around x on which this expression is still negative. Choosing f such that $f^{(n+1)}=0$ outside I, the equality (2.14) leads to a contradiction. Thus (2.13) is satisfied. The lemma is proved. □

REMARK 2.1. Let μ be the (signed) measure such that $\mu(du)=dF(u)$. In [11] can be found a characterization of measures μ such that (2.9) is satisfied for all generalized convex functions f. We note, that the characterization given in [11], in the case of n-convex functions, which are a special case of generalized convex functions, is different from that given in Lemma 2.2. Namely, in place of conditions (2.10)–(2.12) (in Lemma 2.2), in [11] are given so-called moment conditions.

COROLLARY 2.1. *Let μ_1, μ_2 be two signed measures on $\mathscr{B}(\mathbb{R})$, which are concentrated on (a,b), and such that $\int_a^b|x|^n\mu_i(dx)<\infty$, $i=1,2$. Then in order that*

$$\int_a^b f(x)d\mu_1(x)\leqslant\int_a^b f(x)d\mu_2(x)$$

for all continuous n-convex functions $f\colon[a,b]\to\mathbb{R}$, it is necessary and sufficient that μ_1, μ_2 verify the following conditions:

$$\mu_1((a,b))=\mu_2((a,b)), \tag{2.16}$$

$$\int_a^b x^k\mu_1(dx)=\int_a^b x^k\mu_2(dx)\quad \textit{for}\ \ k=1,\ldots,n, \tag{2.17}$$

$$\int_a^b (t-x)_+^n\mu_1(dt)=\int_a^b (t-x)_+^n\mu_2(dt)\quad \textit{for all}\ \ x\in(a,b), \tag{2.18}$$

where $y_+^n=\left\{max\{y,0\}\right\}^n$, $y\in\mathbb{R}$.

Proof. Let F_1, F_2 be the distribution functions corresponding to μ_1, μ_2, respectively. Then $\mu_i(dt)=dF_i(t)$, $i=1,2$. Since μ_1 and μ_2 are concentrated on (a,b), we have $F_1(a)=F_2(a)$. That (2.1) and (2.16) are equivalent is obvious. Put $F=F_2-F_1$. By (2.5) with $f(x)=x$, and taking into account (2.1), it follows that the conditions (2.2) and (2.17) for $k=1$ are equivalent. The equivalence of (2.3) and (2.17) for $k=2,\ldots,n$, can be proved, using (2.6) and (2.17), by induction on k. We omit the proof.

Next, by reversing the order of integration in (2.4), we obtain

$$\begin{aligned}
&(-1)^{n+1}\int_a^x\int_a^{x_{n-1}}\dots\int_a^{x_1}(F_2(t)-F_1(t))dtdx_1\dots dx_{n-1}\\
&=(-1)^{n+1}\int_a^x\int_a^{x_{n-1}}\dots\int_a^{x_1}F(t)dtdx_1\dots dx_{n-1}\\
&=(-1)^{n+1}\int_a^x\frac{(x-t)^n}{n!}dF(t)=(-1)^{n+1}(-1)^n\int_a^x\frac{(t-x)^n}{n!}dF(t)\\
&=-\int_a^x\frac{(t-x)^n}{n!}dF(t)=\int_a^b\frac{(t-x)^n}{n!}dF(t)-\int_a^x\frac{(t-x)^n}{n!}dF(t)\\
&=\int_x^b\frac{(t-x)^n}{n!}dF(t)=\int_a^b\frac{(t-x)_+^n}{n!}dF(t),
\end{aligned}$$

which implies the equivalence of (2.4) and (2.18). The corollary is proved. □

In [8] can be found the following necessary and sufficient conditions for the verification of the $(s+1)$-convex order.

PROPOSITION 2.1. ([8]) *If X and Y are two real valued random variables such that $\mathbb{E}|X|^s<\infty$ and $\mathbb{E}|Y|^s<\infty$, then*

$$\mathbb{E}f(X)\leqslant\mathbb{E}f(Y) \tag{2.19}$$

for all continuous s-convex functions $f\colon\mathbb{R}\to\mathbb{R}$ if, and only if,

$$\mathbb{E}X^k=\mathbb{E}Y^k\quad \text{for } k=1,2,\dots,s, \tag{2.20}$$

$$\mathbb{E}(X-t)_+^s\leqslant\mathbb{E}(Y-t)_+^s\quad \text{for all } t\in\mathbb{R}. \tag{2.21}$$

REMARK 2.2. The inequality (2.21) coincides with (2.19) for the spline function $f(x)=(x-t)_+^s$. Moreover, it is well known that s-convex function has the integral representation, such that the spline functions are the generating functions (see [19]).

REMARK 2.3. Note, that if the measures μ_X, μ_Y, corresponding to the random variables X, Y, respectively, occurring in Proposition 2.1, are concentrated on some interval $[a,b]$, then this proposition is an easy consequence of Corollary 2.1.

Note that Theorem 2.1 can be rewritten in the following form.

THEOREM 2.2. *Let $F_1,F_2\colon[a,b]\to\mathbb{R}$ be two functions with bounded variation such that $F_1(a)=F_2(a)$. Let*

$$\begin{aligned}
&H_0(t_0)=F_2(t_0)-F_1(t_0)\quad \text{for } t_0\in[a,b],\\
&H_k(t_k)=\int_a^{t_{k-1}}H_{k-1}(t_{k-1})dt_{k-1}\quad \text{for } t_k\in[a,b],\ k=1,2,\dots,n.
\end{aligned}$$

Then, in order that

$$\int_a^b f(x)dF_1(x)\leqslant\int_a^b f(x)dF_2(x)$$

for all continuous n-convex functions $f\colon [a,b] \to \mathbb{R}$, it is necessary and sufficient that the following conditions are satisfied:

$$\begin{aligned} H_k(b) &= 0 \quad \text{for } k = 0,1,2,\ldots,n, \\ (-1)^{n+1}H_n(x) &\geqslant 0 \quad \text{for all } x \in (a,b). \end{aligned}$$

REMARK 2.4. The functions $H_1,\ldots,H_n$, that appear in Theorem 2.2, can be obtained from the following formulas

$$H_n(x) = (-1)^{n+1}\int_a^b \frac{(t-x)_+^n}{n!}d(F_2(t)-F_1(t)), \tag{2.22}$$

$$H_{k-1}(x) = H_k^{'}(x), \quad k = 2,3,\ldots,n. \tag{2.23}$$

Note that the function $(-1)^{n+1}H_{n-1}$ that appears in Theorem 2.2 play a role similar to the role of the function $F = F_2 - F_1$ in Lemma 1.2. Consequently, from Theorem 2.2, Lemma 1.2 and Remarks 1.2, 2.4 we obtain immediately the following useful criterion for the verification of higher order convex ordering.

COROLLARY 2.2. *Let $F_1,F_2\colon [a,b] \to \mathbb{R}$ be functions with bounded variation such that $F_1(a) = F_2(a)$, $F_1(b) = F_2(b)$ and $H_k(b) = 0$ $(k = 1,2,\ldots,n)$, where $H_k(x)$ $(k = 1,2,\ldots,n)$ are given by (2.22) and (2.23). Let $a < x_1 < \ldots < x_m < b$ be the points of sign changes of the function H_{n-1} and let $(-1)^{n+1}H_{n-1}(x) \geqslant 0$ for $x \in (a,x_1)$.*

- *If m is even then the inequality*

$$\int_a^b f(x)dF_1(x) \leqslant \int_a^b f(x)dF_2(x), \tag{2.24}$$

  is not satisfied by all continuous n-convex functions $f\colon [a,b] \to \mathbb{R}$.

- *If m is odd, then the inequality (2.24) is satisfied for all continuous n-convex functions $f\colon [a,b] \to \mathbb{R}$ if, and only if,*

$$(-1)^{n+1}H_n(x_2) \geqslant 0,\ (-1)^{n+1}H_n(x_4) \geqslant 0,\ \ldots,\ (-1)^{n+1}H_n(x_{m-1}) \geqslant 0. \tag{2.25}$$

In the numerical analysis are studied some inequalities, which are connected with quadrature operators. These inequalities, called extremalities, are a particular case of the Hermite-Hadamard type inequalities. Many extremalities are known in the numerical analysis (cf. [1], [7], [6] and the references therein). The numerical analysts prove them using the suitable differentiability assumptions. As proved Wąsowicz in the papers [24], [25], [27], for convex functions of higher order some extremalities can be obtained without assumptions of this kind, using only the higher order convexity itself. The support-type properties play here the crucial role. As we show in [20, 21], some extremalities can be proved using a probabilistic characterization.The extremalities, which we study are known, however our method using the Ohlin lemma [16] and the Denuit-Lefèvre-Shaked theorem [8] on sufficient conditions for the convex stochastic ordering seems to be quite easy. It is worth noting that, these theorems do not apply to proving some extremalities (see [20, 21]). In these cases can be useful results given in this paper.

For a function $f:[-1,1]\to\mathbb{R}$ we consider two operators

$$C(f) := \tfrac{1}{3}\Big(f\big(-\tfrac{\sqrt{2}}{2}\big)+f(0)+f\big(\tfrac{\sqrt{2}}{2}\big)\Big),$$

$$\mathscr{L}_4(f) := \tfrac{1}{12}\Big(f(-1)+f(1)\Big)+\tfrac{5}{12}\Big(f\big(-\tfrac{\sqrt{5}}{5}\big)+f\big(\tfrac{\sqrt{5}}{5}\big)\Big),$$

connected with Chebyshev and Lobatto quadratures, respectively. Wąsowicz [24], [26] proved that

$$C(f)\leqslant \mathscr{L}_4(f),\quad \text{if } f \text{ is } 3\text{-convex.} \tag{2.26}$$

The proof given in [24] is rather complicated. This was done using computer software. In [26] can be found a new proof, without the use of any computer software, based on the spline approximation of convex functions of higher order. Using Corollary 2.2, we give a new proof of (2.26), which is simpler than that given in [26].

Since for the random variables X and Y with the distributions

$$\mu_X = \tfrac{1}{3}\Big(\delta_{-\frac{\sqrt{2}}{2}}+\delta_0+\delta_{\frac{\sqrt{2}}{2}}\Big),$$

$$\mu_Y = \tfrac{1}{12}\Big(\delta_{-1}+\delta_1\Big)+\tfrac{5}{12}\Big(\delta_{-\frac{\sqrt{5}}{5}}+\delta_{\frac{\sqrt{5}}{5}}\Big),$$

respectively, we have

$$C(f)=E[f(X)],\quad \mathscr{L}_4(f)=E[f(Y)],$$

it follows that the inequality (2.26) can be rewritten in terms of higher order convex orderings

$$X\leqslant_{4-cx} Y. \tag{2.27}$$

It is worth noting, that Proposition 2.1 of Denuit, Lefèvre and Shaked does not apply to proving (2.27), because the distribution functions F_X and F_Y cross exactly 5-times. We prove the inequality (2.27) by using Corollary 2.2.

We have $F_1=F_X$, $F_2=F_Y$, $H_0=F=F_Y-F_X$. By (2.22) and (2.23), we obtain

$$H_3(x) = \tfrac{1}{72}\Big\{(-1-x)_+^3+(1-x)_+^3+5\Big[\Big(-\tfrac{\sqrt{5}}{5}-x\Big)_+^3+\Big(\tfrac{\sqrt{5}}{5}-x\Big)_+^3\Big]$$
$$-4\Big[(-1-x)_+^3+\Big(-\tfrac{\sqrt{2}}{2}-x\Big)_+^3+(-x)_+^3+\Big(\tfrac{\sqrt{2}}{2}-x\Big)_+^3\Big]\Big\},$$

$$H_2(x) = \tfrac{1}{24}\Big\{-(-1-x)_+^2-(1-x)_+^2-5\Big[\Big(-\tfrac{\sqrt{5}}{5}-x\Big)_+^2+\Big(\tfrac{\sqrt{5}}{5}-x\Big)_+^2\Big]$$
$$+4\Big[(-1-x)_+^2+\Big(-\tfrac{\sqrt{2}}{2}-x\Big)_+^2+(-x)_+^2+\Big(\tfrac{\sqrt{2}}{2}-x\Big)_+^2\Big]\Big\}.$$

Similarly, from the equality $H_1(x) = H_2^{'}(x)$ can be obtained $H_1(x)$. We compute that $x_1 = -1 - \sqrt{5} + 2\sqrt{2}$, $x_2 = 0$, $x_3 = 1 + \sqrt{5} - 2\sqrt{2}$ are the points of sign changes of the function $H_2(x)$. It is not difficult to check that the assumptions of Corollary 2.2 are satisfied. Since

$$(-1)^{3+1} H_3(x_2) = (-1)^{3+1} H_3(0) = \tfrac{1}{72} + \tfrac{\sqrt{5}}{360} - \tfrac{\sqrt{2}}{72} > 0,$$

it follows that the inequalities (2.25) are satisfied. From Corollary 2.2 we conclude that the relation (2.27) hold.

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