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IMPROVED JENSEN'S INEQUALITY

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Abstract. In this article we present refinements of Jensen's inequality and its reversal for convex functions, by adding as many refining terms as we wish. Then as a standard application, we present several refinements and reverses of well known mean inequalities.

1. Introduction

In the sequel, $\mathbb{I}$ will denote an open interval of the real line. A function $f:\mathbb{I}\to\mathbb{R}$ is said to be convex if, for all $x_1,x_2\in\mathbb{I}$ and $\lambda\in(0,1)$, we have

$$f(\lambda x_1+(1-\lambda)x_2)\leqslant \lambda f(x_1)+(1-\lambda)f(x_2).$$

By induction, we obtain the celebrated Jensen's inequality that if $\mathbf{x}=\{x_1,\cdots,x_n\}\subset\mathbb{I}$ and $\mathbf{p}=\{p_1,\cdots,p_n\}\subset(0,1)$ satisfying $\sum_{i=1}^n p_i=1$, then

$$f\left(\sum_{i=1}^n p_ix_i\right)\leqslant\sum_{i=1}^n p_if(x_i). \tag{1}$$

This inequality for convex functions is of extreme significance in the theory of functions.

Refining this inequality by finding intermediate terms or by adding some positive quantities to the left side has taken the attention of numerous researchers. We refer the reader to [1, 2, 5, 6, 7, 8, 13, 14, 17, 18] and their references for the motivation, applications and different refinements of Jensen's inequality.

The main goal of this paper is to refine Jensen's inequality by finding as many refining terms as we wish. More precisely, we prove that given $\mathbf{x}=\{x_1^{(1)},\cdots,x_n^{(1)}\}\subset\mathbb{I},\mathbf{p}=\{p_1^{(1)},\cdots,p_n^{(1)}\}\subset(0,1)$ satisfying $\sum_{i=1}^n p_i^{(1)}=1$, and $N\in\mathbb{N}$, we have

$$f\left(\sum_{i=1}^n p_i^{(1)}x_i^{(1)}\right)+n\sum_{k=1}^N p_{\min}^{(k)}\left(\frac{1}{n}\sum_{i=1}^n f(x_i^{(k)})-f\left(\frac{1}{n}\sum_{i=1}^n x_i^{(k)}\right)\right)\leqslant\sum_{i=1}^n p_i^{(1)}f(x_i^{(1)}),$$

for certain $p_i^{(k)}$ and $x_i^{(k)}$. See Theorem 1, Section 2.1 below for the details.

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Once this refinement is proved, we use it to obtain a reversed version, where we have $(\geqslant)$ instead of $(\leqslant)$ in the above inequality. See Theorem 3, Section 2.2 for the details.

One motivation of this work is the following result, which was proved in [17] as a refinement and a reverse of Jensen's inequality, where one refining term has been found.

LEMMA 1. *Let* $f:\mathbb{I}\longrightarrow\mathbb{R}$ *be convex,* $\{x_1,\cdots,x_n\}\subset\mathbb{I}$ *and* $\{p_1,\cdots,p_n\}\subset(0,1)$ *be such that* $\sum_{i=1}^n p_i=1$. *Then*

$$f\left(\sum_{i=1}^n p_ix_i\right)+np_{\min}\left(\frac{1}{n}\sum_{i=1}^n f(x_i)-f\left(\frac{1}{n}\sum_{i=1}^n x_i\right)\right)\leqslant\sum_{i=1}^n p_if(x_i). \tag{2}$$

and

$$f\left(\sum_{i=1}^n p_ix_i\right)+np_{\max}\left(\frac{1}{n}\sum_{i=1}^n f(x_i)-f\left(\frac{1}{n}\sum_{i=1}^n x_i\right)\right)\geqslant\sum_{i=1}^n p_if(x_i), \tag{3}$$

where $p_{\min}=\min\{p_1,\cdots,p_n\}$ *and* $p_{\max}=\max\{p_1,\cdots,p_n\}$.

Notice that when $p_i=\frac{1}{n}$ for all i, then the above inequalities become sharp equalities.

Another main motivation of this work is the extensive study of means inequalities in the literature. As we shall see, almost all these refinements and reverses will be immediate consequences of our general results for convex functions. See Section 2.3 below.

2. Main results

2.1. Refining Jensen's inequality

Our first result is the following refinement of Jensen's inequality. Before stating this inequality, we justify the used notations. Let $\mathbf{p}^{(1)}=\{p_1^{(1)},\cdots,p_n^{(1)}\}\subset(0,1)$ be a convex sequence, that is $\sum_{i=1}^n p_i^{(1)}=1$, and let $p_{\min}^{(1)}=\min\{p_i^{(1)}:1\leqslant i\leqslant n\}$. Consider $J_1=\{i:p_i^{(1)}=p_{\min}^{(1)}\}$ and let $|J_1|$ be the cardinality of J_1.

Now for $k\geqslant 2$, we define a new sequence $\mathbf{p}^{(k)}$ inductively as follows

$$p_i^{(k)}=\begin{cases} p_i^{(k-1)}-p_{\min}^{(k-1)}, & p_i^{(k-1)}\neq p_{\min}^{(k-1)}\\ \frac{1}{|J_{k-1}|}np_{\min}^{(k-1)}, & p_i^{(k-1)}=p_{\min}^{(k-1)}\end{cases}\ \text{where } J_{k-1}=\left\{i:p_i^{(k-1)}=p_{\min}^{(k-1)}\right\}, \tag{4}$$

and for $k\geqslant 1$, $p_{\min}^{(k)}=\min\{p_1^{(k)},\cdots,p_n^{(k)}\}$.

Moreover, given $\mathbf{x}^{(1)}=\{x_1^{(1)},\cdots,x_n^{(1)}\}\subset\mathbb{I}$, we construct a new sequence $\mathbf{x}^{(k)}$ as follows

$$x_i^{(k)}=\begin{cases} x_i^{(k-1)}, & p_i^{(k-1)}\neq p_{\min}^{(k-1)}\\ \frac{1}{n}\sum_{i=1}^n x_i^{(k-1)}, & p_i^{(k-1)}=p_{\min}^{(k-1)}\end{cases},\ 1\leqslant i\leqslant n. \tag{5}$$

We emphasize that the order of the $\{x_i^{(1)}\}$ follows the order they are associated with the $\{p_i^{(1)}\}$. That is, $x_1^{(1)}$ is the value multiplied with $p_1^{(1)}$, and so on. Keeping these notations in mind, we state our first main result, which refines (2).

THEOREM 1. *Let* $f:\mathbb{I}\longrightarrow\mathbb{R}$ *be convex,* $\{x_1^{(1)},\cdots,x_n^{(1)}\}\subset\mathbb{I}$ *and* $\{p_1^{(1)},\cdots,p_n^{(1)}\}\subset(0,1)$ *be such that* $\sum_{i=1}^n p_i^{(1)}=1$*. Then for every* $N\in\mathbb{N}$*, we have*

$$f\left(\sum_{i=1}^n p_i^{(1)}x_i^{(1)}\right)+n\sum_{k=1}^N p_{\min}^{(k)}\left(\frac{1}{n}\sum_{i=1}^n f(x_i^{(k)})-f\left(\frac{1}{n}\sum_{i=1}^n x_i^{(k)}\right)\right)\leqslant\sum_{i=1}^n p_i^{(1)}f(x_i^{(1)}), \quad (6)$$

where $p_i^{(k)}$ *and* $x_i^{(k)}$ *are as in (4) and (5).*

Proof. We prove this by induction on N. For $N=1$, the result follows from Lemma 1. Now assume that (6) holds for some $N\in\mathbb{N}$. We emphasize here that this means, given any convex sequence $\{q_i^{(1)}:1\leqslant i\leqslant n\}$ and any elements $\{y_i^{(1)}:1\leqslant i\leqslant n\}\subset\mathbb{I}$, we have the inductive step

$$f\left(\sum_{i=1}^n q_i^{(1)}y_i^{(1)}\right)+n\sum_{k=1}^N q_{\min}^{(k)}\left(\frac{1}{n}\sum_{i=1}^n f(y_i^{(k)})-f\left(\frac{1}{n}\sum_{i=1}^n y_i^{(k)}\right)\right)\leqslant\sum_{i=1}^n q_i^{(1)}f(y_i^{(1)}). \quad (7)$$

Then

$$\begin{aligned}
I&:=\sum_{i=1}^n p_i^{(1)}f\left(x_i^{(1)}\right)-n\,p_{\min}^{(1)}\left(\frac{1}{n}\sum_{i=1}^n f\left(x_i^{(1)}\right)-f\left(\frac{1}{n}\sum_{i=1}^n x_i^{(1)}\right)\right)\\
&=\sum_{i=1}^n p_i^{(1)}f\left(x_i^{(1)}\right)-p_{\min}^{(1)}\sum_{i=1}^n f\left(x_i^{(1)}\right)+n\,p_{\min}^{(1)}f\left(\frac{1}{n}\sum_{i=1}^n x_i^{(1)}\right)\\
&=\sum_{\substack{i=1\\ p_i^{(1)}\neq p_{\min}^{(1)}}}^n\left(p_i^{(1)}-p_{\min}^{(1)}\right)f\left(x_i^{(1)}\right)+\sum_{p_i^{(1)}=p_{\min}^{(1)}}\left(\frac{1}{|J_1|}n\,p_{\min}^{(1)}f\left(\frac{1}{n}\sum_{i=1}^n x_i^{(1)}\right)\right)\\
&=\sum_{i=1}^n p_i^{(2)}f\left(x_i^{(2)}\right),
\end{aligned} \quad (8)$$

where the last line is obtained from the definitions of $(p_i^{(k)})$ and $(x_i^{(k)})$ in (4) and (5). Now, for convenience denote $p_i^{(2)}$ by $q_i^{(1)}$ and $x_i^{(2)}$ by $y_i^{(1)}$.

Notice that

$$\begin{aligned}
\sum_{i=1}^n q_i^{(1)}&=\sum_{i=1}^n p_i^{(2)}\\
&=\sum_{i\notin J_1}\left(p_i^{(1)}-p_{\min}^{(1)}\right)+\sum_{i\in|J_1|}\frac{np_{\min}^{(1)}}{|J_1|}
\end{aligned}$$

$$= \sum_{i=1}^{n} p_i^{(1)} - \sum_{i\in J_1} p_i^{(1)} - \sum_{i\notin J_1} p_{\min}^{(1)} + n\, p_{\min}^{(1)}$$
$$= 1 - |J_1|\, p_{\min}^{(1)} - (n - |J_1|) p_{\min}^{(1)} + n\, p_{\min}^{(1)}$$
$$= 1.$$

Consequently, we may apply the inductive step (7) on (8) to get

$$\begin{aligned} I &= \sum_{i=1}^{n} q_i^{(1)} f\left(y_i^{(1)}\right) \\ &\geqslant f\left(\sum_{i=1}^{n} q_i^{(1)} y_i^{(1)}\right) + n \sum_{k=1}^{N} q_{\min}^{(k)} \left(\frac{1}{n}\sum_{i=1}^{n} f(y_i^{(k)}) - f\left(\frac{1}{n}\sum_{i=1}^{n} y_i^{(k)}\right)\right). \end{aligned} \tag{9}$$

Now,

$$\begin{aligned} \sum_{i=1}^{n} q_i^{(1)} y_i^{(1)} &= \sum_{i=1}^{n} p_i^{(2)} x_i^{(2)} \\ &= \sum_{\substack{i=1\\ i\notin J_1}}^{n} \left(p_i^{(1)} - p_{\min}^{(1)}\right) x_i^{(1)} + \sum_{j\in J_1} \left(\frac{n\, p_{\min}^{(1)}}{|J_1|} \sum_{i=1}^{n} \frac{x_i^{(1)}}{n}\right) \\ &= \sum_{\substack{i=1\\ i\notin J_1}}^{n} p_i^{(1)} x_i^{(1)} - \sum_{\substack{i=1\\ i\notin J_1}}^{n} p_{\min}^{(1)} x_i^{(1)} + \sum_{i=1}^{n} p_{\min}^{(1)} x_i^{(1)} \\ &= \sum_{\substack{i=1\\ i\notin J_1}}^{n} p_i^{(1)} x_i^{(1)} + \sum_{\substack{i=1\\ i\in J_1}}^{n} p_{\min}^{(1)} x_i^{(1)} \\ &= \sum_{i=1}^{n} p_i^{(1)} x_i^{(1)}. \end{aligned} \tag{10}$$

Moreover, since $q_i^{(1)} = p_i^{(2)}$ and $y_i^{(1)} = x_i^{(2)}$, we have $q_i^{(k)} = p_i^{(k+1)}$ and $y_i^{(k)} = x_i^{(k+1)}$ for $k \geqslant 1$. Therefore, invoking (10) in (9), we get

$$\begin{aligned} I &= \sum_{i=1}^{n} p_i^{(1)} f\left(x_i^{(1)}\right) - n\, p_{\min}^{(1)} \left(\frac{1}{n}\sum_{i=1}^{n} f\left(x_i^{(1)}\right) - f\left(\frac{1}{n}\sum_{i=1}^{n} x_i^{(1)}\right)\right) \\ &\geqslant f\left(\sum_{i=1}^{n} p_i^{(1)} x_i^{(1)}\right) + n \sum_{k=1}^{N} p_{\min}^{(k+1)} \left(\frac{1}{n}\sum_{i=1}^{n} f(x_i^{(k+1)}) - f\left(\frac{1}{n}\sum_{i=1}^{n} x_i^{(k+1)}\right)\right) \\ &= f\left(\sum_{i=1}^{n} p_i^{(1)} x_i^{(1)}\right) + n \sum_{k=2}^{N+1} p_{\min}^{(k)} \left(\frac{1}{n}\sum_{i=1}^{n} f(x_i^{(k)}) - f\left(\frac{1}{n}\sum_{i=1}^{n} x_i^{(k)}\right)\right). \end{aligned}$$

That is,

$$f\left(\sum_{i=1}^{n} p_i^{(1)} x_i^{(1)}\right) + n \sum_{k=1}^{N+1} p_{\min}^{(k)} \left(\frac{1}{n}\sum_{i=1}^{n} f(x_i^{(k)}) - f\left(\frac{1}{n}\sum_{i=1}^{n} x_i^{(k)}\right)\right) \leqslant \sum_{i=1}^{n} p_i^{(1)} f\left(x_i^{(1)}\right),$$

completing the proof. □

This enables us to write the following inequality for log-convex functions, refining the corresponding result from [17].

COROLLARY 1. *Let* $f:\mathbb{I}\to\mathbb{R}^+$ *be log-convex,* $\{x_1^{(1)},\cdots,x_n^{(1)}\}\subset\mathbb{I}$ *and* $\{p_1^{(1)},\cdots,p_n^{(1)}\}\subset(0,1)$ *be such that* $\sum_{i=1}^n p_i^{(1)}=1$*. Then for every* $N\in\mathbb{N}$

$$\prod_{k=1}^{N}\left(\frac{\prod_{i=1}^n f^{\frac{1}{n}}(x_i^{(k)})}{f\left(\frac{1}{n}\sum_{i=1}^n x_i^{(k)}\right)}\right)^{n\,p_{\min}^{(k)}}\leqslant\frac{\prod_{i=1}^n f^{p_i^{(1)}}(x_i^{(1)})}{f\left(\sum_{i=1}^n p_i^{(1)}x_i^{(1)}\right)}. \tag{11}$$

Proof. Since f is log-convex, $g(x)=\log f(x)$ is convex. Applying Theorem 1 on g implies the required inequality. □

We discuss the equality possibility in (6).

THEOREM 2. *Let* $f:\mathbb{I}\to\mathbb{R}$ *be convex,* $\{p_i^{(1)}\}$ *be a convex sequence and* $\{x_i^{(1)}\}\subset\mathbb{I}$*. For* $k\geqslant 2$*, let* $\{p_i^{(k)}\}$ *be as in (4). If for some* $M\in\mathbb{N}$*,* $\{p_i^{(M)}\}$ *is the constant sequence, that is* $p_i^{(M)}=\frac{1}{n}\forall i$*, then*

$$f\left(\sum_{i=1}^n p_i^{(1)}x_i^{(1)}\right)+n\sum_{k=1}^{M}p_{\min}^{(k)}\left(\frac{1}{n}\sum_{i=1}^n f(x_i^{(k)})-f\left(\frac{1}{n}\sum_{i=1}^n x_i^{(k)}\right)\right)=\sum_{i=1}^n p_i^{(1)}f(x_i^{(1)}).$$

Proof. Observe that the result follows immediately when $M=1$, hence we may assume that $M\geqslant 2$. We prove first that, for $N\in\mathbb{N}$,

$$\sum_{i=1}^n p_i^{(1)}f(x_i^{(1)})-n\sum_{k=1}^{N}p_{\min}^{(k)}\left(\frac{1}{n}\sum_{i=1}^n f(x_i^{(k)})-f\left(\frac{1}{n}\sum_{i=1}^n x_i^{(k)}\right)\right)=\sum_{i=1}^n p_i^{(N+1)}f(x_i^{(N+1)}),$$

by induction on N. When $N=1$, this has been proved in (8). Assuming the truth of our claim for a certain N, we have

$$\begin{aligned}
&\sum_{i=1}^n p_i^{(1)}f(x_i^{(1)})-n\sum_{k=1}^{N+1}p_{\min}^{(k)}\left(\frac{1}{n}\sum_{i=1}^n f(x_i^{(k)})-f\left(\frac{1}{n}\sum_{i=1}^n x_i^{(k)}\right)\right)\\
&=\sum_{i=1}^n p_i^{(1)}f(x_i^{(1)})-n\sum_{k=1}^{N}p_{\min}^{(k)}\left(\frac{1}{n}\sum_{i=1}^n f(x_i^{(k)})-f\left(\frac{1}{n}\sum_{i=1}^n x_i^{(k)}\right)\right)\\
&\quad-np_{\min}^{(N+1)}\left(\frac{1}{n}\sum_{i=1}^n f(x_i^{(N+1)})-f\left(\frac{1}{n}\sum_{i=1}^n x_i^{(N+1)}\right)\right)\\
&=\sum_{i=1}^n p_i^{(N+1)}f(x_i^{(N+1)})-np_{\min}^{(N+1)}\left(\frac{1}{n}\sum_{i=1}^n f(x_i^{(N+1)})-f\left(\frac{1}{n}\sum_{i=1}^n x_i^{(N+1)}\right)\right)\\
&=\sum_{i=1}^n p_i^{(N+2)}f(x_i^{(N+2)}),
\end{aligned}$$

where in the last two lines, we have applied the inductive step first, then the fact that our claim is true when $N = 1$.

In particular, when $N = M - 1$, we have

$$\sum_{i=1}^{n} p_i^{(1)} f(x_i^{(1)}) - n \sum_{k=1}^{M-1} p_{\min}^{(k)} \left(\frac{1}{n} \sum_{i=1}^{n} f(x_i^{(k)}) - f\left(\frac{1}{n} \sum_{i=1}^{n} x_i^{(k)} \right) \right) = \sum_{i=1}^{n} p_i^{(M)} f(x_i^{(M)}), \quad (12)$$

Now, if $p_i^{(M)} = \frac{1}{n}, \forall i$, we have

$$\begin{aligned} & f\left(\sum_{i=1}^{n} p_i^{(M)} x_i^{(M)} \right) + n p_{\min}^{(M)} \left(\frac{1}{n} \sum_{i=1}^{n} f(x_i^{(M)}) - f\left(\frac{1}{n} \sum_{i=1}^{n} x_i^{(M)} \right) \right) \\ &= \sum_{i=1}^{n} p_i^{(M)} f(x_i^{(M)}) \text{ (now use (12))} \\ &= \sum_{i=1}^{n} p_i^{(1)} f(x_i^{(1)}) - n \sum_{k=1}^{M-1} p_{\min}^{(k)} \left(\frac{1}{n} \sum_{i=1}^{n} f(x_i^{(k)}) - f\left(\frac{1}{n} \sum_{i=1}^{n} x_i^{(k)} \right) \right), \end{aligned} \quad (13)$$

which implies

$$f\left(\sum_{i=1}^{n} p_i^{(M)} x_i^{(M)} \right) + n \sum_{k=1}^{M} p_{\min}^{(k)} \left(\frac{1}{n} \sum_{i=1}^{n} f(x_i^{(k)}) - f\left(\frac{1}{n} \sum_{i=1}^{n} x_i^{(k)} \right) \right) = \sum_{i=1}^{n} p_i^{(1)} f(x_i^{(1)}). \quad (14)$$

By induction, it is easy to show that (see for example (10)),

$$\sum_{i=1}^{n} p_i^{(M)} x_i^{(M)} = \sum_{i=1}^{n} p_i^{(1)} x_i^{(1)}, \quad M \geqslant 2,$$

so that (14) becomes

$$f\left(\sum_{i=1}^{n} p_i^{(1)} x_i^{(1)} \right) + n \sum_{k=1}^{M} p_{\min}^{(k)} \left(\frac{1}{n} \sum_{i=1}^{n} f(x_i^{(k)}) - f\left(\frac{1}{n} \sum_{i=1}^{n} x_i^{(k)} \right) \right) = \sum_{i=1}^{n} p_i^{(1)} f(x_i^{(1)}).$$

This completes the proof. □

2.2. Refining the reversed version

Now we use Theorem 1 to obtain the following refined version of (3). Before proceeding, we need to deal with some notations. Let $p_1^{(1)} \leqslant \cdots \leqslant p_n^{(1)}$ be a convex sequence. That is, $\sum_{i=1}^{n} p_i^{(1)} = 1$. We will refer to $\mathbf{p}^{(1)} = \{p_i^{(1)}\}$ as a monotone convex sequence. Assume also that $p_{\max}^{(1)}$ is attained ℓ times, meaning $p_{n-\ell+1}^{(1)} = \cdots = p_n^{(1)} = p_{\max}^{(1)}$. Since $\sum_{i=1}^{n} p_i^{(1)} = 1$, it follows that

$$p_n^{(1)} = \frac{1 - \sum_{j=1}^{n-\ell} p_j^{(1)}}{\ell}. \quad (15)$$

These notations will be used throughout this section.

It should be noted that if $p_i^{(1)} = \frac{1}{n}, \forall i$, then (3) is sharp. Thus, our concern is when the sequence $\mathbf{p}^{(1)}$ is not the constant sequence, so that $\ell \neq n$.

LEMMA 2. *Let $f : \mathbb{I} \to \mathbb{R}$ be convex, $\mathbf{p}^{(1)}$ be a nonconstant monotone convex sequence and $\mathbf{x}^{(1)} = \{x_i^{(1)}\} \subset \mathbb{I}$. Define the new sequences*

$$q_i^{(1)} = \begin{cases} \frac{1-\sum_{j=1}^{n-\ell} p_j^{(1)} - \ell p_i^{(1)}}{n-\ell} = \ell \frac{p_n^{(1)} - p_i^{(1)}}{n-\ell}, & 1 \leqslant i \leqslant n-\ell \\ \frac{1}{\ell}\frac{n}{n-\ell}\sum_{j=1}^{n-\ell} p_j^{(1)}, & n-\ell+1 \leqslant i \leqslant n \end{cases} \tag{16}$$

and

$$y_i^{(1)} = \begin{cases} x_i^{(1)}, & 1 \leqslant i \leqslant n-\ell \\ \frac{1}{n}\sum_{i=1}^{n} x_i^{(1)}, & n-\ell+1 \leqslant i \leqslant n \end{cases}. \tag{17}$$

Then, for $N \in \mathbb{N}$,

$$\begin{aligned} &\sum_{i=1}^{n-\ell} \frac{1-\sum_{j=1}^{n-\ell} p_j^{(1)} - \ell p_i^{(1)}}{n-\ell} f(x_i^{(1)}) + \left(\frac{n}{n-\ell}\sum_{j=1}^{n-\ell} p_j^{(1)}\right) f\left(\frac{1}{n}\sum_{i=1}^{n} x_i^{(1)}\right) \\ &\geqslant f\left(\sum_{i=1}^{n} q_i^{(1)} y_i^{(1)}\right) + M_N(f;\mathbf{x},\mathbf{p}), \end{aligned} \tag{18}$$

where

$$M_N(f;\mathbf{x},\mathbf{p}) = n\sum_{k=1}^{N} q_{\min}^{(k)} \left(\frac{1}{n}\sum_{i=1}^{n} f(y_i^{(k)}) - f\left(\frac{1}{n}\sum_{i=1}^{n} y_i^{(k)}\right)\right),$$

$\{q_i^{(k)}\}$ and $\{y_i^{(k)}\}$ are generated from $\{q_i^{(1)}\}$ and $\{y_i^{(1)}\}$, defined in (16) and (17), via (4) and (5).

Proof. Notice that

$$\begin{aligned} \sum_{i=1}^{n} q_i^{(1)} &= \sum_{i=1}^{n-\ell} \frac{1-\sum_{j=1}^{n-\ell} p_j^{(1)} - \ell p_i^{(1)}}{n-\ell} + \sum_{i=n-\ell+1}^{n} \left(\frac{1}{\ell}\frac{n}{n-\ell}\sum_{j=1}^{n-\ell} p_j^{(1)}\right) \\ &= \frac{1}{n-\ell}\left[(n-\ell) - (n-\ell)\sum_{j=1}^{n-\ell} p_j^{(1)} - \ell\sum_{j=1}^{n-\ell} p_j^{(1)} + n\sum_{j=1}^{n-\ell} p_j^{(1)}\right] \\ &= 1. \end{aligned}$$

That is, $(q_i^{(1)})$ is a convex sequence. Now, direct computations show that

$$\sum_{i=1}^{n} q_i^{(1)} f(y_i^{(1)}) = \sum_{i=1}^{n-\ell} \frac{1-\sum_{j=1}^{n-\ell} p_j^{(1)} - \ell p_i^{(1)}}{n-\ell} f(x_i^{(1)}) + \left(\frac{n}{n-\ell}\sum_{j=1}^{n-\ell} p_j^{(1)}\right) f\left(\frac{1}{n}\sum_{i=1}^{n} x_i^{(1)}\right). \tag{19}$$

Hence, if $f:\mathbb{I}\to\mathbb{R}$ is convex and $\{x_i^{(1)}:1\leqslant i\leqslant n\}\subset\mathbb{I}$, then Theorem 1 implies, for $N\in\mathbb{N}$,

$$f\left(\sum_{i=1}^{n}q_i^{(1)}y_i^{(1)}\right)+n\sum_{k=1}^{N}q_{\min}^{(k)}\left(\frac{1}{n}\sum_{i=1}^{n}f(y_i^{(k)})-f\left(\frac{1}{n}\sum_{i=1}^{n}y_i^{(k)}\right)\right)\leqslant\sum_{i=1}^{n}q_i^{(1)}f(y_i^{(1)}),\quad(20)$$

where $\{q_i^{(k)}\}$ and $\{y_i^{(k)}\}$ are obtained from $\{q_i^{(1)}\}$ and $\{y_i^{(1)}\}$ as in (4) and (5). Thus, this together with (19) imply

$$\begin{aligned}&\sum_{i=1}^{n-\ell}\frac{1-\sum_{j=1}^{n-\ell}p_j^{(1)}-\ell p_i^{(1)}}{n-\ell}f(x_i^{(1)})+\left(\frac{n}{n-\ell}\sum_{j=1}^{n-\ell}p_j^{(1)}\right)f\left(\frac{1}{n}\sum_{i=1}^{n}x_i^{(1)}\right)\\&\geqslant f\left(\sum_{i=1}^{n}q_i^{(1)}y_i^{(1)}\right)+n\sum_{k=1}^{N}q_{\min}^{(k)}\left(\frac{1}{n}\sum_{i=1}^{n}f(y_i^{(k)})-f\left(\frac{1}{n}\sum_{i=1}^{n}y_i^{(k)}\right)\right)\\&=f\left(\sum_{i=1}^{n}q_i^{(1)}y_i^{(1)}\right)+M_N(f;\mathbf{x},\mathbf{p}).\quad\square\end{aligned}\quad(21)$$

Moreover, letting

$$r_1^{(1)}=\frac{\ell}{n},\quad r_2^{(1)}=\frac{n-\ell}{n},\quad z_1^{(1)}=\sum_{i=1}^{n}p_i^{(1)}x_i^{(1)}\quad\text{and}\quad z_2^{(1)}=\sum_{i=1}^{n}\frac{1-\ell p_i^{(1)}}{n-\ell}x_i^{(1)},$$

then $r_1^{(1)}+r_2^{(1)}=1$ and Theorem 1 imply

$$\begin{aligned}&\frac{\ell}{n}f\left(\sum_{i=1}^{n}p_i^{(1)}x_i^{(1)}\right)+\frac{n-\ell}{n}f\left(\sum_{i=1}^{n}\frac{1-\ell p_i^{(1)}}{n-\ell}x_i^{(1)}\right)\\&=r_1^{(1)}f(z_1^{(1)})+r_2^{(1)}f(z_2^{(1)})\\&\geqslant f\left(r_1^{(1)}z_1^{(1)}+r_2^{(1)}z_2^{(1)}\right)+2\sum_{k=1}^{N}r_{\min}^{(k)}\left(\frac{f(z_1^{(k)})+f(z_2^{(k)})}{2}-f\left(\frac{z_1^{(k)}+z_2^{(k)}}{2}\right)\right)\\&=f\left(\frac{\ell}{n}\sum_{i=1}^{n}p_i^{(1)}x_i^{(1)}+\frac{n-\ell}{n}\sum_{i=1}^{n}\frac{1-\ell p_i^{(1)}}{n-\ell}x_i^{(1)}\right)+S_N(f;\mathbf{x},\mathbf{p}),\end{aligned}\quad(22)$$

where

$$S_N(f;\mathbf{x},\mathbf{p})=2\sum_{k=1}^{N}r_{\min}^{(k)}\left(\frac{f(z_1^{(k)})+f(z_2^{(k)})}{2}-f\left(\frac{z_1^{(k)}+z_2^{(k)}}{2}\right)\right),$$

$\{r_i^{(k)}\}$ and $\{z_i^{(k)}\}$ are generated from $\{r_i^{(1)}\}$ and $\{z_i^{(1)}\}$, via (4) and (5).

THEOREM 3. *Let* $f:\mathbb{I}\longrightarrow\mathbb{R}$ *be convex,* $\mathbf{x}=\{x_1^{(1)},\cdots,x_n^{(1)}\}\subset\mathbb{I}$ *and* $\mathbf{p}=\{p_1^{(1)},\cdots,p_n^{(1)}\}\subset(0,1)$ *be a nonconstant monotone convex sequence. Then for* *every $N \in \mathbb{N}$, we have*

$$f\left(\sum_{i=1}^{n} p_i^{(1)} x_i^{(1)}\right) + np_{\max}^{(1)}\left(\frac{1}{n}\sum_{i=1}^{n} f(x_i^{(1)}) - f\left(\frac{1}{n}\sum_{i=1}^{n} x_i^{(1)}\right)\right) \geqslant \sum_{i=1}^{n} p_i^{(1)} f(x_i^{(1)}) + \frac{n}{\ell} S_N(f;\mathbf{x},\mathbf{p}) + \frac{n-\ell}{\ell} M_N(f;\mathbf{x},\mathbf{p}),$$

where ℓ, S_N and M_N are as discussed above.

Proof. For the given parameters, we have

$$\begin{aligned}
I &:= np_{\max}^{(1)}\left(\frac{1}{n}\sum_{i=1}^{n} f(x_i^{(1)}) - f\left(\frac{1}{n}\sum_{i=1}^{n} x_i^{(1)}\right)\right) - \sum_{i=1}^{n} p_i^{(1)} f(x_i^{(1)}) \\
&= np_n^{(1)}\left(\frac{1}{n}\sum_{i=1}^{n} f(x_i^{(1)}) - f\left(\frac{1}{n}\sum_{i=1}^{n} x_i^{(1)}\right)\right) - \sum_{i=1}^{n} p_i^{(1)} f(x_i^{(1)}) \\
&= \sum_{i=1}^{n} (p_n^{(1)} - p_i^{(1)}) f(x_i) - np_n^{(1)} f\left(\frac{1}{n}\sum_{i=1}^{n} x_i^{(1)}\right) \\
&= \sum_{i=1}^{n-\ell} (p_n^{(1)} - p_i^{(1)}) f(x_i) - np_n^{(1)} f\left(\frac{1}{n}\sum_{i=1}^{n} x_i^{(1)}\right) \quad \text{(now use (15))} \\
&= \sum_{i=1}^{n-\ell}\left(\frac{1-\sum_{j=1}^{n-\ell} p_j^{(1)}}{\ell} - p_i^{(1)}\right) f(x_i^{(1)}) - n\frac{1-\sum_{j=1}^{n-\ell} p_j^{(1)}}{\ell} f\left(\frac{1}{n}\sum_{i=1}^{n} x_i^{(1)}\right) \\
&= \frac{1}{\ell}\sum_{i=1}^{n-\ell}\left(1 - \sum_{j=1}^{n-\ell} p_j^{(1)} - \ell p_i^{(1)}\right) f(x_i^{(1)}) + \left(\frac{n}{\ell}\sum_{j=1}^{n-\ell} p_j^{(1)}\right) f\left(\frac{1}{n}\sum_{i=1}^{n} x_i^{(1)}\right) \\
&\quad - \frac{n}{\ell} f\left(\frac{1}{n}\sum_{i=1}^{n} x_i^{(1)}\right) \\
&= \frac{n-\ell}{\ell}\left[\sum_{i=1}^{n-\ell} \frac{1-\sum_{j=1}^{n-\ell} p_j^{(1)} - \ell p_i^{(1)}}{n-\ell} f(x_i^{(1)}) + \left(\frac{n}{n-\ell}\sum_{j=1}^{n-\ell} p_j^{(1)}\right) f\left(\frac{1}{n}\sum_{i=1}^{n} x_i^{(1)}\right)\right] \\
&\quad - \frac{n}{\ell} f\left(\frac{1}{n}\sum_{i=1}^{n} x_i^{(1)}\right) \\
&\geqslant \frac{n-\ell}{\ell} f\left(\sum_{i=1}^{n-\ell} \frac{1-\sum_{j=1}^{n-\ell} p_j^{(1)} - \ell p_i^{(1)}}{n-\ell} x_i^{(1)} + \left(\frac{n}{n-\ell}\sum_{j=1}^{n-\ell} p_j^{(1)}\right) \frac{1}{n}\sum_{i=1}^{n} x_i^{(1)}\right) \\
&\quad + \frac{n-\ell}{\ell} M_N(f;\mathbf{x},\mathbf{p}) - \frac{n}{\ell} f\left(\frac{1}{n}\sum_{i=1}^{n} x_i^{(1)}\right) \quad \text{(by (21))}
\end{aligned}$$

$$= \frac{n-\ell}{\ell} f\left(\sum_{i=1}^{n-\ell} \frac{1-\ell p_i^{(1)}}{n-\ell} x_i^{(1)} + \frac{1-\ell p_n^{(1)}}{n-\ell} \sum_{i=n-\ell+1}^{n} x_i^{(1)}\right) - \frac{n}{\ell} f\left(\frac{1}{n}\sum_{i=1}^{n} x_i^{(1)}\right)$$
$$+ \frac{n-\ell}{\ell} M_N(f;\mathbf{x},\mathbf{p})$$
$$= \frac{n-\ell}{\ell} f\left(\sum_{i=1}^{n} \frac{1-\ell p_i^{(1)}}{n-\ell} x_i^{(1)}\right) - \frac{n}{\ell} f\left(\frac{1}{n}\sum_{i=1}^{n} x_i^{(1)}\right) + \frac{n-\ell}{\ell} M_N(f;\mathbf{x},\mathbf{p}). \tag{23}$$

Now, using (23), we have

$$I' := f\left(\sum_{i=1}^{n} p_i^{(1)} x_i^{(1)}\right) + np_{\max}^{(1)}\left(\frac{1}{n}\sum_{i=1}^{n} f(x_i^{(1)}) - f\left(\frac{1}{n}\sum_{i=1}^{n} x_i^{(1)}\right)\right)$$
$$- \sum_{i=1}^{n} p_i^{(1)} f(x_i^{(1)})$$
$$= f\left(\sum_{i=1}^{n} p_i^{(1)} x_i^{(1)}\right) + \frac{n-\ell}{\ell} f\left(\sum_{i=1}^{n} \frac{1-\ell p_i^{(1)}}{n-\ell} x_i^{(1)}\right) - \frac{n}{\ell} f\left(\frac{1}{n}\sum_{i=1}^{n} x_i^{(1)}\right)$$
$$+ \frac{n-\ell}{\ell} M_N(f;\mathbf{x},\mathbf{p})$$
$$= \frac{n}{\ell}\left[\frac{\ell}{n} f\left(\sum_{i=1}^{n} p_i^{(1)} x_i^{(1)}\right) + \frac{n-\ell}{n} f\left(\sum_{i=1}^{n} \frac{1-\ell p_i^{(1)}}{n-\ell} x_i^{(1)}\right)\right]$$
$$- \frac{n}{\ell} f\left(\frac{1}{n}\sum_{i=1}^{n} x_i^{(1)}\right) + \frac{n-\ell}{\ell} M_N(f;\mathbf{x},\mathbf{p})$$
$$\geqslant \frac{n}{\ell} f\left(\frac{\ell}{n}\sum_{i=1}^{n} p_i^{(1)} x_i^{(1)} + \frac{n-\ell}{n}\sum_{i=1}^{n} \frac{1-\ell p_i^{(1)}}{n-\ell} x_i^{(1)}\right) + \frac{n}{\ell} S_N(f;\mathbf{x},\mathbf{p})$$
$$- \frac{n}{\ell} f\left(\frac{1}{n}\sum_{i=1}^{n} x_i^{(1)}\right) + \frac{n-\ell}{\ell} M_N(f;\mathbf{x},\mathbf{p}) \ \text{(by (22))}$$
$$= \frac{n}{\ell} f\left(\frac{1}{n}\sum_{i=1}^{n} x_i^{(1)}\right) + \frac{n}{\ell} S_N(f;\mathbf{x},\mathbf{p}) - \frac{n}{\ell} f\left(\frac{1}{n}\sum_{i=1}^{n} x_i^{(1)}\right) + \frac{n-\ell}{\ell} M_N(f;\mathbf{x},\mathbf{p})$$
$$= \frac{n}{\ell} S_N(f;\mathbf{x},\mathbf{p}) + \frac{n-\ell}{\ell} M_N(f;\mathbf{x},\mathbf{p}).$$

Thus, we have shown that

$$f\left(\sum_{i=1}^{n} p_i^{(1)} x_i^{(1)}\right) + np_{\max}^{(1)}\left(\frac{1}{n}\sum_{i=1}^{n} f(x_i^{(1)}) - f\left(\frac{1}{n}\sum_{i=1}^{n} x_i^{(1)}\right)\right) - \sum_{i=1}^{n} p_i^{(1)} f(x_i^{(1)})$$
$$\geqslant \frac{n}{\ell} S_N(f;\mathbf{x},\mathbf{p}) + \frac{n-\ell}{\ell} M_N(f;\mathbf{x},\mathbf{p}),$$

which completes the proof. □

Now if f is log-convex, then $\log f$ is convex. Applying Theorem 3 on $\log f$ implies the following result for log-convex functions, refining the corresponding result in [17].

THEOREM 4. *Let $f:\mathbb{I}\longrightarrow\mathbb{R}$ be log-convex, $\mathbf{x}=\{x_1^{(1)},\cdots,x_n^{(1)}\}\subset\mathbb{I}$ and $\mathbf{p}=\{p_1^{(1)},\cdots,p_n^{(1)}\}\subset(0,1)$ be a nonconstant increasing convex sequence. Then for every $N\in\mathbb{N}$, we have*

$$\frac{\prod_{i=1}^n f^{p_i^{(1)}}(x_i^{(1)})}{f\left(\sum_{i=1}^n p_i^{(1)}x_i^{(1)}\right)}$$

$$\leqslant \frac{\prod_{i=1}^n f^{p_i^{(1)}}(x_i^{(1)})}{f\left(\sum_{i=1}^n p_i^{(1)}x_i^{(1)}\right)}\prod_{k=1}^N\left\{\left(\frac{\sqrt{f(z_1^{(k)})f(z_2^{(k)})}}{f\left(\frac{z_1^{(k)}+z_2^{(k)}}{2}\right)}\right)^{\frac{2n}{\ell}r_{\min}^{(k)}}\left(\frac{\prod_{i=1}^n f^{\frac{1}{n}}(y_i^{(k)})}{f\left(\frac{1}{n}\sum_{i=1}^n y_i^{(k)}\right)}\right)^{\frac{n(n-\ell)}{\ell}q_{\min}^{(k)}}\right\}$$

$$\leqslant\left(\frac{\prod_{i=1}^n f^{\frac{1}{n}}(x_i^{(1)})}{f\left(\frac{1}{n}\sum_{i=1}^n x_i^{(1)}\right)}\right)^{n\,p_{\max}^{(1)}}.$$

2.3. Some applications

Among the most important applications of Jensen's inequality and its refinements is the comparison between different means. In this section, we adopt the same notations from before.

2.3.1. The arithmetic-geometric mean (AM-GM) inequality

Let $\{x_i^{(1)}:1\leqslant i\leqslant n\}$ be a given set of positive real numbers and let $\{p_i^{(1)}:1\leqslant i\leqslant n\}$ be a convex sequence. The generalized AM-GM inequality states that

$$\prod_{i=1}^n\left(x_i^{(1)}\right)^{p_i^{(1)}}\leqslant\sum_{i=1}^n p_i^{(1)}x_i^{(1)}. \tag{24}$$

A simple proof of this inequality follows from applying Jensen's inequality using the convex function $f(x)=-\log x$. The following is a multiplicative refinement of this inequality. The proof follows from Theorem 1, using the function $f(x)=-\log x$.

THEOREM 5. *Let $\{x_i^{(1)}:1\leqslant i\leqslant n\}$ be a given set of positive real numbers and let $\{p_i^{(1)}:1\leqslant i\leqslant n\}$ be a convex sequence. Then for $N\in\mathbb{N}$,*

$$\prod_{k=1}^N\left(\frac{\sum_{i=1}^n x_i^{(k)}/n}{\left(\prod_{i=1}^n x_i^{(k)}\right)^{1/n}}\right)^{np_{\min}^{(k)}}\prod_{i=1}^n\left(x_i^{(1)}\right)^{p_i^{(1)}}\leqslant\sum_{i=1}^n p_i^{(1)}x_i^{(1)}. \tag{25}$$

On the other hand, applying Theorem 3 implies the following reverse of the AM-GM inequality. The notations used down are the same of Theorem 3.

THEOREM 6. *Let* $\{x_i^{(1)}: 1 \leqslant i \leqslant n\}$ *be a given set of positive real numbers and let* $\{p_1^{(1)} \leqslant p_2^{(1)} \leqslant \cdots \leqslant p_n^{(1)}\}$ *be a convex sequence. Then for* $N \in \mathbb{N}$,

$$\left(\frac{\left(\prod_{i=1}^{n} x_i^{(1)}\right)^{1/n}}{\sum_{i=1}^{n} \frac{x_i^{(1)}}{n}}\right)^{np_{\max}^{(1)}} \cdot \sum_{i=1}^{n} p_i^{(1)} x_i^{(1)}$$

$$\leqslant \prod_{k=1}^{N} \left(\frac{2\sqrt{z_1^{(k)} z_2^{(k)}}}{z_1^{(k)} + z_2^{(k)}}\right)^{2\frac{n}{\ell} r_{\min}^{(k)}} \prod_{k=1}^{N} \left(\frac{\left(\prod_{i=1}^{n} y_i^{(k)}\right)^{1/n}}{\sum_{i=1}^{n} \frac{y_i^{(k)}}{n}}\right)^{\frac{n(n-\ell)}{\ell} q_{\min}^{(k)}} \cdot \prod_{i=1}^{n} \left(x_i^{(1)}\right)^{p_i^{(1)}}.$$

2.3.2. The Wiener number

In this section, we follow the notations of [4]. Let G be a simple connected graph, with set of vertices $V(G) = \{v_1, \cdots, v_n\}$ and set of edges $E(G) \subset \{\{v_i, v_j\} : v_i, v_j \in G\}$, with cardinality m. The distance $d_G(v_i, v_j)$ between two vertices v_i, v_j is defined as the length of the shortest path from v_i to v_j. The Wiener index of a graph G is defined by

$$W(G) = \sum_{\{v_i, v_j\} \subset G} d_G(v_i, v_j).$$

This index has its applications in chemical graph theory as a topological index of a molecule. In [9, 10], a multiplicative version of the Wiener index was proposed as

$$\pi(G) = \prod_{\{v_i, v_j\} \subset G} d_G(v_i, v_j).$$

Notice that if we let $d(G,k) = \left|\left\{\{v_i, v_j\} : d_G(v_i, v_j) = k\right\}\right|$, we have

$$W(G) = \sum_{k \geqslant 1} k d(G,k) \text{ and } \pi(G) = \prod_{k \geqslant 1} k^{d(G,k)}.$$

In [4], the following relations between the two indices have been shown

$$W(G) - m + 1 - \frac{1}{2} d(d+1) \geqslant A \pi(G)^{\frac{1}{A}} - (d-1)(d!)^{\frac{1}{d-1}},$$

and

$$W(G) - \frac{1}{2} n(n-1) \pi(G)^{\frac{2}{n(n-1)}} \geqslant \frac{1}{2} d(d+1) - d(d!)^{\frac{1}{d}}. \tag{26}$$

where $d = \max d_G(v_i, v_j)$ is the diameter of G and $A = \binom{n}{2} - m$. Our main application here is to give a double sided inequality that describes the relation between $W(G)$ and $\pi(G)$.

THEOREM 7. *Let G be a connected graph of order $|V(G)| = n$. Then, for $N \in \mathbb{N}$,*

$$\prod_{k=1}^{N}\left(\frac{\sum_{i=1}^{d} x_i^{(k)}/d}{\left(\prod_{i=1}^{d} x_i^{(k)}\right)^{1/d}}\right)^{dp_{\min}^{(k)}} \pi(G)^{\frac{2}{n(n-1)}} \leqslant \frac{2}{n(n-1)}W(G),$$

and

$$\left(\frac{2(d!)^{1/d}}{d+1}\right)^{dp_{\max}^{(1)}} \frac{2}{n(n-1)}W(G)$$
$$\leqslant \prod_{k=1}^{N}\left(\frac{2\sqrt{z_1^{(k)} z_2^{(k)}}}{z_1^{(k)}+z_2^{(k)}}\right)^{2\frac{d}{\ell}r_{\min}^{(k)}} \prod_{k=1}^{N}\left(\frac{\left(\prod\limits_{i=1}^{d} y_i^{(k)}\right)^{1/d}}{\sum\limits_{i=1}^{d}\frac{y_i^{(k)}}{d}}\right)^{\frac{d(d-\ell)}{\ell}q_{\min}^{(k)}} \cdot \pi(G)^{\frac{2}{n(n-1)}}.$$

for the parameters found as in Theorems 1 and 3, using the initial values $x_i^{(1)} = i$ and $p_i^{(1)} = \frac{d(G,i)}{\binom{n}{2}}$, $1 \leqslant i \leqslant d$.

Proof. For $1 \leqslant i \leqslant d$, let

$$p_i^{(1)} = \frac{d(G,i)}{\binom{n}{2}}.$$

Then clearly $\sum_{i=1}^{d} p_i^{(1)} = 1$. Applying Theorem 5 implies the first inequality. On the other hand, noting that $x_i^{(1)} = i$ and

$$\frac{\left(\prod\limits_{i=1}^{d} x_i^{(1)}\right)^{1/d}}{\sum\limits_{i=1}^{d}\frac{x_i^{(1)}}{d}} = \frac{2(d!)^{1/d}}{d+1},$$

then applying Theorem 6 imply the second inequality. □

2.3.3. The special case $n = 2$

When $n = 2$ many interesting results about means of two positive numbers, and hence of matrices, can be obtained. This is achieved using certain mean functions.

In the following discussion we quickly mention how our results generalize almost all known results treating means refinements.

In particular, given two positive numbers a and b, the functions $f(x) = a\#_t b$ and $g(t) = a!_t b$ are convex on $[0,1]$, where

$$a\#_t b = a^{1-t}b^t \text{ and } a!_t b = \left((1-t)a^{-1} + tb^{-1}\right)^{-1}.$$

The celebrated Young's inequality state that

$$a\#_t b \leqslant a\nabla_t b, \quad 0 \leqslant t \leqslant 1,$$

where $a\nabla_t b = (1-t)a + tb$. This inequality, though very simple, is of great significance in the theory of inequalities. This inequality has been studied in the literature in various ways. For example, in [12], this inequality was refined as follows

$$a\#_t b + \min\{t, 1-t\}(\sqrt{a} - \sqrt{b})^2 \leqslant a\nabla_t b, \quad 0 \leqslant t \leqslant 1.$$

Direct computations show that this refinement follows from Theorem 1 using $n = 2$, $x_1^{(1)} = 0$, $x_2^{(1)} = 1$, $p_1^{(1)} = 1-t$, $p_2^{(1)} = t$, $N = 1$ and the convex function $f(t) = a\#_t b$. On the other hand, using the same parameters with $N = 2$ implies the results in [23]. Then it can be shown that using arbitrary N, we obtain the complete refinement recently proved in [20].

On the other hand, using the same parameters, with $N = 1$, and applying Corollary 11, we obtain the multiplicative refinement

$$K(h,2)^r a\#_t b \leqslant a\nabla_t b, \quad r = \min\{t, 1-t\}, \quad h = \frac{a}{b}$$

where $K(h,2) = \frac{(h+1)^2}{4h}$ is the Kantorovich constant. This result has been proved in [24].

Furthermore, applying Theorem 3 and Corollary 4 using the same parameters as before, we obtain reversed versions of Young's inequality proved in [12, 16, 23, 24].

Moreover, applying our results to the function $f(t) = a!_t b$, we obtain refinements and reverses in [15] of the well known arithmetic-harmonic and geometric-harmonic mean inequalities

$$a!_t b \leqslant a\#_t b \leqslant a\nabla_t b.$$

Notice that taking larger N implies further refinements.

Further, one can see that many refinements appearing in [3, 11, 21, 22] follow from our results.

We remark that these inequalities have their significance in operator theory, where matrix inequalities can be obtained from the corresponding numerical results. We refer the reader to [11, 12, 15, 16, 19, 20, 21, 22, 23, 24] as a sample of some work treating matrix inequalities, in view of the corresponding scalar ones.

We leave the details of this idea to the interested reader, as it is just an application of our main results.

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