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VARIABLE INTEGRAL AND SMOOTH EXPONENT TRIEBEL–LIZORKIN SPACES ASSOCIATED WITH A NON–NEGATIVE SELF–ADJOINT OPERATOR

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Abstract. In this paper, variable integral and smooth exponent Triebel-Lizorkin spaces associated with a non-negative self-adjoint operator are introduced. Then equivalent norms and atomic decomposition of these new spaces are given.

1. Introduction

Recent years, function spaces associated operators, such as self-adjoint non-negative operators; divergence form elliptic operators; magnetic Schrödinger operators; degenerate elliptic operators, have been introduced; see [3, 4, 5, 6, 13, 14, 19, 21, 22, 23, 31, 42, 43, 44, 45, 46, 47, 48]. Indeed, G. Kerkyacharian and P. Petrushev introduced Besov and Triebel-Lizorkin spaces associated with non-negative self-adjoint operators and gave their Heat kernel characterization and frame decomposition in [28]. For classical Besov and Triebel-Lizorkin spaces, we refer to [36, 37, 38, 39]. In [19] Hu gave their equivalent quasi-norms by Peetre type maximal functions and atomic decompositions. In [13] and [14], Hardy spaces associated with non-negative self-adjoint operators are introduced and their atomic decompositions are given.

Last decades, motivated by the applications in electrorheological fluids [34], image restoration [7] and PDE, variable exponent function spaces have been studied extensively; see [1, 2, 10, 11, 12, 16, 17, 18, 20, 24, 25, 27, 30, 32, 33, 41]. We remark here that the list is not exhausted. The study for variable exponent functions has a long history; see [9, 11, 16, 20]. Stimulated by these literatures, we shall introduce variable integral and smooth exponent Triebel-Lizorkin spaces associated with a non-negative self-adjoint operator. To give their definition, we need to clarify the definition of the underlying spaces.

Throughout the paper, we assume $(\mathscr{X},\rho,\mu)$ is a metric measure space satisfying the conditions: $(\mathscr{X},\rho)$ is a locally compact and arc-connected metric space with

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distance $\rho(\cdot,\cdot)$ and μ is a positive Radon measure which obeys the following volume doubling condition

$$0<\mu(B(x,2r))\leqslant c_0\mu(B(x,r))<\infty \text{ for all } x\in\mathscr{X} \text{ and } r>0,$$

where $B(x,r)$ is the open ball centered at x with radius r and c_0 is a constant. From the doubling condition it follows that

$$\mu(B(x,\lambda r))\leqslant c_0\lambda^d\mu(B(x,r))<\infty \text{ for all } x\in\mathscr{X} \text{ and } r>0, \text{ and } \lambda>1,$$

where $d=\log_2 c_0>0$ is a constant playing the role of a dimension.

By $\operatorname{supp} f$ we denote the support of the function f, i.e. the closure of its non zero set.

Let $\mathscr{L}$ be a self-adjoint non-negative operator on $L^2(\mathscr{X},\mathrm{d}\mu)$ such that the associated semigroup $P_t=e^{-t\mathscr{L}}$ consists of integral operators with heat kernel $p_t(x,y)$ obeying the following conditions:

(a) Gaussian upper bound: for $x,y\in\mathscr{X}$, $t>0$,

$$|p_t(x,y)|\leqslant\frac{C_1\exp\{-c_2\rho^2(x,y)/t\}}{\sqrt{\mu(B(x,\sqrt{t}))\mu(B(y,\sqrt{t}))}}.$$

(b) Hölder continuity: there exists a constant $\alpha>0$ for $x,y\in\mathscr{X}$, $t>0$ and $\rho(y,y')\leqslant\sqrt{t}$ such that

$$|p_t(x,y)-p_t(x,y')|\leqslant C_1\left(\frac{\rho(y,y')}{\sqrt{t}}\right)^{\alpha}\frac{\exp\{-c_2\rho^2(x,y)/t\}}{\sqrt{\mu(B(x,\sqrt{t}))\mu(B(y,\sqrt{t}))}}.$$

We denote the domain of $\mathscr{L}$ by $\operatorname{dom}(\mathscr{L})$. We also denote $\mathbb{N}_0=\mathbb{N}\cup\{0\}$, and $V_r(x):=\mu(B(x,r))$, for any $x\in\mathscr{X}$ and $r>0$.

DEFINITION 1. (i) If $\mu(\mathscr{X})<\infty$, the test function class $\mathscr{D}$ is defined as the collection of all functions $\phi\in\cap_{n\in\mathbb{N}}\operatorname{dom}(\mathscr{L}^n)$ with the topology induced by the family of seminorms

$$\mathscr{P}_n(\phi):=\|\mathscr{L}^n\phi\|_{L^2(\mathscr{X},\mathrm{d}\mu)},\ n\in\mathbb{N}_0.$$

(ii) If $\mu(\mathscr{X})=+\infty$, the class $\mathscr{D}$ is defined as the collection of all functions $\phi\in\cap_{n\in\mathbb{N}}\operatorname{dom}(\mathscr{L}^n)$ with the topology induced by the family of seminorms

$$\mathscr{P}_{n,l}(\phi):=\sup_{x\in\mathscr{X}}(1+\rho(x,x_0))^l|\mathscr{L}^n\phi(x)|<\infty,\ n,l\in\mathbb{N}_0,$$

where $x_0\in\mathscr{X}$ is a fixed point.

In either case $\mathscr{D}$ is a Fréchet space; see [28]. Moreover, in the case $\mu(\mathscr{X})=\infty$, the class $\mathscr{D}$ is independent of the choice of x_0, that means different choices of x_0 in Definition 1 yield the same class $\mathscr{D}$ with equivalent topology. Therefore, we may choose and fix a point $x_0\in\mathscr{X}$ in the sequel.

The set of all continuous linear functionals on $\mathscr{D}$ is denoted by $\mathscr{D}'$. The duality between the spaces is denoted by the map $(\cdot,\cdot): \mathscr{D}'\times\mathscr{D}\to\mathbb{C}$.

For the reader's conveniences, we shall use the same notation as in [28].

$$D_{\delta,\sigma}(x,y) := [V_\delta(x)V_\delta(y)]^{-1/2}\left(1+\frac{\rho(x,y)}{\delta}\right)^{-\sigma}, \quad x,y\in\mathscr{X}.$$

Lemma 2.1 in [28] says that for $\sigma>d$ and $\delta>0$

$$\int_{\mathscr{X}}\left(1+\frac{\rho(x,y)}{\delta}\right)^{-\sigma} \mathrm{d}(y) \leqslant CV_\delta(x), \quad x\in\mathscr{X}. \tag{1}$$

Formula (2.7) in [28] says that if $0<p<\infty$ and $\sigma>d(1/2+1/p)$ then

$$\|D_{\delta,\sigma}(x,\cdot)\|_{L^2(\mathscr{X},\mathrm{d}\mu(y))} \leqslant C[V_\delta(x)]^{1/p-1}, \quad x\in\mathscr{X}. \tag{2}$$

A function $f:[0,\infty)\to\mathbb{C}$ is said to belong to the class $\mathscr{S}([0,\infty))$, if $f\in C^\infty((0,\infty))$ $\cap C([0,\infty))$, and for any $k\in\mathbb{N}_0$, $f^{(k)}$ decays rapidly at infinity and $\lim_{\lambda\to 0^+} f^{(k)}(\lambda)$ exists. Then Borel's theorem (p.55 in [29]) concerning the existence of smooth functions with arbitrary Maclaurin series implies that $\mathscr{S}([0,\infty))=\mathscr{S}(\mathbb{R})_{[0,\infty)}$.

DEFINITION 2. Let (ϕ_0,ϕ) be a pair of functions in $\mathscr{S}([0,\infty))$ and M be an integer. The pair (ϕ_0,ϕ) is said to be in the class $\mathscr{A}_M([0,\infty))$ if

$$|\phi_0(\lambda)|>0 \ \text{ on } \ [0,4\varepsilon), \tag{3}$$

and

$$|\phi(\lambda)|>0 \ \text{ on } \ (\varepsilon/4,4\varepsilon) \tag{4}$$

for some $\varepsilon>0$, and if $(\cdot)^{-M}\phi(\cdot)\in\mathscr{S}([0,\infty))$.

In the sequel, given any pair (ϕ_0,ϕ) of functions in $\mathscr{S}([0,\infty))$, we denote the system $\{\phi_j\}$ of functions in $\mathscr{S}([0,\infty))$ by setting

$$\phi_j(\lambda) := \phi(2^{-2j}\lambda) \ \text{ for } \ j\geqslant 1 \ \text{ and } \ \lambda\in[0,\infty). \tag{5}$$

Let $p(\cdot)$ be a measurable function on $\mathscr{X}$, denote

$$p^- := \operatorname*{ess\,inf}_{x\in\mathscr{X}} p(x), \qquad p^+ := \operatorname*{ess\,sup}_{x\in\mathscr{X}} p(x).$$

Denote by $\mathscr{P}(\mathscr{X})$ the set of measurable functions $p(\cdot)$ on $\mathscr{X}$ such that $p^->1$ and $p^+<\infty$, and $\mathscr{P}^0(\mathscr{X})$ the set of measurable functions $p(\cdot)$ on $\mathscr{X}$ such that $p^->0$ and $p^+<\infty$. If $p(\cdot)$ is a measurable function on $\mathscr{X}$ such that $p^->0$, denote by $L^{p(\cdot)}(\mathscr{X})$ the set of measurable functions f on $\mathscr{X}$ such that for some $\lambda>0$,

$$\int_{\mathscr{X}}\left(\frac{|f(x)|}{\lambda}\right)^{p(x)} \mathrm{d}x<\infty$$

with the norm

$$\|f\|_{L^{p(\cdot)}} := \inf\left\{\lambda > 0 : \int_{\mathscr{X}} \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} \mathrm{d}x \leqslant 1\right\}.$$

$(L^{p(\cdot)}(\mathscr{X}), \|\cdot\|_{L^{p(\cdot)}})$ becomes a Banach function space when $p^- \geqslant 1$. It is known that if $p(\cdot) \in \mathscr{P}^0(\mathscr{X})$ and $0 < p_0 < p^-$, then for each $f \in L^{p(\cdot)}(\mathscr{X})$,

$$\|f\|_{L^{p(\cdot)}} = \||f|^{p_0}\|^{1/p_0}_{L^{p(\cdot)/p_0}}.$$

Now we introduce variable integral and smooth exponent Trieble-Lizorkin spaces associated to the operator $\mathscr{L}$.

DEFINITION 3. Assume that $p(\cdot), q(\cdot), \alpha(\cdot) \in \mathscr{P}^0(\mathscr{X})$. Let $\phi_0,\ \phi$ be functions in $\mathscr{S}([0,\infty))$ and assume (3) and (4). The variable exponent Triebel-Lizorkin space $F^{\alpha(\cdot),\mathscr{L}}_{p(\cdot),q(\cdot)}(\mathscr{X})$ denotes the set of all $f \in \mathscr{D}'$ such that

$$\|f\|_{F^{\alpha(\cdot),\mathscr{L}}_{p(\cdot),q(\cdot)}(\mathscr{X})} := \left\|\left\{2^{j\alpha(\cdot)}\phi_j(\mathscr{L})f\right\}_{j=0}^{\infty}\right\|_{L^{p(\cdot)}(\ell^{q(\cdot)})} < \infty,$$

where $L^{p(\cdot)}(\ell^{q(\cdot)})$ are the spaces of all sequences $\{g_j\}$ of measurable functions on $\mathscr{X}$ with finite quasi-norms

$$\|\{g_j\}_{j=0}^{\infty}\|_{L^{p(\cdot)}(\ell^{q(\cdot)})} = \|\|\{g_j\}_{j=0}^{\infty}\|_{\ell^{q(\cdot)}}\|_{L_{p(\cdot)}} = \left\|\left(\sum_{j=0}^{\infty} |g_j|^{q(\cdot)}\right)^{\frac{1}{q(\cdot)}}\right\|_{L^{p(\cdot)}}.$$

To assure that the space in Definition 3 is independent of the choice of pair (ϕ_0, ϕ), we shall put suitable conditions on $p(\cdot), q(\cdot)$ and $\alpha(\cdot)$. This is the main task in the next section. In Section 3 we shall give atomic decomposition of $F^{\alpha(\cdot),\mathscr{L}}_{p(\cdot),q(\cdot)}(\mathscr{X})$. Finally, $a \lesssim b$ means that there exists a positive constant C such that $a \leqslant Cb$. If $a \lesssim b$ and $b \lesssim a$, then we denote $a \approx b$. Letter C will denote various positive constants which may change from line to line.

2. Preliminaries

Let us start with recalling the structure of $\mathscr{X}$.

LEMMA 1. (see [8]) *There exists a collection* $\{Q^k_\alpha : k \in \mathbb{Z}, \alpha \in I_k\}$ *of open subsets of* $\mathscr{X}$, *where* I_k *is some index set (possibly finite), and constants* $\delta \in (0,1)$ *and* $A_1, A_2 > 0$, *such that* (i) $\mu(\mathscr{X} \setminus \bigcup_{\alpha \in I_k} Q^k_\alpha) = 0$ *for each fixed* k *and* $Q^k_\alpha \bigcap Q^k_\beta = \emptyset$ *if* $\alpha \neq \beta$;

(ii) *for any* α, β, k, l *with* $l \geqslant k$, *either* $Q^k_\alpha \subset Q^l_\beta$ *or* $Q^k_\alpha \bigcap Q^l_\beta = \emptyset$;

(iii) *for each* (k,α) *and* $l < k$, *there exists a unique* β *such that* $Q^k_\alpha \subset Q^l_\beta$;

(iv) $\mathrm{diam}(Q^k_\alpha) \leqslant A_1\delta^k$, *where* $\mathrm{diam}(Q^k_\alpha) := \sup\{\rho(x,y) : x,y \in Q^k_\alpha\}$;

(v) *each* Q^k_α *contains some ball* $B(z^k_\alpha, A_2\delta^k)$, *where* $z^k_\alpha \in \mathscr{X}$.

The set Q_α^k can be thought of as a dyadic cube on $\mathscr{X}$ with diameter roughly δ^k and centered at z_α^k. We denote by $\mathscr{D}$ the family of all dyadic cubes on $\mathscr{X}$. For $k \in \mathbb{Z}$, we set $\mathscr{D}_k = \{Q_\alpha^k : \alpha \in I_k\}$, so that $\mathscr{D} = \bigcup_{k\in\mathbb{Z}} \mathscr{D}_k$. For any dyadic cube $Q = Q_\alpha^k$, we denote by $z_Q := z_\alpha^k$ the "center" of Q. From (iv) and (v), one has $B(z_Q, A_2\delta^k) \subset Q \subset B(z_Q, A_1\delta^k)$. For convenience, for $\lambda > 1$, we denote $\lambda Q = B(z_Q, \lambda A_1 \delta^k)$. Then by the doubling condition, we have $\mu(\lambda Q) \approx \mu(Q)$ for fixed λ. In the sequel, we assume without loss of generality that $\delta = \frac{1}{2}$. If this is not the case, we need to replace 2^j in Definition 3 by δ^{-j} and make some other necessary changes.

Likewise to classical Besov and Triebel-Lizorkin spaces, our key tool is the boundedness of Hardy-Littlewood maximal operator on Lebesgue spaces. Let $L^1_{\text{loc}}(\mathscr{X})$ be the collection of all locally integrable functions on $\mathscr{X}$. Given a function $f \in L^1_{\text{loc}}(\mathscr{X})$, we denote the mean-value of f, defined on a set A of finite, non-zero measure by

$$M_A f := \fint_A f(x)\mathrm{d}\mu(x) =: \frac{1}{\mu(A)} \int_A f(x)\mathrm{d}\mu(x).$$

The Hardy-Littlewood maximal operator $\mathscr{M}$ is defined on $L^1_{\text{loc}}(\mathscr{X})$ by

$$\mathscr{M} f(x) := \sup_{r>0} \fint_{B(x,r)} |f(y)|\mathrm{d}\mu(y), \ \forall x \in \mathscr{X}.$$

Denote by $\mathscr{B}(\mathscr{X})$ the set of $p(\cdot) \in \mathscr{P}(\mathscr{X})$ such that $\mathscr{M}$ is bounded on $L^{p(\cdot)}(\mathscr{X})$. In [1], Adamowicz, Harjulehto and Hästö gave a sufficient condition for $p(\cdot)$ belonging to $\mathscr{B}(\mathscr{X})$. To state their result, we need to recall some notations.

DEFINITION 4. Let $p(\cdot) \in C(\mathscr{X})$. $p(\cdot)$ is called locally log-Hölder's continuous, abbreviated $p(\cdot) \in C^{\log}_{\text{loc}}(\mathscr{X})$, if there exists $c_{\log} > 0$ such that for all $x, y \in \mathscr{X}$

$$|p(x) - p(y)| \leqslant \frac{c_{\log}}{\log(e + 1/\rho(x,y))}.$$

The exponent $p(\cdot)$ is said to be globally log-Hölder's continuous, abbreviated to $p(\cdot) \in C^{\log}(\mathscr{X})$, with base point $x_0 \in \mathscr{X}$ if it is locally log-Hölder's continuous and there exists p_∞ such that for all $x \in \mathscr{X}$

$$|p(x) - p_\infty| \leqslant \frac{c_{\log}}{\log(e + \rho(x, x_0))}.$$

We define a class of exponent $p(\cdot)$ whose reciprocal is log-Hölder continuous:

$$\mathscr{P}^{\log}_{\mathrm{d}}(\mathscr{X}) := \{p(\cdot) : \ \mathscr{X} \to (0,\infty] \mid 1/p(\cdot) \text{ is log-Hölder continuous}\}.$$

By $c(p)$ we denote the log-Hölder constant of $1/p(\cdot)$.

DEFINITION 5. Let $(\mathscr{X}, \rho, \nu)$ be a metric measure space. A function $p(\cdot) : \mathscr{X} \to [1,\infty]$ is said belonging to $\mathscr{P}^{\log}_\nu(\mathscr{X})$ if there exists $C > 0$ such that

$$\nu(B)^{\frac{1}{p_B^+} - \frac{1}{p_B^-}} \leqslant C$$

for every ball $B \subset \mathscr{X}$ and there exists $p_\infty \in [1,\infty]$ such that

$$1 \in L^{s(\cdot)}(\mathscr{X}), \ where\ \frac{1}{s(x)} := \left|\frac{1}{p(x)} - \frac{1}{p_\infty}\right| \ for\ any\ x \in \mathscr{X}.$$

Since we assume μ is doubling, from Theorem 1.4 in [1] we have the following lemma.

LEMMA 2. *If $p(\cdot) \in \mathscr{P}_{\mathrm{d}}^{\log}(\mathscr{X})$ with $p^- \geqslant 1$, then $p(\cdot) \in \mathscr{P}_{\mu}^{\log}(\mathscr{X})$.*

LEMMA 3. (Theorem 1.7 in [1]) *Let $\mathscr{X}$ be a quasi-metric measure space, $p(\cdot) \in \mathscr{P}_{\mu}^{\log}(\mathscr{X})$ with $1 < p^-$. If $\mathscr{M} : L^{p^-}(\mathscr{X}) \to L^{p^-}(\mathscr{X})$ is bounded for the constant exponent p^-, then there is a positive constant C independent of f such that*

$$\|\mathscr{M}f\|_{L^{p(\cdot)}} \leqslant C\|f\|_{L^{p(\cdot)}}, \ \forall f \in L^{p(\cdot)}(\mathscr{X}).$$

Since we suppose μ is doubling, $\mathscr{M}$ is bounded on $L^p(\mathscr{X})$ for constant $p \in (1,\infty)$. Hence by Lemmas 2 and 3, $\mathscr{P}_{\mathrm{d}}^{\log}(\mathscr{X}) \subset \mathscr{B}(\mathscr{X})$.

To our purpose, we need further results of the class $\mathscr{P}_{\mu}^{\log}(\mathscr{X})$. Indeed, we shall use the method in [17].

LEMMA 4. (Lemma A.3 in [1]) *Let $p(\cdot) \in \mathscr{P}_{\mu}^{\log}(\mathscr{X})$. Define $q \in \mathscr{P}_{\mathrm{d}}^{\log}(\mathscr{X} \times \mathscr{X})$ by*

$$\frac{1}{q(x,y)} := \max\left\{0, \frac{1}{p(x)} - \frac{1}{p(y)}\right\} \ \forall x,y \in \mathscr{X}.$$

Then for any $\gamma > 0$ there exists $\beta \in (0,1)$ depending on $c_{\log}$ such that

$$\left(\beta \fint_B |f(y)|\mathrm{d}\mu(y)\right)^{p(x)} \leqslant \fint_B |f(y)|^{p(y)}\mathrm{d}\mu(y) + \fint_B \gamma^{q(x,y)}\chi_{\{0<f(y)\leqslant 1\}}\mathrm{d}\mu(y),$$

for all $f \in L^{p(\cdot)}(\mathscr{X}) + L^\infty(\mathscr{X})$ with $\|f\|_{L^{p(\cdot)}(\mathscr{X})+L^\infty(\mathscr{X})} \leqslant 1$ and every ball $B \subset \mathscr{X}$ and all $x \in B$.

By Lemma 4 we have the following Lemma.

LEMMA 5. *Let $p(\cdot) \in \mathscr{P}_{\mu}^{log}(\mathscr{X})$ with $1 \leqslant p^- \leqslant p^+ < \infty$. Then for any $k \geqslant 0$ there exists $\beta \in (0,1)$ such that*

$$\left(\beta \fint_B |f(y)|\mathrm{d}\mu(y)\right)^{p(x)} \leqslant \fint_B |f(y)|^{p(y)}\mathrm{d}\mu(y) + h_B(x),$$

for every ball $B \subset \mathscr{X}$, and all $x \in B$ and $f \in L^{p(\cdot)}(\mathscr{X}) + L^\infty(\mathscr{X})$ with $\|f\|_{L^{p(\cdot)}(\mathscr{X})+L^\infty(\mathscr{X})} \leqslant 1$, where

$$h_B(x) := \min\{\mu(B)^k, 1\}\left((e+\rho(x,x_0))^{-k} + \fint_B (e+\rho(y,x_0))^{-k}\mathrm{d}y\right) := \min\{\mu(B)^k, 1\}h(x).$$

Here, $h(\cdot) \in$ weak-$L^1(\mathscr{X}) \cap L^\infty(\mathscr{X})$, β depends on $p(\cdot)$ only via the constant of log-Hölder continuity of $1/p(\cdot)$.

Proof. We shall use the same idea as in the proof of Lemma 3.3 in [15].

Let $\gamma := \exp(-K)$ for some $K > 0$ and let q as in Lemma 4. We define

$$\rho_{q(x,y)}(t) := \begin{cases} t^{q(x,y)}, & \text{for } 0 < q(x,y) < \infty \\ 0, & \text{for } q(x,y) = \infty,\ t \in (0,1] \end{cases}.$$

We have $\rho_{q(x,y)}(\gamma) = \rho_{q(x,y)/2}(\gamma) \cdot \rho_{q(x,y)/2}(\gamma)$. We shall show that $\rho_{q(x,y)/2}(\gamma) \leqslant \min\{\mu(B)^k, 1\}$ and $\rho_{q(x,y)/2}(\gamma) \leqslant (e+\rho(x,x_0))^{-k} + (e+\rho(y,x_0))^{-k}$ for a suitable k. Then the claim follows from Lemma 4.

If $q(x,y) = \infty$, then $\rho_{q(x,y)}(\gamma) = \rho_\infty(\gamma) = 0$ and there is nothing to prove. So we can assume that $\rho_{q(x,y)}(\gamma) < \infty$.

The local log-Hölder continuity of $1/p(\cdot)$ implies that for each $x, y \in \mathscr{X}$

$$\left|\frac{1}{q(x,y)}\right| \leqslant \left|\frac{1}{p(x)} - \frac{1}{p(y)}\right| \leqslant \frac{c_{\log}}{\log(e + 1/\rho(x,y))}.$$

Hence we get

$$\gamma^{\frac{q(x,y)}{2}} \leqslant \exp\left(\frac{K \log \mu(B)}{2c_{\log}}\right) = \mu(B)^{\frac{K}{2c_{\log}}} \leqslant \mu(B)^k$$

for $K \geqslant 2kc_{\log}$ and $\mu(B) \leqslant 1$. If $\mu(B) > 1$, then we use $\rho_{q(x,y)/2}(t) \leqslant 1$ which follows from $t < 1$. Hence, we get $\rho_{q(x,y)/2}(\gamma) \leqslant \min\{\mu(B)^k, 1\}$ for $K \geqslant 2kc_{\log}$.

Define $s(\cdot)$ by $\frac{1}{s(x)} = |\frac{1}{p(x)} - \frac{1}{p_\infty}|$. Then $\frac{1}{q(x,y)} \leqslant \frac{1}{s(x)} + \frac{1}{s(y)} \leqslant 2\max\{\frac{1}{s(x)}, \frac{1}{s(y)}\}$. So $q(x,y) \geqslant \frac{1}{2}\min\{s(x), s(y)\}$ and

$$\gamma^{\frac{q(x,y)}{2}} \leqslant \gamma^{\frac{1}{4}\min\{s(x),s(y)\}} \leqslant \gamma^{\frac{s(x)}{4}} + \gamma^{\frac{s(y)}{4}}.$$

Due to the decay condition on $\frac{1}{p(\cdot)}$ at infinity, $\frac{1}{s(x)} \leqslant \frac{c'_{\log}}{\log(e+\rho(x,x_0))}$ and $\frac{1}{s(y)} \leqslant \frac{c'_{\log}}{\log(e+\rho(y,x_0))}$. This implies that

$$\gamma^{\frac{s(x)}{4}} \leqslant \exp\left(-\frac{K \log(e+\rho(x,x_0))}{4c'_{\log}}\right) = (e+\rho(x,x_0))^{\frac{K}{4c'_{\log}}} \leqslant (e+\rho(x,x_0))^{-k}$$

and similar $\gamma^{\frac{s(y)}{4}} \leqslant (e+\rho(y,x_0))^{-k}$ for $K \geqslant 4kc'_{\log}$. Thus

$$\gamma^{\frac{q(x,y)}{2}} \leqslant \gamma^{\frac{s(x)}{4}} + \gamma^{\frac{s(y)}{4}} \leqslant (e+\rho(x,x_0))^{-k} + (e+\rho(y,x_0))^{-k}. \quad \square$$

For $t \in \mathbb{N}_0$, $m \in \mathbb{Z}$, denote

$$\theta_{t,m}(x,z) := [V_{2^{-t}(z)}(1 + 2^t \rho(x,z))^m]^{-1},$$

let $g \in L^1_{\mathrm{loc}}$, denote

$$g_t * \theta_{t,m}(x) := \int_{\mathscr{X}} \frac{|g_t(z)|}{V_{2^{-t}(z)}(1 + 2^t\rho(x,z))^m} \mathrm{d}\mu(z).$$

The doubling condition yields

$$\theta_{t,m}(x,z) = [V_{2^{-t}(z)}(1 + 2^t\rho(x,z))^m]^{-1} \approx 2^{td}(1 + 2^t\rho(x,z))^{-m}.$$

LEMMA 6. *For every $m > d$ there exists $c = c(m,d) > 0$ such that*

$$g_t * \theta_{t,m}(x) \leqslant c \sum_{j\geqslant 0} 2^{-j(m-d)} \sum_{Q\in\mathscr{D}_{t-j}} \chi_{3Q}(x) M_Q g_t$$

for all $t \in \mathbb{N}_0, g_t \in L^1_{\mathrm{loc}}, x \in \mathscr{X}$.

Proof. Fix $t \in \mathbb{N}_0$, $g \in L^1_{loc}$ and $x, y \in \mathscr{X}$. If $\rho(x,y) \leqslant 2^{-t}$, then we choose $Q \in \mathscr{D}_t$ containing y. If $\rho(x,y) > 2^{-t}$, then we choose $j \in \mathbb{N}_0$ such that $2^{j-t} \leqslant \rho(x,y) \leqslant 2^{j-t+1}$ and let $Q \in \mathscr{D}_{t-j}$ be the cube containing y. Note that, in either case, $x \in 3Q$. Thus we conclude that

$$2^{td}(1+2^t\rho(x,z))^{-m} \leqslant c2^{-j(m-d)}\chi_{3Q}(x)\frac{\chi_Q(y)}{\mu(Q)}.$$

Next we multiply this inequality by $|g_t(y)|$ and integrate with respect to y over $\mathscr{X}$. This gives $|g_t * \theta_{t,m}(x)| \lesssim 2^{-j(m-d)}|M_Q g_t|\chi_{3Q}(x)$, which clearly implies the claim. □

The following lemma is the estimate for vector-valued setting in variable Lebesgue spaces on homogeneous type. Its proof is similar to that of Corollary 2.1 in [10], because the method for variable Lebesgue spaces on Euclidean spaces also holds for variable Lebesgue spaces on homogeneous type. So we omit the detail here.

LEMMA 7. *If $p(\cdot) \in \mathscr{B}(\mathscr{X})$, and $1 < q \leqslant \infty$, then there is a constant C such that*

$$\|\{\mathscr{M} f_j\}_{j=0}^{\infty}\|_{L^{p(\cdot)}(\ell^q)} \leqslant C\|\{f_j\}_{j=0}^{\infty}\|_{L^{p(\cdot)}(\ell^q)}$$

holds for all locally integrable functions $\{f_j\}_{j=0}^{\infty}$ on $\mathscr{X}$.

LEMMA 8. *Let $p(\cdot)$, $q(\cdot) \in \mathscr{P}_{\mathrm{d}}^{\log}(\mathscr{X})$, $1 < p^- \leqslant p^+ < \infty$, $1 < q^- \leqslant q^+ < \infty$, and $(p/q)^- \cdot q^- > 1$. If $m > d$, then there is a positive constant C such that*

$$\|\|\{g_t * \theta_{t,m}\}_{t=0}^{\infty}\|_{\ell^{q(\cdot)}}\|_{L^{p(\cdot)}} \leqslant C\|\|\{g_t\}_{t=0}^{\infty}\|_{\ell^{q(\cdot)}}\|_{L^{p(\cdot)}}$$

for every sequence $\{g_t\}_{t\in\mathbb{N}_0}$ of L^1_{loc}-functions.

Proof. By homogeneity, it suffices to consider the case

$$\|\|\{g_t\}_{t=0}^{\infty}\|_{\ell^{q(\cdot)}}\|_{L^{p(\cdot)}} \leqslant 1.$$

Then particular, for every $t \in \mathbb{N}_0$,

$$\int_{\mathscr{X}} |g_t(x)|^{p(x)} \mathrm{d}\mu(x) \leqslant 1. \tag{6}$$

Using Lemma 6 and Hölder's inequality we estimate

$$\begin{aligned}
&\int_{\mathscr{X}} \Big(\sum_{t\geqslant 0} |g_t * \theta_{t,m}(x)|^{q(x)} \Big)^{\frac{p(x)}{q(x)}} \, \mathrm{d}\mu(x) \\
&\quad \leqslant \int_{\mathscr{X}} \Bigg(\sum_{t\geqslant 0} \Big(\sum_{j\geqslant 0} 2^{-j(m-d)} \sum_{Q\in\mathscr{D}_{t-j}} \chi_{3Q}(x) M_Q g_t \Big)^{q(x)} \Bigg)^{\frac{p(x)}{q(x)}} \, \mathrm{d}\mu(x) \\
&\quad \leqslant C \int_{\mathscr{X}} \Bigg(\sum_{t\geqslant 0} \sum_{j\geqslant 0} 2^{-j(m-d)} \Big(\sum_{Q\in\mathscr{D}_{t-j}} \chi_{3Q}(x) M_Q g_t \Big)^{q(x)} \Bigg)^{\frac{p(x)}{q(x)}} \, \mathrm{d}\mu(x) \\
&\quad \leqslant C \int_{\mathscr{X}} \Big(\sum_{t\geqslant 0} \sum_{j\geqslant 0} 2^{-j(m-d)} \sum_{Q\in\mathscr{D}_{t-j}} \chi_{3Q}(x) (M_Q g_t)^{q(x)} \Big)^{\frac{p(x)}{q(x)}} \, \mathrm{d}\mu(x).
\end{aligned}$$

For the last inequality we used the fact that the innermost sum contains only a finite, uniformly bounded number of non-zero terms.

It follows from (6) and $p(x) \geqslant \frac{q(x)}{q^-}$ that $\|g_t\|_{L^{\frac{q(\cdot)}{q^-}}} \leqslant C$. Thus, by Lemma 5

$$(M_Q g_t)^{\frac{q(\cdot)}{q^-}} \leqslant C M_Q(|g_t|^{\frac{q(\cdot)}{q^-}}) + C \min\{\mu(Q), 1\} h(x)$$

for all $Q \in \mathscr{D}_{t-j}$ and $x \in Q$. Combining this with the estimate above, we have

$$\begin{aligned}
&\int_{\mathscr{X}} \Big(\sum_{t\geqslant 0} |g_t * \theta_{t,m}(x)|^{q(x)} \Big)^{\frac{p(x)}{q(x)}} \, \mathrm{d}\mu(x) \\
&\quad \leqslant C \int_{\mathscr{X}} \Big(\sum_{t\geqslant 0} \sum_{j\geqslant 0} 2^{-j(m-d)} \sum_{Q\in\mathscr{D}_{t-j}} \chi_{3Q}(x) [M_Q(|g_t|^{\frac{q(x)}{q^-}})]^{q^-} \Big)^{\frac{p(x)}{q(x)}} \, \mathrm{d}\mu(x) \\
&\qquad + C \int_{\mathscr{X}} \Big(\sum_{t\geqslant 0} \sum_{j\geqslant 0} 2^{-j(m-d)} \sum_{Q\in\mathscr{D}_{t-j}} \chi_{3Q}(x) (\min\{\mu(Q), 1\} h(x))^{q^-} \Big)^{\frac{p(x)}{q(x)}} \, \mathrm{d}\mu(x) \\
&\quad := I + II.
\end{aligned}$$

Now we easily estimate that

$$\begin{aligned}
I &\leqslant C \int_{\mathscr{X}} \Big(\sum_{t\geqslant 0} [\mathscr{M}(|g_t|^{\frac{q(x)}{q^-}})(x)]^{q^-} \sum_{j\geqslant 0} 2^{-j(m-d)} \sum_{Q\in\mathscr{D}_{t-j}} \chi_{3Q}(x) \Big)^{\frac{p(x)}{q(x)}} \, \mathrm{d}\mu(x) \\
&\leqslant C \int_{\mathscr{X}} \Big(\sum_{t\geqslant 0} [\mathscr{M}(|g_t|^{\frac{q(x)}{q^-}})(x)]^{q^-} \Big)^{\frac{p(x)}{q(x)}} \, \mathrm{d}\mu(x).
\end{aligned}$$

In Lemmas 3 and 7 with $(p/q)^{-}\cdot q^{-}>1$ and $q^{-}>1$, implies that the last expression is bounded since

$$\int_{\mathscr{X}}\left(\sum_{t\geqslant 0}(|g_t(x)|^{\frac{q(x)}{q^{-}}})^{q^{-}}\right)^{\frac{p(x)}{q(x)}}\mathrm{d}\mu(x)=\int_{\mathscr{X}}\left(\sum_{t\geqslant 0}|g_t(x)|^{q(x)}\right)^{\frac{p(x)}{q(x)}}\mathrm{d}\mu(x)\leqslant 1.$$

To estimate II, by the condition $m>d$ we note the inequality

$$\begin{aligned}\sum_{t\geqslant 0}\sum_{j\geqslant 0}2^{-j(m-d)}&\sum_{Q\in\mathscr{D}_{t-j}}\chi_{3Q}(x)\min\{\mu(Q),1\}^{q^{-}}\\&\lesssim\sum_{t\geqslant 0}\sum_{j\geqslant 0}2^{-j(m-d)}\min\{2^{d(j-t)q^{-}},1\}\\&\lesssim\sum_{j\geqslant 0}2^{-j(m-d)}(j+\sum_{t\geqslant j}2^{d(j-t)q^{-}})\\&\lesssim\sum_{j\geqslant 0}2^{-j(m-d)}(j+1)\\&\leqslant C.\end{aligned}$$

Therefore, we estimate II as follows:

$$\begin{aligned}II&=C\int_{\mathscr{X}}\left(\sum_{t\geqslant 0}\sum_{j\geqslant 0}2^{-j(m-d)}\sum_{Q\in\mathscr{D}_{t-j}}\chi_{3Q}(x)(\min\{\mu(Q),1\}h(x))^{q^{-}}\right)^{\frac{p(x)}{q(x)}}\mathrm{d}\mu(x)\\&\lesssim\int_{\mathscr{X}}\left(h(x)^{q^{-}}\sum_{t\geqslant 0}\sum_{j\geqslant 0}2^{-j(m-d)}\sum_{Q\in\mathscr{D}_{t-j}}\chi_{3Q}(x)\min\{\mu(Q),1\}\right)^{\frac{p(x)}{q(x)}}\mathrm{d}\mu(x)\\&\lesssim\int_{\mathscr{X}}h(x)^{\frac{p(x)}{q(x)}q^{-}}\mathrm{d}\mu(x).\end{aligned}$$

Since $(p/q)^{-}\cdot q^{-}>1$ and $h\in$ weak-$L^1\cap L^\infty$, the last expression is bounded. □

THEOREM 1. *Let $p(\cdot)$, $q(\cdot)\in\mathscr{P}_d^{\log}(\mathscr{X})$ with $1<p^{-}\leqslant p^{+}<\infty$ and $1<q^{-}\leqslant q^{+}<\infty$. If $m>d$, then there exits a positive constant C such that*

$$\|\|\{g_t*\theta_{t,m}\}_{t=0}^{\infty}\|_{\ell^{q(\cdot)}}\|_{L^{p(\cdot)}}\leqslant C\|\|\{g_t\}_{t=0}^{\infty}\|_{\ell^{q(\cdot)}}\|_{L^{p(\cdot)}}$$

holds for every sequence $\{g_t\}_{t\in\mathbb{N}_0}$ of L^1_{loc}-functions.

Proof. We shall use the idea in the proof of Theorem 3.2 in [17]. Because of $(\mathscr{X},\rho)$ is a locally compact metric space, we can choose a finite cover $\{\Omega_i\}_{i=1}^k$ of $\mathscr{X}$ with the following properties:

(i) each $\Omega_i\subset\mathscr{X}, 1\leqslant i\leqslant k$, is open;

(ii) the sets $\{\Omega_i\}_{i=1}^k$ cover $\mathscr{X}$, i.e., $\bigcup_{i=1}^k\Omega_i=\mathscr{X}$;

(iii) non-continuous sets are separated in the sense that $\rho(\Omega_i,\Omega_j)>0$ if $|i-j|>1$; and

(iv) we have $(p/q)^-_{A_i}q^-_{A_i}>1$ for $1\leqslant i\leqslant k$, where $A_i=\bigcup_{j=i-1}^{i+1}\Omega_j$ (with the convention that $\Omega_0=\Omega_{k+1}=\emptyset$).

Let us choose an integer l so that $2^l\leqslant\min_{|i-j|>1}3\rho(\Omega_i,\Omega_j)<2^{l+1}$. Since there are only finitely many indexes, the third condition guarantees that such an l exists.

Next we split the problem and work with the domains Ω_i. In each of these we argue as in Lemma 6 to conclude that

$$\begin{aligned}\int_{\mathscr{X}}\Big(\sum_{t\geqslant0}|g_t*\theta_{t,m}(x)|^{q(x)}\Big)^{\frac{p(x)}{q(x)}}\,\mathrm{d}\mu(x)&\leqslant\sum_{i=1}^k\int_{\Omega_i}\Big(\sum_{t\geqslant0}|g_t*\theta_{t,m}(x)|^{q(x)}\Big)^{\frac{p(x)}{q(x)}}\,\mathrm{d}\mu(x)\\
&\lesssim\sum_{i=1}^k\int_{\Omega_i}\Big(\sum_{t\geqslant0}\sum_{j\geqslant0}2^{-j(m-d)}\sum_{Q\in\mathscr{D}_{t-j}}\chi_{3Q}(x)(M_Qg_t)^{q(x)}\Big)^{\frac{p(x)}{q(x)}}\,\mathrm{d}\mu(x)\\
&\lesssim\sum_{i=1}^k\int_{\Omega_i}\Big(\sum_{t\geqslant0}\sum_{j=0}^{t+l}2^{-j(m-d)}\sum_{Q\in\mathscr{D}_{t-j}}\chi_{3Q}(x)(M_Qg_t)^{q(x)}\Big)^{\frac{p(x)}{q(x)}}\,\mathrm{d}\mu(x)\\
&\quad+\sum_{i=1}^k\int_{\Omega_i}\Big(\sum_{t\geqslant0}\sum_{j\geqslant t+l}2^{-j(m-d)}\mathscr{M}g_t(x)^{q(x)}\Big)^{\frac{p(x)}{q(x)}}\,\mathrm{d}\mu(x).\end{aligned}$$

The first integral on the right-hand side is treated as in the proof of Lemma 8. This is possible, since the cubes in this integral are always in A_i and $(p/q)^-_{A_i}q^-_{A_i}>1$. So it remains only to bound

$$\int_{\Omega_i}\Big(\sum_{t\geqslant0}\sum_{j\geqslant t+l}2^{-j(m-d)}\mathscr{M}g_t(x)^{q(x)}\Big)^{\frac{p(x)}{q(x)}}\,\mathrm{d}\mu(x)\leqslant\int_{\Omega_i}\Big(\sum_{t\geqslant0}2^{-t(m-d)}\mathscr{M}g_t(x)^{q(x)}\Big)^{\frac{p(x)}{q(x)}}\,d\mu(\mathrm{x}).$$

For a non-negative sequence (a_i) we have

$$\Big(\sum_{i\geqslant0}2^{-i(m-d)}a_i\Big)^r\leqslant\begin{cases}C(r)\sum_{i\geqslant0}2^{-i(m-d)}a_i^r, & r\geqslant1\\ \sum_{i\geqslant0}2^{-i(m-d)r}a_i^r, & r\leqslant1.\end{cases}$$

We apply this estimate for $r=\frac{p(x)}{q(x)}$ and conclude that

$$\begin{aligned}\int_{\Omega_i}\Big(\sum_{t\geqslant0}2^{-t(m-d)}\mathscr{M}g_t(x)^{q(x)}\Big)^{\frac{p(x)}{q(x)}}\,\mathrm{d}\mu(x)&\leqslant C\sum_{t\geqslant0}2^{-(m-d)t\min\{1,(\frac{p}{q})^-\}}\int_{\Omega_i}\mathscr{M}g_t(x)^{p(x)}\mathrm{d}\mu(x)\\
&\leqslant C\sum_{t\geqslant0}2^{-(m-d)t\min\{1,(\frac{p}{q})^-\}}\\
&\leqslant C,\end{aligned}$$

where in the second inequality we used the boundedness of the maximal operator and since $\int |g_t(x)|^{p(x)} \mathrm{d}\mu(x) \leqslant 1$. This complete the proof. $\square$

The following lemma is a variant of Lemma 6.1 in [17].

LEMMA 9. *Let $\alpha(\cdot)$ be in $C^{\log}_{\mathrm{loc}}(\mathscr{X})$ and have a limit at infinity. There exist $s \in (d,\infty)$ such that if $m > s$, then*

$$2^{\nu\alpha(x)}\theta_{\nu,2m}(x,y) \leqslant C2^{\nu\alpha(y)}\theta_{\nu,m}(x,y)$$

for all $x,y \in \mathscr{X}$.

Proof. Choose $k \in \mathbb{N}_0$ as small as possible subject to the condition that $\rho(x,y) \leqslant 2^{-\nu+k}$. Then $1+2^{\nu}\rho(x,y) \approx 2^k$. Firstly, we have

$$\frac{\theta_{\nu,2m}(x,y)}{\theta_{\nu,m}(x,y)} \leqslant C(1+2^k)^{-m} \leqslant c2^{-km}.$$

On the other hand, the log-Hölder's continuity of $\alpha(\cdot)$ implies that

$$2^{\nu(\alpha(x)-\alpha(y))} \geqslant 2^{-\nu c_{\log}/\log(e+1/\rho(x,y))} \geqslant 2^{-kc_{\log}}\rho(x,y)^{-c_{\log}/\log(e+1/\rho(x,y))} \geqslant c2^{-kc_{\log}}.$$

Hence, the claim follows from these estimates provided we choose $m \geqslant c_{\log}$. $\square$

LEMMA 10. (Lemma 3.7 in [19]) *Suppose φ_0, φ are functions in $\mathscr{S}([0,\infty)$ such that $(\cdot)^{-M}\varphi(\cdot) \in \mathscr{S}([0,\infty)$ for some integer $M \geqslant 1$ and that*

$$\sum_{j=0}^{\infty} \varphi_j(\lambda) = 1, \quad \lambda \in [0,\infty)$$

where $\varphi_j(\cdot) := \varphi(2^{-2j}\cdot)$ for $j \geqslant 1$. Then for any $f \in \mathscr{D}'$

$$f = \sum_{j=0}^{\infty} \varphi_j(\mathscr{L})f \quad \text{in } \mathscr{D}'.$$

LEMMA 11. (Lemma 3.6 in [19]) *Let (ϕ_0, ϕ) be a pair of functions in $\mathscr{S}([0,\infty))$ satisfying (3) and (4). Then there exists another pair (ψ_0, ψ) of functions in $\mathscr{S}([0,\infty))$ satisfying (3) and (4) such that*

$$\operatorname{supp}\psi_0 \subset [0,4\varepsilon], \ \ \operatorname{supp}\psi \subset [\varepsilon/4, 4\varepsilon],$$

and for any $\lambda \in [0,\infty)$

$$\phi_0(\lambda)\psi_0(\lambda) + \sum_{j=1}^{\infty} \phi(2^{-2j}\lambda)\psi(2^{-2j}\lambda) = 1.$$

LEMMA 12. (Lemma 9 in [26]) *Let $p(\cdot),q(\cdot)$ are positive functions on $\mathscr{X}$ such that $0<q^-\leqslant q^+<\infty$. For any sequence $\{g_j\}_{j=0}^{\infty}$ of nonnegative measurable functions on $\mathscr{X}$ denote*

$$G_j(x):=\sum_{k=0}^{\infty}2^{-|k-j|\delta}g_k(x),\qquad x\in\mathscr{X}.$$

Then there is a positive constant $C=C(q(\cdot),\delta)$ such that

$$\|\{G_j\}_{j=0}^{\infty}\|_{L^{p(\cdot)}(\ell^{q(\cdot)})}\leqslant C\|\{g_k\}_{k=0}^{\infty}\|_{L^{p(\cdot)}(\ell^{q(\cdot)})}. \tag{7}$$

Given a couple of (φ_0,φ) of function in $\mathscr{S}([0,\infty))$, a distribution $f\in\mathscr{D}'$, and a positive number $a>0$, we define the system of the Peetre type maximal functions by

$$(\varphi_j^*f)_a(x):=\sup_{z\in\mathscr{X}}\frac{|\varphi_j(\mathscr{L})f(z)|}{(1+2^j\rho(x,z))^a},\ \ x\in\mathscr{X},\ j\in\mathbb{N}_0.$$

THEOREM 2. *Let $p(\cdot),\ q(\cdot)\in\mathscr{P}_{\mathrm{d}}^{\log}(\mathscr{X})$ with $0<p^-,q^-$ and $p^+,q^+<\infty$. Let $\alpha(\cdot)$ be in $C_{\mathrm{loc}}^{\log}(\mathscr{X})$ and have a limit at infinity. Assume $\varphi_0,\varphi\in\mathscr{S}([0,\infty))$ satisfying conditions (3) and (4).*

If $a>\frac{2d}{\min\{1,p^-,q^-\}}$, then there exists a constant $C>0$ such that for all $f\in\mathscr{D}'$

$$\|\{2^{j\alpha(\cdot)}(\varphi_j^*f)_a\}_{j=0}^{\infty}\|_{L^{p(\cdot)}(l^{q(\cdot)})}\leqslant C\|\{2^{j\alpha(\cdot)}\varphi_j(\mathscr{L})f\}_{j=0}^{\infty}\|_{L^{p(\cdot)}(l^{q(\cdot)})}.$$

Proof. By the proof of Theorem 3.3 in [19] for $l\in\mathbb{N}_0$ and $r\in(0,1)$ such that $ar>2d$ we can get

$$\{(\varphi_l^*f)_a(x)\}^r\lesssim\sum_{j=0}^{\infty}2^{-2jSr}2^{jd}\int\frac{|\varphi_{j+l}(\mathscr{L})f(z)|^r}{V_{2^{-l}(z)}(1+2^l\rho(x,z))^{ar}}\mathrm{d}\mu(z).$$

Since for $j,k\in\mathbb{N}_0$,

$$\frac{1}{V_{2^{-l}(z)}(1+2^l\rho(x,z))^{ar}}\lesssim\frac{2^j2^{-jar}}{V_{2^{-(l+j)}(z)}(1+2^{(l+j)}\rho(x,z))^{ar}},$$

it follows that

$$\begin{aligned}\{(\varphi_l^*f)_a(x)\}^r&\lesssim\sum_{j=0}^{\infty}2^{-2jSr}2^{jar}\int\frac{|\varphi_{j+l}(\mathscr{L})f(z)|^r}{V_{2^{-(l+j)}(z)}(1+2^{(l+j)}\rho(x,z))^{ar}}\mathrm{d}\mu(z)\\&=\sum_{j=0}^{\infty}2^{-2jSr}2^{jar}\theta_{(l+j),ar}*[|\varphi_{j+l}(\mathscr{L})f|]^r(x).\end{aligned}$$

Then by Lemma 9 we obtain

$$\begin{aligned}\{2^{-l\alpha(x)}(\varphi_j^*f)_a(x)\}^r&\lesssim\sum_{j=0}^{\infty}2^{-j(2Sr-\alpha(x)r-ar)}2^{-(j+l)r\alpha(x)}\theta_{l,ar}*[|\varphi_{j+l}(\mathscr{L})f|]^r(x)\\&\lesssim\sum_{j=0}^{\infty}2^{-j(2Sr-\alpha(x)r-ar)}\theta_{l,ar/2}*\left[\left|2^{-(j+l)\alpha(\cdot)}\varphi_{j+l}(\mathscr{L})f\right|\right]^r(x)\end{aligned}$$

$$= \sum_{j=l}^{\infty} 2^{-(j-l)(2Sr-\alpha(x)r-ar)} \theta_{l,ar/2} * \left[\left|2^{-j\alpha(\cdot)}\varphi_j(\mathscr{L})f\right|\right]^r(x)$$
$$\leqslant \sum_{j=0}^{\infty} 2^{-|j-l|(2Sr-\alpha(x)r-ar)} \theta_{l,ar/2} * \left[\left|2^{-j\alpha(\cdot)}\varphi_j(\mathscr{L})f\right|\right]^r(x).$$

We now choose and fix the integer S such that $2S > \|\alpha\|_{L^\infty} - a$. Then applying Lemma 12 in spaces $L^{p(\cdot)/r}(l^{q(\cdot)/r})$ with $r < \min\{p^-, q^-\}$ and Theorem 1 gives

$$\|\{2^{l\alpha(\cdot)}(\varphi_l^* f)_a\}_{l=0}^\infty\|_{l^{q(\cdot)}(L^{p(\cdot)})} \leqslant C\|\{2^{j\alpha(\cdot)}\theta_{l,ar} * [\varphi_j(\mathscr{L})f]\}_{j=0}^\infty\|_{L^{p(\cdot)}(l^{q(\cdot)})}$$
$$\leqslant C\|\{2^{j\alpha(\cdot)}\varphi_j(\mathscr{L})f\}_{j=0}^\infty\|_{L^{p(\cdot)}(l^{q(\cdot)})}. \quad \square$$

THEOREM 3. *Assume that $p(\cdot), q(\cdot)$ are positive functions on $\mathscr{X}$ such that $0 < q^- \leqslant q^+ < \infty$. Let $\alpha(\cdot)$ be in $C_{\text{loc}}^{\log}(\mathscr{X})$ and have a limit at infinity. If $a > 0$, $M > \alpha^+/2$ and $(\varphi_0, \varphi), (\tilde{\varphi}_0, \tilde{\varphi}) \in \mathscr{A}_M([0,\infty))$, then there exists a constant $C > 0$ such that for all $f \in \mathscr{D}'$*

$$\|\{2^{j\alpha(\cdot)}(\varphi_j^* f)_a\}_{j=0}^\infty\|_{L^{p(\cdot)}(l^{q(\cdot)})} \leqslant C\|\{2^{j\alpha(\cdot)}(\tilde{\varphi}_j{}^* f)_a\}_{j=0}^\infty\|_{L^{p(\cdot)}(l^{q(\cdot)})}.$$

Proof. By the proof of Theorem 3.4 in [19] we obtain

$$2^{l\alpha(x)}(\tilde{\varphi}_l{}^* f)_a(x) \lesssim \sum_{j=0}^{\infty} 2^{-2|j-l|\delta}[2^{j\alpha(x)}(\varphi_j^* f)_a(x)]$$

Applying Lemma 12, we get

$$\|\{2^{j\alpha(\cdot)}(\varphi_j^* f)_a\}_{j=0}^\infty\|_{L^{p(\cdot)}(l^{q(\cdot)})} \leqslant C\|\{2^{j\alpha(\cdot)}(\tilde{\varphi}_j{}^* f)_a\}_{j=0}^\infty\|_{L^{p(\cdot)}(l^{q(\cdot)})}. \quad \square$$

From Theorems 2 and 3, we obtain the following Peetre maximal function characterization of Triebel-Lizorkin spaces.

COROLLARY 1. Let $p(\cdot),\ q(\cdot) \in \mathscr{P}_d^{\log}(\mathscr{X})$ with $0 < p^-, q^-$ and $p^+, q^+ < \infty$. Let $\alpha(\cdot)$ be in $C_{\text{loc}}^{\log}(\mathscr{X})$ and have a limit at infinity. Let $M > \alpha^+/2$ and let (ϕ_0, ϕ), $(\psi_0, \psi) \in \mathscr{A}_M([0,\infty))$.

If $a > \frac{2d}{\min\{1,p^-,q^-\}}$, then

$$\|\{2^{\alpha(\cdot)j}\phi_{j,a}^*(\mathscr{L})f\}_{j=0}^\infty\|_{L^{p(\cdot)}(\ell^{q(\cdot)})}, \quad \|\{2^{j\alpha(\cdot)}\psi_{j,a}^*(\mathscr{L})f\}_{j=0}^\infty\|_{L^{p(\cdot)}(\ell^{q(\cdot)})},$$

$$\|\{2^{\alpha(\cdot)j}\phi_j(\mathscr{L})f\}_{j=0}^\infty\|_{L^{p(\cdot)}(\ell^{q(\cdot)})},$$

and

$$\|\{2^{j\alpha(\cdot)}\psi_j(\mathscr{L})\}_{j=0}^\infty\|_{L^{p(\cdot)}(\ell^{q(\cdot)})}$$

are equivalent quasi-norms on $F_{p(\cdot),q(\cdot)}^{\alpha(\cdot),\mathscr{L}}(\mathscr{X})$.

3. Atomic decomposition

In this section, we shall give the atomic decomposition of $F^{\alpha(\cdot),\mathscr{L}}_{p(\cdot),q(\cdot)}(\mathscr{X})$. To do so, we need some notation firstly.

DEFINITION 6. Assume that $p(\cdot),q(\cdot)$ are positive functions and $\alpha(\cdot)$ is a non-negative function on $\mathscr{X}$. The sequence space $f^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$ consists all sequence $w = \{w_Q\}_{Q\in\cup_{k\geqslant 0}\mathscr{D}_k}$ such that

$$\|w\|_{f^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} := \|\|2^{k\alpha(x)}\sum_{Q\in\mathscr{D}_k}(|w_Q|[\mu(Q)^{-1/2}]\chi_Q)\|_{l_k^{q(\cdot)}}\|_{L^{p(\cdot)}}.$$

Here χ_Q is the characteristic function of Q.

DEFINITION 7. Let $K,S\in\mathbb{N}_0$, and let Q be a dyadic cube in $\mathscr{D}_k$ with $k\in\mathbb{N}_0$. In the case $k\geqslant 1$, a function $a_Q\in L^2(\mathscr{X},\mathrm{d}\mu)$ is said to be a (K,S)-atom for Q if a_Q satisfies the following conditions for $m=K$ and also for $m=-S$.

(i) $a_Q\in D(\mathscr{L}^m)$;

(ii) $\mathrm{supp}(\mathscr{L}^m a_Q)\subset B(z_Q,(A_1+1)2^{-k})$;

(iii) $\sup_{x\in\mathscr{X}}|\mathscr{L}^m a_Q(x)|\leqslant 2^{2km}[\mu(Q)^{-1/2}]$.

In the case $k=0$, a function a_Q is said to be a (K,S)-atom for Q if it satisfies above (i)–(iii) only for $m=K$.

LEMMA 13. (Lemma 4.7 in [19]) *Let $M\in\mathbb{N}$ (resp. $M=0$). There exists a function $\psi\in\mathscr{S}([0,\infty)$ such that the following conditions holds.*

(i) $\lambda^m\psi(\lambda)\in\mathscr{S}([0,\infty))$.

(ii) *There exists $\varepsilon>0$ such that $|\psi(\lambda)|>0$ on $\{\varepsilon/4<\lambda<4\varepsilon\}$ (resp. $|\psi(\lambda)|>0$ on $\{0<\lambda<4\varepsilon\}$).*

(iii) *For all integer $k\geqslant -M$ and for all $j\in\mathbb{N}_0$,*

$$\mathrm{supp}\,K_{(2^{-2j}\mathscr{L})^k\psi(2^{-2j}\mathscr{L})}\subset\{(x,y)\in\mathscr{X}\times\mathscr{X}:\rho(x,y)<2^{-j}\},$$

where $K_{(2^{-2j}\mathscr{L})^k\psi(2^{-2j}L)}$ is the integral kernel of the operator $(2^{-2j}\mathscr{L})^k\psi(2^{-2j}\mathscr{L})$.

(iv) *For all integer $k\geqslant -M$, there exists a constant $c=c(k)$ (depending on k) such that for all $j\in\mathbb{N}_0$*

$$|K_{(2^{-2j}\mathscr{L})^k\psi(2^{-2j}\mathscr{L})}(x,y)|\leqslant c[V_{2^{-j}}(x)]^{-1}.$$

LEMMA 14. (Lemma 4.6 in [19]) *Suppose $K,S\in\mathbb{N}_0$, Q is a dyadic cube in $\mathscr{D}_k$ with $k\in\mathbb{N}_0$, and a_Q is an (K,S)-atoms for Q. Suppose further that $\phi_0,\phi\in\mathscr{S}([0,\infty))$, such that $\lambda^{-\max(K,S)}\varphi(\lambda)\in\mathscr{S}([0,\infty))$. Then the following estimate holds.*

$$|\varphi_j(\mathscr{L})a_Q(x)|\lesssim\begin{cases}2^{2(j-k)S}[\mu(Q)]^{1/2}D_{2^{-j},N}(x,z_Q), & j\leqslant k\\ 2^{2(k-j)S}[\mu(Q)]^{1/2}D_{2^{-k},N}(x,z_Q), & j>k,\end{cases}$$

where $N>0$ can be taken arbitrarily large.

Now, it is the position to state the decomposition.

THEOREM 4. *Assume $p(\cdot)$, $q(\cdot) \in \mathscr{P}_{\mathrm{d}}^{\log}(\mathscr{X})$ such that $0 < p^-, q^-$ and $p^+, q^+ < \infty$. Let $\alpha(\cdot)$ be in $C_{\mathrm{loc}}^{\log}(\mathscr{X})$ and have a limit at infinity. Let $K, S \in \mathbb{N}_0$ such that $K > \frac{1}{2}\alpha^+$ and $S > \frac{d}{2\min(1,p^-,q^-)} - \frac{1}{2}\alpha^-$. Then there is a constant $C > 0$ such that for every sequence (K,S)-atoms $\{a_Q\}_{Q \in \cup_{k \geqslant 0} \mathscr{D}_k}$*

$$\left\| \sum_{k=0}^{\infty} \sum_{Q \in \mathscr{D}_k} w_Q a_Q \right\|_{F^{\alpha(\cdot),\mathscr{L}}_{p(\cdot),q(\cdot)}(\mathscr{X})} \leqslant C \|w\|_{f^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}.$$

Conversely, there is a constant C' such that given any distribution $f \in F^{\alpha(\cdot),\mathscr{L}}_{p(\cdot),q(\cdot)}(\mathscr{X})$, there exists a sequence of (K,S)-atoms $\{a_Q\}_{Q \in \cup_{k \geqslant 0} \mathscr{D}_k}$ and a sequence of complex scalars $w = \{w_Q\}_{Q \in \cup_{k \geqslant 0} \mathscr{D}_k}$ such that

$$f = \sum_{k=0}^{\infty} \sum_{Q \in \mathscr{D}_k} w_Q a_Q,$$

where the sum converges in $\mathscr{D}'$, and moreover,

$$\|w\|_{f^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \leqslant C' \|f\|_{F^{\alpha(\cdot),\mathscr{L}}_{p(\cdot),q(\cdot)}(\mathscr{X})}. \tag{8}$$

Proof. We shall use the method in [19], which goes back to [40] and [35]. Let $K, S \in \mathbb{N}_0$ such that $K > \frac{1}{2}\alpha^+$ and $S > \frac{d}{2\min(1,p^-)} - \frac{1}{2}\alpha^-$. Let $(\varphi, \varphi_0) \in \mathscr{A}_M([0,\infty))$ with $M \geqslant \max\{K,S\}$. Then by Lemma 14 we have

$$\begin{aligned}
2^{j\alpha(x)} & \left| \varphi(2^{-2j}\mathscr{L}) \left(\sum_{k=0}^{\infty} \sum_{Q \in \mathscr{D}_k} w_Q a_Q \right)(x) \right| \\
& \leqslant 2^{j\alpha(x)} \sum_{k=0}^{\infty} \sum_{Q \in \mathscr{D}_k} |w_Q| |\varphi(2^{-2j}L) a_Q| \\
& \lesssim 2^{j\alpha(x)} \sum_{k=0}^{j} \sum_{Q \in \mathscr{D}_k} 2^{2(j-k)S} |w_Q| [\mu(Q)]^{1/2} D_{2^{-j},N}(x, z_Q) \\
& \quad + 2^{j\alpha(x)} \sum_{k=j+1}^{\infty} \sum_{Q \in \mathscr{D}_k} 2^{2(k-j)K} |w_Q| [\mu(Q)]^{1/2} D_{2^{-k},N}(x, z_Q) \\
& \lesssim \sum_{k=0}^{j} 2^{2(j-k)S + (j-k)\alpha(x)} \sum_{Q \in \mathscr{D}_k} 2^{k\alpha(x)} |w_Q| [\mu(Q)]^{1/2} D_{2^{-j},N}(x, z_Q) \\
& \quad + \sum_{k=j+1}^{\infty} 2^{2(k-j)K + (j-k)\alpha(x)} \sum_{Q \in \mathscr{D}_k} 2^{k\alpha(x)} |w_Q| [\mu(Q)]^{1/2} D_{2^{-k},N}(x, z_Q)
\end{aligned}$$

where $N > 0$ can be taken arbitrarily large.

Now let us set

$$\mathscr{S}_0 := \left\{Q \in \mathscr{D}_k : \rho(z_Q, x) < A_1 2^{-(j\wedge k)}\right\},$$
$$\mathscr{S}_m := \left\{Q \in \mathscr{D}_k : A_1 2^{m-1} 2^{-(j\wedge k)} \leqslant \rho(z_Q, x) < A_1 2^m 2^{-(j\wedge k)}\right\},\ m \in \mathbb{N},$$
$$B_m := \left\{z \in \mathscr{X} : \rho(z, x) < A_1 2^{m+1} 2^{-(j\wedge k)}\right\},\ m \in \mathbb{N},$$

where the notation $j\wedge k$ denotes $\min\{j,k\}$, and A_1 is a constant as in Definition 7. Choose and fix $0 < r < \min\{1, p^-, q^-\}$ such that $2S + \alpha^- - d/r > 0$. This is possible since $S > \frac{d}{2\min(1,p^-,q^-)} - \frac{1}{2}\alpha^-$. Then take $N > 2d/r$. By Lemma 9

$$\begin{aligned}
&\sum_{Q\in\mathscr{D}_k} 2^{k\alpha(x)}|w_Q|[\mu(Q)]^{1/2}(1+2^{j\wedge k}\rho(z_Q,x))^{-N} \\
&\lesssim \sum_{m=0}^{\infty}\sum_{Q\in\mathscr{S}_m} 2^{k\alpha(x)}|w_Q|[\mu(Q)]^{1/2}(1+A_1 2^{m-1})^{-N} \\
&\lesssim \left(\sum_{m=0}^{\infty}\sum_{Q\in\mathscr{S}_m} 2^{kr\alpha(x)}|w_Q|^r[\mu(Q)]^{-r/2}(1+A_1 2^{m-1})^{-Nr}\right)^{1/r} \\
&= \left(\sum_{m=0}^{\infty}\int_{\mathscr{X}}\sum_{Q\in\mathscr{S}_m} 2^{kr\alpha(x)}|w_Q|^r[\mu(Q)]^{-r/2}(1+A_1 2^{m-1})^{-Nr}[\mu(Q)]^{-1}\chi_Q(z)\,\mathrm{d}\mu(z)\right)^{1/r} \\
&\lesssim 2^{kd/r}\left(\sum_{m=0}^{\infty}\int_{B_m}\sum_{Q\in\mathscr{S}_m} 2^{kr\alpha(z)}|w_Q|^r[\mu(Q)]^{-r/2}(1+2^{j\wedge k}\rho(z,x))^{-Nr/2}\chi_Q(z)\,\mathrm{d}\mu(z)\right)^{1/r} \\
&\lesssim 2^{kd/r}\left(\int_{\mathscr{X}}\sum_{m=0}^{\infty}\sum_{Q\in\mathscr{S}_m} 2^{kr\alpha(z)}|w_Q|^r[\mu(Q)]^{-r/2}(1+2^{j\wedge k}\rho(z,x))^{-Nr/2}\chi_Q(z)\,\mathrm{d}\mu(z)\right)^{1/r} \\
&\lesssim 2^{kd/r}\left(\int_{\mathscr{X}}\sum_{Q\in\mathscr{D}_k} 2^{kr\alpha(z)}|w_Q|^r[\mu(Q)]^{-r/2}(1+2^{j\wedge k}\rho(z,x))^{-Nr/2}\chi_Q(z)\,\mathrm{d}\mu(z)\right)^{1/r} \\
&= 2^{kd/r}2^{-j\wedge kd/r}\left(\theta_{j\wedge k,Nr/2} * \left(\sum_{Q\in\mathscr{D}_k}\left(2^{kr\alpha(\cdot)}|w_Q|^r[\mu(Q)]^{-r/2}\chi_Q\right)(x)\right)\right)^{1/r}.
\end{aligned}$$

Then we have

$$\begin{aligned}
&2^{j\alpha(x)}\left|\varphi(2^{-2j}\mathscr{L})\left(\sum_{k=0}^{\infty}\sum_{Q\in\mathscr{D}_k} w_Q a_Q\right)(x)\right| \\
&\quad\lesssim \sum_{k=0}^{j} 2^{(j-k)(2S+\alpha(x)-d/r)}\left(\theta_{j,Nr/2} * \left(\sum_{Q\in\mathscr{D}_k} 2^{kr\alpha(\cdot)}[\mu(Q)]^{-r/2}|w_Q|^r\chi_Q\right)(x)\right)^{1/r} \\
&\qquad + \sum_{k=j+1}^{\infty} 2^{(k-j)(2K-\alpha(x))}\left(\theta_{k,Nr/2} * \left(\sum_{Q\in\mathscr{D}_k} 2^{kr\alpha(\cdot)}[\mu(Q)]^{-r/2}|w_Q|^r\chi_Q\right)(x)\right)^{1/r}
\end{aligned}$$

$$\leqslant \sum_{k=0}^{\infty} \gamma(j-k) \left(\theta_{j\wedge k, Nr/2} * \left(\sum_{Q\in\mathscr{D}_k} 2^{kr\alpha(\cdot)} [\mu(Q)]^{-r/2} |w_Q|^r \chi_Q \right)(x) \right)^{1/r},$$

where the map $\gamma : \mathbb{Z} \to \mathbb{R}$ is defined by

$$\gamma(j) = \begin{cases} 2^{j(2S+\alpha^- - d/r)}, & j \leqslant 0, \\ 2^{-j(2K-\alpha^+)}, & j > 0. \end{cases}$$

Rise the inequality to the power $q(x)$ and sum over $j \in \mathbb{N}_0$; then raise to the power $1/q(x)$ and take $\|\cdot\|_{L^{p(\cdot)}(\mathscr{X},d\mu)}$ norm in $\mathscr{X}$. We obtain

$$\left\| \sum_{k=0}^{\infty} \sum_{Q\in\mathscr{D}_k} w_Q a_Q \right\|_{F^{\alpha(\cdot),\mathscr{L}}_{p(\cdot),q(\cdot)}(\mathscr{X})}$$
$$\lesssim \left\| \left\| \left\{ \sum_{k=0}^{\infty} \gamma(j-k) \left\{ \theta_{j\wedge k, Nr/2} * \left(\sum_{Q\in\mathscr{D}_k} (2^{k\alpha(\cdot)} |w_Q| [\mu(Q)^{-1/2}] \chi_Q)^r \right) \right\}^{1/r} \right\}_{j=0}^{\infty} \right\|_{\ell^{q(\cdot)}} \right\|_{L^{p(\cdot)}(\mathscr{X})}.$$

By the proof of Lemma 12 we arrived at

$$\left\| \left\{ \sum_{k=0}^{\infty} \gamma(j-k) \left\{ \theta_{j\wedge k, Nr/2} * \left(\sum_{Q\in\mathscr{D}_k} (2^{k\alpha(\cdot)} |w_Q| [\mu(Q)^{-1/2}] \chi_Q)^r \right) \right\}^{1/r} \right\}_{j=0}^{\infty} \right\|_{\ell^{q(\cdot)}}$$
$$\leqslant \left\{ \sum_{j=0}^{\infty} \gamma(j)^{1\wedge q(\cdot)} \right\}^{\frac{1}{1\wedge q(\cdot)}} \left\| \left\{ \theta_{j\wedge k, Nr/2} * \left(\sum_{Q\in\mathscr{D}_k} (2^{k\alpha(\cdot)} |w_Q| [\mu(Q)^{-1/2}] \chi_Q)^r \right) \right\}_{k=0}^{\infty} \right\|_{\ell^{q(\cdot)/r}}^{\frac{1}{r}}$$

And we note that the first term is a finite quantity since $2K - \alpha^+ > 0$ and $2S + \alpha^- - d/r > 0$. By Theorem 1 we conclude that

$$\left\| \sum_{k=0}^{\infty} \sum_{Q\in\mathscr{D}_k} w_Q a_Q \right\|_{F^{\alpha(\cdot),\mathscr{L}}_{p(\cdot),q(\cdot)}(\mathscr{X})}$$
$$\lesssim \left\| \left\| \left\{ \theta_{j\wedge k, Nr/2} * \left(\sum_{Q\in\mathscr{D}_k} (2^{k\alpha(\cdot)} |w_Q| [\mu(Q)^{-1/2}] \chi_Q)^r \right) \right\}_{k=0}^{\infty} \right\|_{\ell^{q(\cdot)/r}}^{\frac{1}{r}} \right\|_{L^{p(\cdot)}(\mathscr{X})}$$
$$\lesssim \left\| \left\| \left\{ \theta_{j\wedge k, Nr/2} * \left(\sum_{Q\in\mathscr{D}_k} (2^{k\alpha(\cdot)} |w_Q| [\mu(Q)^{-1/2}] \chi_Q)^r \right) \right\}_{k=0}^{\infty} \right\|_{\ell^{q(\cdot)/r}} \right\|_{L^{p(\cdot)/r}(\mathscr{X})}^{1/r}$$
$$\lesssim \left\| \left\| \left\{ \sum_{Q\in\mathscr{D}_k} (2^{k\alpha(\cdot)} |w_Q| [\mu(Q)^{-1/2}] \chi_Q)^r \right\}_{k=0}^{\infty} \right\|_{\ell^{q(\cdot)/r}} \right\|_{L^{p(\cdot)/r}(\mathscr{X})}^{1/r}$$

$$= \left\| \left\| \left\{ \sum_{Q\in\mathscr{D}_k} (2^{k\alpha(\cdot)}|w_Q|[\mu(Q)^{-1/2}]\chi_Q) \right\}_{k=0}^{\infty} \right\|_{\ell^{q(\cdot)}} \right\|_{L^{p(\cdot)}(\mathscr{X})} = \|w\|_{f^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}.$$

We now turn to the converse of the statement. Let $K,S\in\mathbb{N}_0$. We choose $\psi\in\mathscr{S}([0,\infty))$ (resp. $\psi_0\in\mathscr{S}([0,\infty))$ such that ψ (resp. ψ_0) satisfies (i)–(iv) in Lemma 13 with $M\geqslant S$ (resp. $M=0$). In particular, the couple (ψ_0,ψ) satisfied (3) and (4). Hence, by Lemma 11 it is possible to find $\psi_0\in\mathscr{S}([0,\infty))$ such that $\mathrm{supp}\varphi_0\subset[0,4\varepsilon],\mathrm{supp}\varphi\subset[\varepsilon/4,4\varepsilon]$, $|\varphi_0(\lambda)|>0$ on $\{0\leqslant\lambda<4\varepsilon\}$, $|\varphi_0(\lambda)|>0$ $\{\varepsilon/4\leqslant\lambda<4\varepsilon\}$, and

$$\sum_{j=0}^{\infty}\psi_j(\lambda)\varphi_j(\lambda)=1 \text{ for all } \lambda\in[0,\infty), \tag{9}$$

where we used the convention (5). Clearly $(\varphi_0,\varphi)\in\mathscr{A}_M([0,\infty))$ for all $M\in\mathbb{Z}$. Specially, (φ_0,φ) can be used to define $L^{p(\cdot)}(\mathscr{X},\mathrm{d}\mu)$. From (9) and Lemma 10, it follows that for all $f\in\mathscr{D}'$

$$f=\sum_{j=0}^{\infty}\psi_j(\mathscr{L})\varphi_j(\mathscr{L})f,\quad f\in\mathscr{D}'. \tag{10}$$

If $Q\in\mathscr{D}_0$, we set

$$\widetilde{w}_Q:=[\mu(Q)]^{1/2}(\sup_{y\in Q}|\varphi_0(\mathscr{L})f(y)|)\left(\sup_{x\in\mathscr{X}}\int_Q|K_{(2^{-2j}L)^k\psi_0(\mathscr{L})}(x,y)|\mathrm{d}\mu(y)\right),$$

$$\widetilde{a}_Q:=\frac{1}{\widetilde{w}_Q}\int_Q K_{\psi_0(\mathscr{L})}(x,y)\varphi_0(\mathscr{L})f(y)\mathrm{d}\mu(y).$$

while if $Q\in\mathscr{D}_j$ with $j\geqslant 1$, we set

$$\widetilde{w}_Q:=[\mu(Q)]^{1/2}(\sup_{y\in Q}|\varphi_j(\mathscr{L})f(y)|)\left(\max_{m\in\{K,-S\}}\sup_{x\in\mathscr{X}}\int_Q|K_{(2^{-2j}L)^m\psi_j(\mathscr{L})}(x,y)|dy\right),$$

$$\widetilde{a}_Q:=\frac{1}{\widetilde{w}_Q}\int_Q K_{\psi_j(\mathscr{L})}(\cdot,y)\varphi_j(\mathscr{L})f(y)\mathrm{d}\mu(y).$$

Then it follows from (10) that

$$\begin{aligned} f&=\sum_{j=0}^{\infty}\int_{\mathscr{X}}K_{\psi_j(\mathscr{L})}(\cdot,y)\varphi_j(\mathscr{L})f(y)\mathrm{d}\mu(y)\\ &=\sum_{j=0}^{\infty}\sum_{Q\in\mathscr{D}_j}\int_Q K_{\psi_j(\mathscr{L})}(\cdot,y)\varphi_j(\mathscr{L})f(y)\mathrm{d}\mu(y)\\ &=\sum_{j=0}^{\infty}\sum_{Q\in\mathscr{D}_j}\widetilde{w}_Q\widetilde{a}_Q, \end{aligned}$$

where the sum converges in $\mathscr{D}'$.

Since $\widetilde{a}_Q$ can be express as $\widetilde{a}_Q = \frac{1}{\widetilde{w}_Q}\psi_j(\mathscr{L})[(\varphi_j(\mathscr{L})f)\chi_Q]$, and since ψ (resp. ψ_0) satisfies the condition (i) in Lemma 13 with $M \geqslant S$ (resp. $M = 0$), we have $\widetilde{a}_Q \in D(\mathscr{L}^K)\cap D(\mathscr{L}^{-S})$ (resp. $\widetilde{a}_Q \in D(\mathscr{L}^K)$) whenever $Q \in \mathscr{D}_j$ with $j \geqslant 1$ (resp. $j = 0$). Moreover, if $Q \in \mathscr{D}_j$ with $j \geqslant 1$ (resp. $j = 0$), then

$$L^m\widetilde{a}_Q = \frac{1}{\widetilde{w}_Q}L^m\psi_j(\mathscr{L})[(\varphi_j(\mathscr{L})f)\chi_Q] = \frac{2^{2jm}}{\widetilde{w}_Q}\int_Q K_{(2^{-2j}L)^m\psi_j(\mathscr{L})}(\cdot,y)\varphi_j(\mathscr{L})f(y)\mathrm{d}\mu(y)$$

holds for $m \in \{K,-S\}$ (resp. $M = K$). Therefore, by using the conditions (i)–(iv) in Lemma 13 it deduces that for any $Q \in \cup_{j\geqslant 0}\mathscr{D}_j$, $\widetilde{a}_Q$ is a (K,S)-atom multiplied by a constant independent of Q.

Now, for any $Q \in \cup_{j\geqslant 0}\mathscr{D}_j$, we set $w_Q := c\widetilde{w}_Q$ and $a_Q := c\widetilde{a}_Q$, where $c > 0$ is a sufficiently large constant independent of Q. Then $\widetilde{a}_Q$ is a (K,S)-atom, and moreover,

$$f = \sum_{j=0}^{\infty}\sum_{Q\in\mathscr{D}_j} w_Q a_Q,$$

where the sum converges in $\mathscr{D}'$.

It remains to prove (8). Indeed, by our choice of ψ_0,ψ and by the conditions (iii) and (iv) in Lemma 13, we have

$$\operatorname{supp}K_{(2^{-2j}L)^m\psi_j(\mathscr{L})} \in \{(x,y) \in \mathscr{X}\times\mathscr{X} : \rho(x,y) < 2^{-j}\},$$
$$|K_{(2^{-2j}L)^m\psi_j(\mathscr{L})}(x,y)| \leqslant C[V_{2^{-j}}(x)]^{-1},$$

both of which holds for $m \in \{K,-S\}$ (resp. $M = K$) if $j \geqslant 1$ (resp. $j = 0$). In the last inequality C is a positive constant independent of j. Hence, for all $Q \in \cup_{j\geqslant 0}\mathscr{D}_j$, we have

$$\begin{aligned}|w_Q| &\lesssim [\mu(Q)]^{1/2}(\sup_{y\in Q}|\varphi_j(\mathscr{L})f(y)|)\sup_{\rho(x,z_Q)\leqslant(A_1+1)2^{-j}}\int_Q[V_{2^{-j}}(x)]^{-1}\mathrm{d}\mu(y)\\ &\lesssim [\mu(Q)]^{1/2}(\sup_{y\in Q}|\varphi_j(\mathscr{L})f(y)|).\end{aligned}$$

Now, we choose $a > \frac{2d}{\min(1,p^-,q^-)}$, and note that

$$\begin{aligned}\sum_{Q\in\mathscr{D}_k}2^{k\alpha(x)}|w_Q|[\mu(Q)^{-1/2}]\chi_Q(x) &\lesssim \sum_{Q\in\mathscr{D}_k}\sup_{y\in Q}2^{j\alpha(x)}|\varphi_j(\mathscr{L})f(y)|\chi_Q(x)\\ &\lesssim \sup_{y\in B(x,2A_1 2^{-j})}2^{j\alpha(x)}|\varphi_j(\mathscr{L})f(y)|\\ &\lesssim \sup_{y\in\mathscr{X}}\frac{2^{j\alpha(x)}|\varphi_j(\mathscr{L})f(y)|}{(1+2^j\rho(x,y))^a}\\ &= 2^{j\alpha(x)}(\varphi_j^*f)_a(x),\end{aligned}$$

which along with Theorems 3 and 2 and the fact $(\varphi_0,\varphi) \in \mathscr{A}_M([0,\infty))$ for all $M \in \mathbb{Z}$ yields

$$\|w\|_{f^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \leqslant C'\|f\|_{F^{\alpha(\cdot),\mathscr{L}}_{p(\cdot),q(\cdot)}(\mathscr{X})}.$$

Hence, we finish the proof of Theorem 4. $\square$

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