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GENERALIZED VON NEUMANN–JORDAN CONSTANT $C_{NJ}^{(p)}(X)$ FOR THE REGULAR OCTAGON SPACE

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Abstract. In this paper, we study the exact values of the generalized von Neumann-Jordan constant $C_{NJ}^{(p)}(X)$ for X being the regular octagon space. We give that $C_{NJ}^{(p)}(X) = 1+(\sqrt{2}-1)^p$ is valid for $p \geqslant 2$, and $C_{NJ}^{(p)}(X) = 2^{2-p}[1+(\sqrt{2}-1)^{\frac{p}{p-1}}]^{p-1}$ for $1 < p \leqslant 2$.

1. Introduction

In order to study the geometric structure of a Banach space, geometric constants play an important role. In many geometric constants, the von Neuman-Jordan constant $C_{NJ}(X)$ is widely treated. In [1], as a generalization of the von Neuman-Jordan constant, a new geometric constant called the generalized von Neumann-Jordan constant $C_{NJ}^{(p)}(X)$ was introduced. The authors proved that the $C_{NJ}^{(p)}(X)$ was strongly connected with geometric structure, such as uniformly non-square, uniformly normal structure. Hence it's necessary to compute the $C_{NJ}^{(p)}(X)$ for some concrete spaces.

Throughout this paper, let $X = (X, \|\cdot\|)$ be a real Banach spaces. We will use B_X, S_X and $ex(B_X)$ to denote unit ball, unit sphere of X and the set of extreme points of B_X, respectively.

Recall that the von Neumann-Jordan constant $C_{NJ}(X)$ of a Banach space X was introduced by Clarkson [2], as the smallest constant C for which,

$$\frac{1}{C} \leqslant \frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} \leqslant C,$$

holds for all $x, y \in X$.

The properties of $C_{NJ}(X)$ have been investigated in many papers (see for instances [3]–[11]).

Recently, a generalized form of this constant was introduced as following

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DEFINITION 1.1. The generalized von Neumann-Jordan constant $C_{NJ}^{(p)}(X)$ is defined by [1]

$$C_{NJ}^{(p)}(X) := \sup\left\{ \frac{\|x+y\|^p + \|x-y\|^p}{2^{p-1}(\|x\|^p + \|y\|^p)} : x,y \in X, (x,y) \neq (0,0) \right\},$$

where $1 \leqslant p < \infty$.

It's equivalent to

$$C_{NJ}^{(p)}(X) = \sup\{ \frac{\|x+ty\|^p + \|x-ty\|^p}{2^{p-1}(1+t^p)} : x,y \in S_X, 0 \leqslant t \leqslant 1\},$$

where $1 \leqslant p < \infty$.

Now let us collect some properties of this constant (see [1]):

(i) $1 \leqslant C_{NJ}^{(p)}(X) \leqslant 2$;

(ii) X is uniformly non-square if and only if $C_{NJ}^{(p)}(X) < 2$;

(iii) Let $r \in (1,2]$ and $\frac{1}{r} + \frac{1}{r'} = 1$. Then for $X = L_r[0,1]$,

(1) if $1 < p \leqslant r$ then $C_{NJ}^{(p)}(X) = 2^{2-p}$ and if $r < p \leqslant r'$ then $C_{NJ}^{(p)}(X) = 2^{\frac{p}{r}-p+1}$,

(2) if $r' < p < \infty$ then $C_{NJ}^{(p)}(X) = 1$.

In this paper, for any $p > 1$, we obtain the exact values of the generalized von Neumann-Jordan constant $C_{NJ}^{(p)}(X)$ for X being the regular octagon space.

2. Main results

Firstly, in order to get our main result, we need following lemmas.

LEMMA 2.1. *Let* $p \geqslant 2.11$. *Then*

$$(\sqrt{2}+1)^{p-1} + 1 \leqslant (2\sqrt{2})^{p-1}[1 + (\sqrt{2}-1)^p]. \tag{2.1}$$

Proof. Obviously, (2.1) is equivalent to

$$\frac{(\sqrt{2}+1)^p + 1}{\sqrt{2}+1} + \frac{\sqrt{2}}{\sqrt{2}+1} \leqslant (2\sqrt{2})^{p-1}\frac{(\sqrt{2}+1)^p + 1}{(\sqrt{2}+1)^p} = \frac{(4-2\sqrt{2})^{p-1}}{\sqrt{2}+1}[(\sqrt{2}+1)^p + 1].$$

So, we only need to prove the following (2.2)

$$[(4-2\sqrt{2})^{p-1} - 1][1 + (\sqrt{2}+1)^p] \geqslant \sqrt{2}. \tag{2.2}$$

Now by $p \geqslant 2.11$, we have

$$\begin{aligned}[(4-2\sqrt{2})^{p-1} - 1][1 + (\sqrt{2}+1)^p] &\geqslant [(4-2\sqrt{2})^{1.11} - 1][1 + (\sqrt{2}+1)^{2.11}] \\ &= 1.426\cdots \geqslant \sqrt{2},\end{aligned}$$

as desired. □

LEMMA 2.2. *Let* $2 \leqslant p \leqslant 2.11$ *and* $x \in [\sqrt{2}, 1.735]$. *Then*

$$(x+1)^{p-1} - (\sqrt{2}-1)[x-(\sqrt{2}-1)]^{p-1} < 2^{p-1}[1+(\sqrt{2}-1)^p]. \tag{2.3}$$

Proof. Letting $h(x) = (\frac{x+1}{2})^{p-1} - (\sqrt{2}-1)(\frac{x-(\sqrt{2}-1)}{2})^{p-1}$, we can easily find the function $h(x)$ is increasing on $[\sqrt{2}, 1.735]$. So by

$$\begin{aligned} h(1.735) &\leqslant \left(\frac{2.735}{2}\right)^{1.11} - (\sqrt{2}-1)\left(\frac{2.735-\sqrt{2}}{2}\right)^{1.11} \\ &= 1.154\cdots < 1.155\cdots = 1+(\sqrt{2}-1)^{2.11} \\ &\leqslant 1+(\sqrt{2}-1)^p, \end{aligned}$$

we know that (2.3) is valid. □

LEMMA 2.3. *Let* $p \geqslant 2$ *and* $t \in [0, 1.735^{-1}]$. *Then*

$$\frac{(1+t)^{p-1} + (1-(\sqrt{2}-1)t)^{p-1}}{2^{p-1}} \leqslant 1+(\sqrt{2}-1)^p. \tag{2.4}$$

Proof. Clearly,

$$\begin{aligned} \frac{(1+t)^{p-1}+(1-(\sqrt{2}-1)t)^{p-1}}{2^{p-1}} &\leqslant \left(\frac{1+1.735^{-1}}{2}\right)^{p-1} + \left(\frac{1-(\sqrt{2}-1)\times 1.735^{-1}}{2}\right)^{p-1} \\ &\leqslant 0.79^{p-1} + 0.381^{p-1}. \end{aligned}$$

On the other hand, by $p \geqslant 2$, we have

$$\begin{aligned} &(\sqrt{2}+1)^{p-1}(1-0.79^{p-1}) - ((\sqrt{2}+1)\times 0.381)^{p-1} + \sqrt{2}-1 \\ &\geqslant (\sqrt{2}+1)\times 0.21 - (\sqrt{2}+1)\times 0.381 + \sqrt{2}-1 \\ &= 0.0013\cdots \geqslant 0. \end{aligned}$$

Then,

$$1-0.79^{p-1} - 0.381^{p-1} + (\sqrt{2}-1)^p \geqslant 0,$$

which implies (2.4). □

LEMMA 2.4. *Let* $p \geqslant 2$ *and* $\tau \in [0, \frac{\sqrt{2}}{2}]$. *Then*

$$2^{p-1}(1+\tau^p)(1+(\sqrt{2}-1)^p) \geqslant (1+\tau)^p + (1-(\sqrt{2}-1)\tau)^p. \tag{2.5}$$

Proof. Letting $f(\tau) = 2^{p-1}(1+\tau^p)(1+(\sqrt{2}-1)^p) - (1+\tau)^p - (1-(\sqrt{2}-1)\tau)^p$, we have $f(0) = 2^{p-1}(1+(\sqrt{2}-1)^p) - 2 \geqslant 0$ and

$$\begin{aligned} f\left(\frac{\sqrt{2}}{2}\right) &= \left(\frac{1}{\sqrt{2}}\right)^p \{2^{p-1}(1+(\sqrt{2})^p)(1+(\sqrt{2}-1)^p) - 1 - (\sqrt{2}+1)^p\} \\ &\geqslant \left(\frac{1}{\sqrt{2}}\right)^p \{(1+\sqrt{2})^p(1+(\sqrt{2}-1)^p)) - 1 - (\sqrt{2}+1)^p\} = 0. \end{aligned}$$

Hence, to complete the proof of (2.5), it is enough to show that $f(t_0) \geqslant 0$ for any possible stationary point $t_0 \in (0, \frac{\sqrt{2}}{2})$. Now suppose that $f(\tau)$ have a stationary point $t_0 \in (0, \frac{\sqrt{2}}{2})$, then we have

$$(1+t_0)^{p-1} = 2^{p-1}t_0^{p-1}(1+(\sqrt{2}-1)^p) + (\sqrt{2}-1)(1-(\sqrt{2}-1)t_0)^{p-1}, \quad (2.6)$$

that is,

$$\begin{aligned}&(1+t_0)^{p-1} + (1-(\sqrt{2}-1)t_0)^{p-1}\\ &= 2^{p-1}t_0^{p-1}(1+(\sqrt{2}-1)^p) + \sqrt{2}(1-(\sqrt{2}-1)t_0)^{p-1}.\end{aligned} \quad (2.7)$$

Case I. $p \geqslant 2.11$. Applying Lemma 2.1 and through (2.6), (2.7), we have

$$\begin{aligned}f(t_0) &= 2^{p-1}(1+t_0^p)[1+(\sqrt{2}-1)^p] - (1+t_0)^p - [1-(\sqrt{2}-1)t_0]^p\\ &= 2^{p-1}(1+t_0^p)[1+(\sqrt{2}-1)^p] - (1+t_0)\{2^{p-1}t_0^{p-1}[1+(\sqrt{2}-1)^p]\\ &\quad +(\sqrt{2}-1)[1-(\sqrt{2}-1)t_0]^{p-1}\} - [1-(\sqrt{2}-1)t_0]^p\\ &= 2^{p-1}[1+(\sqrt{2}-1)^p] - 2^{p-1}t_0^{p-1}(1+(\sqrt{2}-1)^p) - \sqrt{2}[1-(\sqrt{2}-1)t_0]^{p-1}\\ &= 2^{p-1}[1+(\sqrt{2}-1)^p] - (1+t_0)^{p-1} - [1-(\sqrt{2}-1)t_0]^{p-1}\\ &\geqslant 2^{p-1}[1+(\sqrt{2}-1)^p] - (1+\tfrac{1}{\sqrt{2}})^{p-1} - [1-(\sqrt{2}-1)\tfrac{1}{\sqrt{2}}]^{p-1}\\ &\geqslant 0\end{aligned}$$

Case II. $2 \leqslant p \leqslant 2.11$. By (2.6),

$$(1+t_0^{-1})^{p-1} - (\sqrt{2}-1)(t_0^{-1}-(\sqrt{2}-1))^{p-1} = 2^{p-1}(1+(\sqrt{2}-1)^p).$$

Hence by Lemma 2.2, we get $t_0 \leqslant 1.735^{-1}$. Therefore by Lemma 2.3 we have

$$f(t_0) = 2^{p-1}(1+(\sqrt{2}-1)^p) - (1+t_0)^{p-1} - (1-(\sqrt{2}-1)t_0)^{p-1} \geqslant 0. \quad \square$$

LEMMA 2.5. *Let X be any Banach space, then for any $\alpha, \beta \in [0,1]$ and any $x_1, x_2, y_1, y_2 \in B_X$ with $x = \alpha x_1 + (1-\alpha)x_2$, $y = \beta y_1 + (1-\beta)y_2$, we have*

$$\|x+ty\|^p + \|x-ty\|^p \leqslant \max\{\|x_i+ty_j\|^p + \|x_i-ty_j\|^p : i,j = 1,2\}.$$

Proof. By Hölder inequality, for any $\alpha, \beta \in [0,1]$ and any $x_1, x_2, y_1, y_2 \in B_X$ with $x = \alpha x_1 + (1-\alpha)x_2$, $y = \beta y_1 + (1-\beta)y_2$, we have

$$\begin{aligned}&\|x+ty\|^p + \|x-ty\|^p\\ &= \|\alpha(x_1+ty) + (1-\alpha)(x_2+ty)\|^p + \|\alpha(x_1-ty) + (1-\alpha)(x_2-ty)\|^p\\ &\leqslant \alpha\|x_1+ty\|^p + (1-\alpha)\|x_2+ty\|^p + \alpha\|x_1-ty\|^p + (1-\alpha)\|x_2-ty\|^p\\ &= \alpha[\|\beta(x_1+ty_1) + (1-\beta)(x_1+ty_2)\|^p + \|\beta(x_1-ty_1) + (1-\beta)(x_1-ty_2)\|^p]\\ &\quad +(1-\alpha)[\|\beta(x_2+ty_1) + (1-\beta)(x_2+ty_2)\|^p + \|\beta(x_2-ty_1) + (1-\beta)(x_2-ty_2)\|^p]\\ &\leqslant \alpha\beta[\|x_1+ty_1\|^p + \|x_1-ty_1\|^p] + \alpha(1-\beta)[\|x_1+ty_2\|^p + \|x_1-ty_2\|^p]\\ &\quad +(1-\alpha)\beta[\|x_2+ty_1\|^p + \|x_2-ty_1\|^p] + (1-\alpha)(1-\beta)[\|x_2+ty_2\|^p + \|x_2-ty_2\|^p]\\ &\leqslant \max\{\|x_i+ty_j\|^p + \|x_i-ty_j\|^p : i,j = 1,2\}. \quad \square\end{aligned}$$

THEOREM 2.1. *Let $p \geqslant 2$ and X be the regular octagon space which is $\mathbb{R}^2$ endowed with the norm*

$$\|x\| = \max\{|x_1| + (\sqrt{2}-1)|x_2|, |x_2| + (\sqrt{2}-1)|x_1|\}.$$

Then

$$C_{NJ}^{(p)}(X) = 1 + (\sqrt{2}-1)^p. \tag{2.8}$$

Proof. Firstly, we can prove that

$$\|x+ty\|^p + \|x-ty\|^p \leqslant \max\{(1+t)^p + (1-(\sqrt{2}-1)t)^p, \\ (1+(\sqrt{2}-1)^p)(1+t)^p, 2(1+(\sqrt{2}-1)t)^p\} \tag{2.9}$$

for any $x, y \in ex(B_X)$ and every $t \in [0,1]$.

Since $ex(B_X) = \{(\pm 1, 0), (0, \pm 1), (\pm\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (\pm\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})\}$ and we can change x into $-x$ or y into $-y$. So we may assume that $x, y = (0,1)$, $(1,0)$ or $(\pm\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

Case I. $x = (1,0)$.

I_a. If $y = (1,0)$, then

$$\|x+ty\|^p + \|x-ty\|^p = (1+t)^p + (1-t)^p \leqslant (1+t)^p + (1-(\sqrt{2}-1)t)^p.$$

Hence, (2.9) is valid for this case.

I_b. If $y = (0,1)$, then

$$\|x+ty\|^p + \|x-ty\|^p = 2(1+(\sqrt{2}-1)t)^p.$$

I_c. If $y = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, then we can get

$$\|x+ty\|^p + \|x-ty\|^p = (1+t)^p + (1-(\sqrt{2}-1)t)^p$$

is valid for $t \in [0, \frac{1}{\sqrt{2}}]$, and

$$\|x+ty\|^p + \|x-ty\|^p = (1+(\sqrt{2}-1)^p)(1+t)^p.$$

is valid for $t \in [\frac{1}{\sqrt{2}}, 1]$. Hence (2.9) is also valid.

Case II. $x = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

II_a. If $y = (1,0)$ or $y = (0,1)$, then $\|x+ty\| = 1+t$ and $\|x-ty\| = \max\{1-(\sqrt{2}-1)t, (\sqrt{2}-1)(1+t)\}$.

II_b If $y = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, then $||x+ty|| = 1+t$, and $\|x-ty\| = 1-t$.

II_c If $y = (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, then $\|x+ty\| = \|x-ty\| = 1+(\sqrt{2}-1)t$.

Case III. $x = (0,1)$ or $x = (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$. These cases can be prove similarly as above. Hence, we can show (2.9) is valid by the case I. Applying Lemma 2.5, we have (2.9) is also valid for any $x, y \in S_X$ and every $t \in [0,1]$.

Now letting $\psi(t)=\frac{2(1+(\sqrt{2}-1)t)^p}{2^{p-1}(1+t^p)}$, we have

$$\psi'(t)=\frac{2p(1+(\sqrt{2}-1)t)^{p-1}}{2^{p-1}(1+t^p)^2}\{\sqrt{2}-1-t^{p-1}\}.$$

So,

$$\begin{aligned}\max\nolimits_{t\in[0,1]}\psi(t) &= \psi((\sqrt{2}-1)^{\frac{1}{p-1}})\\ &= 2^{2-p}(1+(\sqrt{2}-1)^{\frac{p}{p-1}})^{p-1}\\ &\leqslant 1+(\sqrt{2}-1)^p.\end{aligned}$$

Noting that $(1+(\sqrt{2}-1)^p)(1+t)^p \geqslant (1+t)^p+(1-(\sqrt{2}-1)t)^p$ if and only if $t\in[\frac{1}{\sqrt{2}},1]$, we can get

$$C_{NJ}^{(p)}(X)\leqslant 1+(\sqrt{2}-1)^p.$$

by Lemma 2.4. Finally, taking $x_0=(1,0)$ and $y_0=(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$, we have

$$C_{NJ}^{(p)}(X)\geqslant\frac{\|x_0+y_0\|^p+\|x_0-y_0\|^p}{2^p}=1+(\sqrt{2}-1)^p.$$

This completes the proof of (2.8). □

Let's turn to consider the case $1<p\leqslant 2$. Firstly, we need the following lemma.

LEMMA 2.6. *Let* $\alpha\in[0,1]$, *then*

$$\frac{3^\alpha}{2^\alpha}+\frac{2}{2^\alpha}\geqslant\frac{5}{2}, \tag{2.10}$$

and

$$(\sqrt{2})^\alpha+(\sqrt{2}-1)^\alpha+(2-\sqrt{2})^\alpha\geqslant\sqrt{2}+1. \tag{2.11}$$

Proof. Letting $f(\alpha)=3^\alpha+2-\frac{5}{2}\times 2^\alpha$, we have

$$f'(\alpha)=2^\alpha\left[\left(\frac{3}{2}\right)^\alpha\ln 3-\frac{5}{2}\ln 2\right]\leqslant 2^\alpha\left[\frac{3}{2}\ln 3-\frac{5}{2}\ln 2\right]<0,$$

and hence $f(\alpha)\geqslant f(1)=0$ for $\alpha\in[0,1]$.

Next, we consider the function $g(x)=x^\alpha+(x-1)^\alpha+(2-x)^\alpha-x$. By applying $g''(x)=\alpha(\alpha-1)[x^{\alpha-2}+(x-1)^{\alpha-2}+(2-x)^{\alpha-2}]\leqslant 0$ on $(1,2)$, we get $g(x)$ is a concave function on $[1,2]$. Therefore by $g(1)=1$ and (2.10) we have,

$$\frac{g(\sqrt{2})-g(1)}{\sqrt{2}-1}\geqslant\frac{g(1.5)-g(1)}{1.5-1}=\frac{(\frac{3}{2})^\alpha+\frac{2}{2^\alpha}-\frac{5}{2}}{0.5}\geqslant 0.$$

This completes the proof of (2.11). □

By the proof of Theorem 2.1, we have

THEOREM 2.2. *Let* $1 < p \leqslant 2$ *and* X *be the regular octagon space. Then*

$$C_{NJ}^{(p)}(X) = 2^{2-p}(1+(\sqrt{2}-1)^{\frac{p}{p-1}})^{p-1}.$$

Proof. Taking $f(t) = \frac{(1+t)^p+(1-(\sqrt{2}-1)t)^p}{2^{p-1}(1+t^p)}$, from $f'(t_0) = 0$, we can find t_0 must satisfies

$$\frac{(1+t)^{p-1}}{\sqrt{2}-1+t^{p-1}} = \frac{(1-(\sqrt{2}-1)t)^{p-1}}{1-t^{p-1}}. \tag{2.12}$$

On $[0, \frac{1}{\sqrt{2}}]$, we note the function $\phi(t) =: \frac{(1+t)^{p-1}}{\sqrt{2}-1+t^{p-1}}$ is decreasing from $\sqrt{2}+1$ to $\frac{(\sqrt{2}+1)^{p-1}}{(\sqrt{2})^{p-1}(\sqrt{2}-1)+1}$, and the function $\varphi(t) =: \frac{(1-(\sqrt{2}-1)t)^{p-1}}{1-t^{p-1}}$ is increasing from 1 to $\frac{1}{(\sqrt{2})^{p-1}-1}$ (easy to see $\frac{(\sqrt{2}+1)^{p-1}}{(\sqrt{2})^{p-1}(\sqrt{2}-1)+1} < \frac{1}{(\sqrt{2})^{p-1}-1}$). So the solution of (2.12) is unique. Now by $(\sqrt{2}+1)^{\frac{1}{p-1}} + \sqrt{2} \geqslant 2+\sqrt{2}$ and (2.11) for $\alpha = p-1$, we have

$$\begin{aligned}
&\left(1-\frac{2}{(\sqrt{2}+1)^{\frac{1}{p-1}}+\sqrt{2}}\right)^{p-1} \frac{1+(\sqrt{2})^{p-1}}{\sqrt{2}+1-(\sqrt{2})^{p-1}} \\
&\geqslant \left(1-\frac{2}{2+\sqrt{2}}\right)^{p-1} \frac{1+(\sqrt{2})^{p-1}}{\sqrt{2}+1-(\sqrt{2})^{p-1}} \\
&= \frac{(\sqrt{2}-1)^{p-1}+(2-\sqrt{2})^{p-1}}{\sqrt{2}+1-(\sqrt{2})^{p-1}} \\
&\geqslant 1.
\end{aligned}$$

Hence,

$$\begin{aligned}
\phi(\sqrt{2}(\sqrt{2}-1)^{\frac{1}{p-1}}) &= \frac{(1+\sqrt{2}(\sqrt{2}-1)^{\frac{1}{p-1}})^{p-1}}{\sqrt{2}-1+(\sqrt{2})^{p-1}(\sqrt{2}-1)} \\
&\leqslant \frac{[1-\sqrt{2}(\sqrt{2}-1)^{\frac{p}{p-1}}]^{p-1}}{1-(\sqrt{2})^{p-1}(\sqrt{2}-1)} = \varphi(\sqrt{2}(\sqrt{2}-1)^{\frac{1}{p-1}}).
\end{aligned}$$

So we must have $t_0 \leqslant \sqrt{2}(\sqrt{2}-1)^{\frac{1}{p-1}}$ by $\phi(0) > \varphi(0)$. And also

$$\begin{aligned}
\max_{t\in[0,\frac{1}{\sqrt{2}}]} f(t) = f(t_0) &= \frac{(1+t_0)^{p-1}+(1-(\sqrt{2}-1)t_0)^{p-1}}{2^{p-1}} \\
&\leqslant 2^{2-p}\left[1+(\sqrt{2}-1)\frac{t_0}{\sqrt{2}}\right]^{p-1} \\
&\leqslant 2^{2-p}(1+(\sqrt{2}-1)^{\frac{p}{p-1}})^{p-1}.
\end{aligned}$$

Finally, for the function $\psi(t)$ which is in the proof of Theorem 2.1, we also have

$$\max_{t\in[0,1]} \psi(t) = 2^{2-p}(1+(\sqrt{2}-1)^{\frac{p}{p-1}})^{p-1}.$$

To complete the proof, we only need to note for $1 < p \leqslant 2$

$$[1+(\sqrt{2}-1)^p]^{\frac{1}{p-1}} \leqslant 2^{\frac{2-p}{p-1}}[1+(\sqrt{2}-1)^{\frac{p}{p-1}}]. \quad \square$$

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