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ON HARDY'S INEQUALITIES FOR THE SPECIAL HERMITE EXPANSIONS

JINSEN XIAO AND JIANXUN HE

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Abstract. This article presents two types of Hardy's inequalities for the special Hermite expansions. The proofs are mainly based on an estimate of atomic functions deduced by the horizontal Taylor formula of the Heisenberg group.

1. Introduction

The well known Hardy's inequalities on $\mathbb{C}$ state that if $f(z)=\sum_{k=0}^{\infty}a_kz^k\in H^p(\mathbb{C})$, $0<p\leqslant 1$, then one has the following results for the Taylor coefficients:

$$|a_k|\leqslant ck^{1/p-1}\|f\|_{H^p}$$

and

$$\sum_{k=0}^{\infty}(k+1)^{p-2}|a_k|^p\leqslant c_p\|f\|_{H^p}^p,$$

where c_p depends only on p (see Theorems 6.2 and 6.4 in [6]).

Since 1990s, considerable attentions have been paid to study different types of Hardy's inequalities. Generalizations of these results have been established for Hermite expansions [3, 7, 8, 10, 13], for Laguerre expansions [7, 8, 14], and for special Hermite expansions [12, 13, 16]. More works related to this topic can be found in [2, 4, 9, 15]. Now in this paper we aim to extend Hardy's inequalities for the special Hermite expansions to the atomic Hardy spaces $H_a^p(\mathbb{C}^n)$ with $0<p\leqslant 1$. Explicitly, we shall prove the following

THEOREM 1. *Suppose $f\in H_a^p(\mathbb{C}^n)$ with $0<p\leqslant 1$. Then there exists a constant c such that*

$$\left|\hat{f}(\alpha,\beta)\right|\leqslant c(2|\alpha|+n)^{\tau}\|f\|_{H_a^p(\mathbb{C}^n)},$$

where $\tau=n(1/p-1)$.

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From this theorem we can easily obtain

COROLLARY 1. *Suppose* $f \in H_a^p(\mathbb{C}^n)$ *with* $0 < p \leqslant 1$. *Then there exists a constant* c *such that*

$$\left|\hat{f}(\alpha,\beta)\right| \leqslant c(|\alpha|+|\beta|+n)^{\tau}\|f\|_{H_a^p(\mathbb{C}^n)},$$

where $\tau = n(1/p-1)$.

Moreover, we will show that

THEOREM 2. *Suppose* $f \in H_a^p(\mathbb{C}^n)$ *with* $0 < p \leqslant 1$. *Then there exists a constant* c *such that*

$$\sum_{\alpha\in\mathbb{N}^n}\sum_{\beta\in\mathbb{N}^n}\left|\hat{f}(\alpha,\beta)\right|^p(|\alpha|+|\beta|+n)^{-\sigma} \leqslant c\|f\|^p_{H_a^p(\mathbb{C}^n)},$$

where $\sigma = 3n(2-p)/2$.

We note that this theorem is the same as Theorem 3.2 stated in [13] without proof. The key point to prove this theorem is an estimate for the Heisenberg left-invariant vectors (where in [13] is for the normal derivatives) on the special Hermite functions, which is deduced by the horizontal Taylor formula with integral remainder on the Heisenberg group.

2. Preliminaries

We begin by recalling some notions from the article [11], which has laid the foundation for the Heisenberg group. The $(2n+1)$-dimensional Heisenberg group $\mathbb{H}^n$ is a Lie group structure on $\mathbb{C}^n \times \mathbb{R}$ with the multiplication law

$$(z,t)\circ(z',t') = (z+z',t+t'+2\mathrm{Im}z\overline{z'}),$$

where $z\overline{z'} = \sum_{j=1}^n z_j\overline{z'_j}$. For $(z,t) \in \mathbb{H}^n$, its homogeneous norm is $|(z,t)| = (|z|^4+|t|^2)^{1/4}$. The set $B_r(z_0,t_0) = \{(z,t)\in\mathbb{H}^n : |(z_0,t_0)^{-1}\circ(z,t)| < r\}$ is called the ball of radius r centered at (z_0,t_0), and whose measure is given by $|B_r(z_0,t_0)| = cr^Q$, where $Q = 2n+2$ is the homogeneous dimension of $\mathbb{H}^n$.

The Lie algebra $\mathscr{G}$ of $\mathbb{H}^n$, which admits a stratification by $\mathscr{G} = V_1 \oplus V_2$, is generated by the left-invariant vector fields

$$T = \partial/\partial t,\ Z_j = \partial/\partial z_j + i\bar{z}_j\partial/\partial t, \bar{Z}_j = \partial/\partial\bar{z}_j - iz_j\partial/\partial t, 1 \leqslant j \leqslant n.$$

The horizontal layer is the first layer V_1 generated by $Z_j, \bar{Z}_j, 1 \leqslant j \leqslant n$.

Now let $\sigma = (\sigma_1,\cdots,\sigma_{2n},\sigma_{2n+1}) = (1,\cdots,1,2)$, $I = (i_1,\cdots,i_k)$ with $i_1,\cdots,i_k \in \{1,2,\cdots,2n+1\}$ and use the notation $\sigma(I) = \sigma_{i_1}+\cdots+\sigma_{i_k}$ to denote the homogeneous length of I. We also write $X_j = Z_j$, $X_{n+j} = \bar{Z}_j, 1 \leqslant j \leqslant n$, $X_{2n+1} = T$ and use the notation $X_I = X_{i_1}\cdots X_{i_k}$ to denote the left-invariant differential operator.

A function P on $\mathbb{H}^n$ is called a polynomial if $P\circ\exp$ is a polynomial on $\mathscr{G}$. Every polynomial on $\mathbb{H}^n$ can be written uniquely as a finite sum

$$P=\sum_J a_J\eta^J,\ a_J\in\mathbb{C},\ \eta^J=\eta_1^{j_1}\cdots\eta_{2n+1}^{j_{2n+1}},$$

where $\eta_j=\zeta_j\circ\log$, $\zeta_1,\cdots,\zeta_{2n+1}$ is the basis for $\mathscr{G}^*$ dual to the basis $X_1,\cdots,X_{2n+1}$ for $\mathscr{G}$. The monomial η^J is homogeneous of degree $d(J)=\sum_{i=1}^{2n}j_i+2j_{2n+1}$ and the homogeneous degree of P is given by $d(P)=\max\{d(J):a_J\neq 0\}$. For $s\in\mathbb{N}=\{0,1,2,\cdots\}$, we denote by $\mathscr{P}_s$ the space of polynomials whose homogeneous degree $\leqslant s$. Let $(z,t)\in\mathbb{H}^n$ and $f\in C^{s+1}(\mathbb{H}^n)$, the left Taylor polynomial of f at (z,t) of homogeneous degree s is the unique $P_s(f,(z,t))\in\mathscr{P}_s$ such that $X_IP_s(f,(z,t))(0)=X_If(z,t)$ for $\sigma(I)\leqslant s$. Arena *et al.* [1] established the explicit Taylor polynomial on the Heisenberg group, but unfortunately, there are no comments on the remainders. However, Bonfiglioli [5] obtained the following horizontal Taylor formula with integral remainder for the Heisenberg group:

THEOREM A. *Let* $(z',t'),(z,0)\in\mathbb{H}^n$ *and suppose that* $f\in C^{k+1}(k\in\mathbb{N})$, *then*

$$\begin{aligned}&f\big((z',t')\circ(z,0)\big)\\ &=f(z',t')+\sum_{l=1}^{k}\sum_{\substack{I=(i_1,\dots,i_l)\\ i_1,\dots,i_l\leqslant 2n}}\frac{X_If(z',t')}{l!}\xi_{i_1}\cdots\xi_{i_l}\\ &\quad+\sum_{\substack{I=(i_1,\dots,i_{k+1})\\ i_1,\dots,i_{k+1}\leqslant 2n}}\xi_{i_1}\cdots\xi_{i_{k+1}}\int_0^1(X_If)\Big((z',t')\circ\exp\Big(\sum_{j\leqslant 2n}s\xi_jX_j\Big)\Big)\frac{(1-s)^k}{k!}ds,\end{aligned}$$

where the notation $\log(z,0)=\sum_{j\leqslant 2n}\xi_jX_j$.

Now we define the Fock space $\mathscr{H}_\lambda$ consisting of all holomorphic functions F on $\mathbb{C}^n$ such that

$$\|F\|_{\mathscr{H}_\lambda}^2=\left(\frac{2\lambda}{\pi}\right)^n\int_{\mathbb{C}^n}|F(\xi)|^2e^{-2\lambda|\xi|^2}d\xi d\bar{\xi}<\infty.$$

Then $\mathscr{H}_\lambda$ is a Hilbert space with an orthogonal basis $\{E_\alpha^\lambda(\xi)=(\sqrt{2\lambda}\xi)^\alpha/\sqrt{\alpha!},\alpha\in\mathbb{N}^n\}$. For $\lambda\in\mathbb{R}^*=\mathbb{R}\backslash\{0\}$, the Fock representation Π_λ of $\mathbb{H}^n$ acts on $\mathscr{H}_{|\lambda|}$ by

$$\Pi_\lambda(z,t)F(\xi)=\begin{cases}e^{i\lambda t+2\lambda(\xi z-|z|^2/2)}F(\xi-\bar{z}), & \text{if } \lambda>0,\\ e^{i\lambda t-2\lambda(\xi\bar{z}-|z|^2/2)}F(\xi-z), & \text{if } \lambda<0.\end{cases}$$

Then $\Pi_\lambda(z,t)=\Pi_\lambda(z)e^{i\lambda t}$ is an irreducible unitary representation of $\mathbb{H}^n$ on $\mathscr{H}_{|\lambda|}$. Set $\Phi_{\alpha,\beta}^\lambda(z,t)=\big(\Pi_\lambda(z,t)E_\alpha^\lambda,E_\beta^\lambda\big)_{\mathscr{H}_\lambda}$ and $\Pi(z)=\Pi_1(z)$, then $\Phi_{\alpha,\beta}(z)=\big(\Pi(z)E_\alpha^1,E_\beta^1\big)_{\mathscr{H}_1}$ is called the special Hermite function and $\{\Phi_{\alpha,\beta}(z)\}$ forms an orthonormal basis for $L^2(\mathbb{C}^n)$. Let $L=-\Delta_z+\frac{1}{4}|z|^2+iN$, where $N=\sum_{j=1}^n(x_j\partial/\partial y_j-y_j\partial/\partial x_j)$, then $\Phi_{\alpha,\beta}(z)$ are eigenfunctions of L with eigenvalues $2|\alpha|+n$, and of the operators $-\Delta_z+\frac{1}{4}|z|^2$ with $|\alpha|+|\beta|+n$.

Note that Π is an irreducible projective representation of $\mathbb{C}^n$ into the Fock space $\mathscr{H}_1$ such that

$$\Pi(z+w) = \Pi(z)\Pi(w)e^{-i2\mathrm{Im}z\bar{w}}.$$

Given a function $f \in L^1(\mathbb{C}^n)$, the Weyl transform of f is a bounded operator on $\mathscr{H}_1$ defined by

$$\hat{f} = \int_{\mathbb{C}^n} f(z)\Pi(z)dzd\bar{z}.$$

And we have the following special Hermite expansions for functions in $L^2(\mathbb{C}^n)$:

$$f(z) = \sum_{\alpha\in\mathbb{N}^n}\sum_{\beta\in\mathbb{N}^n}\hat{f}(\alpha,\beta)\Phi_{\alpha,\beta}(z),$$

where each coefficient $\hat{f}(\alpha,\beta)$ is defined by

$$\hat{f}(\alpha,\beta) = \int_{\mathbb{C}^n} f(z)\bar{\Phi}_{\alpha,\beta}(z)dzd\bar{z}.$$

Moreover, we have

$$\sum_{\alpha\in\mathbb{N}^n}\sum_{\beta\in\mathbb{N}^n}\left|\hat{f}(\alpha,\beta)\right|^2 = \|\hat{f}\|_{HS}^2 = \|f\|_2^2.$$

For various results related to these expansions readers can refer to [17, 18].

Now for $0 < p \leqslant 1 \leqslant q \leqslant \infty$, $p \neq q$, $s \in \mathbb{N}$ and $s \geqslant [Q(1/p-1)]$, a function a is a $(p,q,s)-$atom with the center z_0 if

(i) $\mathrm{supp}(a) \subset B_r(z_0)$;

(ii) $\|a\|_q \leqslant |B_r(z_0)|^{1/q-1/p}$;

(iii) $\int_{\mathbb{C}^n} a(z)P(z-z_0)e^{i2\mathrm{Im}z\bar{z}_0}dzd\bar{z} = 0$ for any polynomial P whose degree $\leqslant s$.

For $0 < p \leqslant 1$, a tempered distribution f is said to be an element of the atomic Hardy space $H_a^p(\mathbb{C}^n)$ if it can be characterized by the decomposition

$$f(z) = \sum_{j=1}^{\infty}\lambda_j a_j(z),$$

where a_j's are $(p,q,s)-$atoms and $\sum_{j=1}^{\infty}|\lambda_j|^p < \infty$. Moreover, the space $H_a^p(\mathbb{C}^n)$ can be made into a metric space by means of the quasi-norm defined by

$$\|f\|_{H_a^p}^p = \inf\{\Sigma_{j=1}^{\infty}|\lambda_j|^p : f = \sum_{j=1}^{\infty}\lambda_j a_j\}.$$

Note that the cancelation condition of $(p,q,s)-$atom is defined in consideration of the Weyl transform, thus the atomic Hardy space $H_a^p(\mathbb{C}^n)$ defined above is different from the ordinary Hardy space $H^p(\mathbb{C}^n)$ for $0 < p < 1$ (see [13]).

3. Proofs of the main results

Before giving the proofs of the main results, we need some crucial lemmas. Throughout this article, we will adopt the convention that c denotes constants which may be different from one statement to another.

LEMMA 1. *For any* $(z,t) \in \mathbb{H}^n$, *and* $\alpha \in \mathbb{N}^n$, *we have*

$$\sum_{\beta \in \mathbb{N}^n} \left| X_I \Phi^\lambda_{\alpha,\beta}(z,t) \right|^2 \leqslant c_I \big((2|\alpha| + n)|\lambda| \big)^{\sigma(I)}.$$

Proof. We expand $\Pi_\lambda(w) E^\lambda_\alpha(\xi)$ in terms of E^λ_β by

$$\Pi_\lambda(w) E^\lambda_\alpha(\xi) = \sum_\beta \left(\Pi_\lambda(w) E^\lambda_\alpha, E^\lambda_\beta \right)_{\mathscr{H}_\lambda} E^\lambda_\beta(\xi),$$

from which we get

$$\begin{aligned} \Phi^\lambda_{\alpha,\alpha}((-z,-t)\circ(z,t)) &= \big(\Pi_\lambda(z,t) E^\lambda_\alpha,\, \Pi_\lambda(z,t) E^\lambda_\alpha \big)_{\mathscr{H}_\lambda} \\ &= \sum_{\beta \in \mathbb{N}^n} \big(\Pi_\lambda(z,t) E^\lambda_\alpha, E^\lambda_\beta \big)_{\mathscr{H}_\lambda} \big(E^\lambda_\beta,\, \Pi_\lambda(z,t) E^\lambda_\alpha \big)_{\mathscr{H}_\lambda} \\ &= \sum_{\beta \in \mathbb{N}^n} \left| \Phi^\lambda_{\alpha,\beta}(z,t) \right|^2. \end{aligned}$$

Note that $\Phi^\lambda_{\alpha,\alpha}((-z,-t)\circ(z,t)) = \Phi^\lambda_{\alpha,\alpha}(0,0) = 1$ and hence we have

$$\sum_{\beta \in \mathbb{N}^n} \left| \Phi^\lambda_{\alpha,\beta}(z,t) \right|^2 = 1.$$

Then we can easily deduce this lemma by the facts that: $X_{2n+1}\Phi^\lambda_{\alpha,\beta}(z,t) = i\lambda \Phi^\lambda_{\alpha,\beta}(z,t)$ and for $1 \leqslant j \leqslant n$,

$$X_j \Phi^\lambda_{\alpha,\beta}(z,t) = \begin{cases} \big((2\alpha_j + 2)|\lambda|\big)^{1/2} \Phi^\lambda_{\alpha+e_j,\beta}(z,t), & \text{if } \lambda > 0, \\ -\big(2\alpha_j|\lambda|\big)^{1/2} \Phi^\lambda_{\alpha-e_j,\beta}(z,t), & \text{if } \lambda < 0, \end{cases}$$

$$X_{j+n} \Phi^\lambda_{\alpha,\beta}(z,t) = \begin{cases} -\big(2\alpha_j|\lambda|\big)^{1/2} \Phi^\lambda_{\alpha-e_j,\beta}(z,t), & \text{if } \lambda > 0, \\ \big((2\alpha_j + 2)|\lambda|\big)^{1/2} \Phi^\lambda_{\alpha+e_j,\beta}(z,t), & \text{if } \lambda < 0, \end{cases}$$

where $\{e_1, \cdots, e_n\}$ is the standard basis of $\mathbb{R}^n$. $\square$

LEMMA 2. *Let $P_k(\Phi_{\alpha,\gamma},0)$ be the horizontal Taylor polynomial of $\Phi_{\alpha,\gamma}$ at origin of homogeneous degree k. Then for $(z,0),(z_0,0)\in\mathbb{H}^n$ we have*

$$\sum_{\beta\in\mathbb{N}^n}\Big|\sum_{\gamma}\Phi_{\gamma,\beta}(z_0,0)\Big(\Phi_{\alpha,\gamma}(z,0)-P_k(\Phi_{\alpha,\gamma},0)(z,0)\Big)\Big|^2\leqslant c_k|z|^{2k+2}\big(2|\alpha|+n\big)^{k+1}.$$

Proof. For the special Hermite functions we find that

$$\Phi_{\alpha,\beta}(z_0+z)=\sum_{\gamma}\Phi_{\gamma,\beta}(z_0)\Phi_{\alpha,\gamma}(z)e^{i2\mathrm{Im}z\bar{z}_0}.$$

Hence

$$\Big|\sum_{\gamma}\Phi_{\gamma,\beta}(z_0,0)X_I\Phi_{\alpha,\gamma}(z,0)\Big|=\big|X_I\Phi_{\alpha,\beta}((z_0,0)(z,0))\big|.$$

Then for $(z,0)=\exp(\sum_{j\leqslant 2n}\xi_jX_j)$, by the horizontal Taylor formula and Lemma 1 we have

$$\begin{aligned}
&\sum_{\beta\in\mathbb{N}^n}\Big|\sum_{\gamma}\Phi_{\gamma,\beta}(z_0)\Big(\Phi_{\alpha,\gamma}(z,0)-P_k(\Phi_{\alpha,\gamma},0)(z,0)\Big)\Big|^2\\
&=\sum_{\beta\in\mathbb{N}^n}\Big|\sum_{\gamma}\Phi_{\gamma,\beta}(z_0)\Big(\sum_{\substack{I=(i_1,\ldots,i_{k+1})\\ i_1,\ldots,i_{k+1}\leqslant 2n}}\xi_{i_1}\cdots\xi_{i_{k+1}}\int_0^1X_I\Phi_{\alpha,\gamma}\big(\exp\big(\sum_{j\leqslant 2n}s\xi_jX_j\big)\big)\frac{(1-s)^k}{k!}ds\Big)\Big|^2\\
&\leqslant\sum_{\beta\in\mathbb{N}^n}\Big(\sum_{\substack{I=(i_1,\ldots,i_{k+1})\\ i_1,\ldots,i_{k+1}\leqslant 2n}}\big|\xi_{i_1}\cdots\xi_{i_{k+1}}\big|^2\Big)\\
&\quad\times\Big(\sum_{\substack{I=(i_1,\ldots,i_{k+1})\\ i_1,\ldots,i_{k+1}\leqslant 2n}}\Big|\int_0^1\sum_{\gamma}\Phi_{\gamma,\beta}(z_0,0)X_I\Phi_{\alpha,\gamma}\Big(\exp\Big(\sum_{j\leqslant 2n}s\xi_jX_j\Big)\Big)\frac{(1-s)^k}{k!}ds\Big|^2\Big)\\
&\leqslant c_k'|z|^{2k+2}\sum_{\substack{I=(i_1,\ldots,i_{k+1})\\ i_1,\ldots,i_{k+1}\leqslant 2n}}\int_0^1\sum_{\beta\in\mathbb{N}^n}\Big|X_I\Phi_{\alpha,\beta}((z_0,0)(z_s,0))\Big|^2ds\int_0^1\frac{(1-s)^{2k}}{(k!)^2}ds\\
&\leqslant c_k|z|^{2k+2}\big(2|\alpha|+n\big)^{k+1},
\end{aligned}$$

where we have used the notation $(z_s,0)=\exp(\sum_{j\leqslant 2n}s\xi_jX_j)$ and the fact $|\xi_{i_l}|\leqslant c|(z,0)|=c|z|$ (see Lemma 3 of [5]) in the second inequality. □

LEMMA 3. *Let a be the $(p,2,s)$-atom with the center z_0. Then we have*
(i) $\sum_{\beta\in\mathbb{N}^n}\big|\hat{a}(\alpha,\beta)\big|^2\leqslant c(2|\alpha|+n)^{s+1}\|a\|_2^{2-\frac{s+1+n}{n(1/p-1/2)}}$;
(ii) $\sum_{\beta\in\mathbb{N}^n}\big|\hat{a}(\alpha,\beta)\big|^2\leqslant c\|a\|_2^{2-\frac{1}{1/p-1/2}}$.

Proof. For the first estimate, assume that $P_s(\Phi_{\alpha,\gamma},0)(z,0)$ is the Taylor polynomial of $\Phi_{\alpha,\gamma}(z,0)$ at origin of homogeneous degree s. By the property (ii) of p-atom, i.e., $\|a\|_2^{-1/[2n(1/p-1/2)]} \geqslant r$, together with the property (iii) of p-atom and Lemma 2 we have

$$\begin{aligned}
&\sum_{\beta\in\mathbb{N}^n} \left|\hat{a}(\alpha,\beta)\right|^2\\
&= \sum_{\beta}\left|\int_{B_r(0)} \bar{a}(z_0+z)\Phi_{\alpha,\beta}(z_0+z)dzd\bar{z}\right|^2\\
&= \sum_{\beta}\left|\int_{B_r(0)} \bar{a}(z_0+z)\sum_{\gamma}\Phi_{\gamma,\beta}(z_0)\Phi_{\alpha,\gamma}(z)e^{i2\mathrm{Im}z\bar{z}_0}dzd\bar{z}\right|^2\\
&= \sum_{\beta}\left|\sum_{\gamma}\Phi_{\gamma,\beta}(z_0)\int_{B_r(0)} \bar{a}(z_0+z)\Big(\Phi_{\alpha,\gamma}(z,0)-P_s(\Phi_{\alpha,\gamma},0)(z,0)\Big)e^{i2\mathrm{Im}z\bar{z}_0}dzd\bar{z}\right|^2\\
&\leqslant \|a\|_2^2\sum_{\beta}\int_{B_r(0)}\left|\sum_{\gamma}\Phi_{\gamma,\beta}(z_0)\Big(\Phi_{\alpha,\gamma}(z,0)-P_s(\Phi_{\alpha,\gamma},0)(z,0)\Big)\right|^2 dzd\bar{z}\\
&\leqslant c\|a\|_2^2(2|\alpha|+n)^{s+1}\int_{B_r(0)}|z|^{2s+2}dzd\bar{z}\\
&\leqslant c(2|\alpha|+n)^{s+1}\|a\|_2^{2-\frac{2s+2+2n}{2n(1/p-1/2)}}.
\end{aligned}$$

For the second estimate, by Lemma 1 and the property (ii) of $p-$atom we get

$$\begin{aligned}
\sum_{\beta\in\mathbb{N}^n}\left|\hat{a}(\alpha,\beta)\right|^2 &= \sum_{\beta}\left|\int_{B_r}\bar{a}(z)\Phi_{\alpha,\beta}(z)dzd\bar{z}\right|^2\\
&\leqslant \sum_{\beta}\int_{B_r}|\Phi_{\alpha,\beta}(z)|^2dzd\bar{z}\|a\|_2^2\\
&\leqslant cr^{2n}\|a\|_2^2\\
&\leqslant c\|a\|_2^{2-\frac{1}{(1/p-1/2)}}.
\end{aligned}$$

Hence the proof of this lemma is completed. □

Proof of Theorem 1. For $f \in H_a^p(\mathbb{C}^n)$, it follows that $f=\sum_{j=1}^{\infty}\lambda_j a_j$, where a_j's are the $(p,2,s)$-atoms and $\sum_{j=1}^{\infty}|\lambda_j|^p<\infty$. Then we get

$$\left|\hat{f}(\alpha,\beta)\right| \leqslant \sum_{j=1}^{\infty}|\lambda_j|\left|\hat{a}_j(\alpha,\beta)\right|.$$

Hence to prove this theorem, it is enough to show that for any j,

$$\left|\hat{a}_j(\alpha,\beta)\right| \leqslant c(2|\alpha|+n)^{\frac{2n}{2}(\frac{1}{p}-1)}.$$

In order to do this we first assume that $r^{-2} \leqslant 2|\alpha| + n$. By (ii) of Lemma 3 we obtain

$$\left|\hat{a}_j(\alpha,\beta)\right|^2 \leqslant c\|a_j\|_2^{2-\frac{1}{1/p-1/2}} \leqslant c(2|\alpha|+n)^{2n(\frac{1}{p}-1)}.$$

Next, if $r^{-2} > 2|\alpha| + n$, then by (i) of Lemma 3 we get

$$\begin{aligned}\left|\hat{a}_j(\alpha,\beta)\right|^2 &\leqslant c(2|\alpha|+n)^{s+1}\|a_j\|_2^{2-\frac{s+1+n}{n(1/p-1/2)}} \\ &\leqslant c(2|\alpha|+n)^{s+1}(2|\alpha|+n)^{2n(\frac{1}{p}-1)-s-1} \\ &= c(2|\alpha|+n)^{2n(\frac{1}{p}-1)}.\end{aligned}$$

As desired, we finish the proof of this theorem. □

Proof of Theorem 2. Again using the fact that $f = \sum_{j=1}^{\infty} \lambda_j a_j$, where a_j's are the $(p,2,s)$-atoms and $\sum_{j=1}^{\infty} |\lambda_j|^p < \infty$, we have

$$\begin{aligned}&\sum_{\alpha,\beta} \left|\hat{f}(\alpha,\beta)\right|^p (|\alpha|+|\beta|+n)^{-\sigma} \\ &\leqslant \sum_{j=1}^{\infty} |\lambda_j|^p \sum_{\alpha,\beta} \left|\hat{a}_j(\alpha,\beta)\right|^p (|\alpha|+|\beta|+n)^{-\sigma} \\ &\leqslant \sum_{j=1}^{\infty} |\lambda_j|^p \Big(\sum_{|\alpha|+|\beta|+n>r^{-2}} + \sum_{|\alpha|+|\beta|+n\leqslant r^{-2}} \Big) \left|\hat{a}_j(\alpha,\beta)\right|^p (|\alpha|+|\beta|+n)^{-\sigma} \\ &= \sum_{j=1}^{\infty} |\lambda_j|^p (S_1 + S_2),\end{aligned}$$

from which we see that to get this theorem it suffices to show that S_1 and S_2 are bounded. For the term S_1, by Hölder's inequality we have

$$\begin{aligned}S_1 &= \sum_{|\alpha|+|\beta|+n>r^{-2}} \left|\hat{a}_j(\alpha,\beta)\right|^p (|\alpha|+|\beta|+n)^{-\sigma} \\ &\leqslant \left(\sum_{\alpha} \sum_{\beta} \left|\hat{a}_j(\alpha,\beta)\right|^{p\frac{2}{p}} \right)^{\frac{p}{2}} \left(\sum_{(|\alpha|+|\beta|+n)>r^{-2}} (|\alpha|+|\beta|+n)^{-\frac{2\sigma}{2-p}} \right)^{\frac{2-p}{2}} \\ &\leqslant c\|a_j\|_2^p \left(\sum_{k+n>r^{-2}} \big(k+n\big)^{-\frac{2\sigma}{2-p}+2n-1} \right)^{\frac{2-p}{2}} \\ &\leqslant c r^{np-2n} r^{2\sigma-2n(2-p)} \\ &= c,\end{aligned}$$

where the last inequality follows from the fact $\|a_j\|_2 \leqslant c r^{2n(\frac{1}{2}-\frac{1}{p})}$.

Now for the term S_2. Thanks to (i) of Lemma 3, we have

$$
\begin{aligned}
S_2 &= \sum_{|\alpha|+|\beta|+n\leqslant r^{-2}} \left|\hat{a}_j(\alpha,\beta)\right|^p (|\alpha|+|\beta|+n)^{-\sigma} \\
&= \sum_{|\alpha|\leqslant r^{-2}} \sum_{|\beta|\leqslant r^{-2}-|\alpha|-n} \left|\hat{a}_j(\alpha,\beta)\right|^p (|\alpha|+|\beta|+n)^{-\sigma} \\
&\leqslant \sum_{|\alpha|\leqslant r^{-2}} \left(\sum_{|\beta|\leqslant r^{-2}-|\alpha|-n} \left|\hat{a}_j(\alpha,\beta)\right|^{p\cdot\frac{2}{p}} \right)^{\frac{p}{2}} \left(\sum_{|\beta|\leqslant r^{-2}-|\alpha|-n} (|\alpha|+|\beta|+n)^{-\sigma\cdot\frac{2}{2-p}} \right)^{\frac{2-p}{2}} \\
&\leqslant \sum_{|\alpha|\leqslant r^{-2}} \left(\sum_{\beta} \left|\hat{a}_j(\alpha,\beta)\right|^2 \right)^{\frac{p}{2}} \left(\sum_{|\beta|\leqslant r^{-2}-|\alpha|-n} (|\alpha|+|\beta|+n)^{-\sigma\cdot\frac{2}{2-p}} \right)^{\frac{2-p}{2}} \\
&\leqslant c\|a_j\|_2^{p(1-\frac{s+1+n}{2n(1/p-1/2)})} \sum_{|\alpha|\leqslant r^{-2}} (2|\alpha|+n)^{\frac{p(s+1)}{2}} \left(\sum_{|\beta|\leqslant r^{-2}-|\alpha|-n} (|\alpha|+|\beta|+n)^{-\sigma\cdot\frac{2}{2-p}} \right)^{\frac{2-p}{2}} \\
&\leqslant c\|a_j\|_2^{p(1-\frac{s+1+n}{2n(1/p-1/2)})} \sum_{|\alpha|\leqslant r^{-2}} \left(\sum_{|\beta|\leqslant r^{-2}-|\alpha|-n} (|\alpha|+|\beta|+n)^{(\frac{p(s+1)}{2}-\sigma)\cdot\frac{2}{2-p}} \right)^{\frac{2-p}{2}} \\
&\leqslant c\|a_j\|_2^{p(1-\frac{s+1+n}{2n(1/p-1/2)})} \left(\sum_{k+n\leqslant r^{-2}} (k+n)^{\left(2n-1+[(\frac{p(s+1)}{2}-\sigma)\frac{2}{2-p}]\right)} \right)^{\frac{2-p}{2}} \left(\sum_{|\alpha|\leqslant r^{-2}} 1 \right)^{\frac{p}{2}} \\
&\leqslant cr^{2np(\frac{1}{2}-\frac{1}{p})\left(1-\frac{s+1+n}{2n(1/p-1/2)}\right)} r^{2\sigma-p(s+1)-2n(2-p)} r^{-np} \\
&= c.
\end{aligned}
$$

This ends the proof. □

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Jinsen Xiao
School of Sciences
Guangdong University of Petrochemical Technology
Maoming 525000, P. R. China
e-mail: jinsenxiao@yahoo.com

Coresponding author:
Jianxun He
School of Mathematics and Information Sciences
Guangzhou University
Guangzhou 510006, P. R. China
e-mail: hejianxun@gzhu.edu.cn