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ON THE COMPOSITION OF FUNCTIONS IN MULTIDIMENSIONAL BESOV SPACES

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Abstract. For the composition operator $T_f : g \mapsto f \circ g$ we find a class of functions $f : \mathbb{R} \to \mathbb{R}$ for which there exists a family of positive constants $c(f,t)$, $t > 0$, such that the estimate

$$\|T_f(g)\|_{B^s_{p,q}(\mathbb{R}^n)} \leqslant c(f,t)\|g\|_{B^s_{p,q}(\mathbb{R}^n)}$$

holds, for all $g \in W^1_\infty \cap B^s_{p,q}(\mathbb{R}^n)$ satisfying $\|\nabla g\|_\infty \leqslant t$ (or $g \in L_\infty \cap B^s_{p,q}(\mathbb{R}^n)$ with $\|g\|_\infty \leqslant t$ and $[s] = 1$). We establish this assertion, for all $f \in B^{s_1,\ell oc}_{p,\infty}(\mathbb{R})$ with $s_1 > 1 + 1/p$, in the case $1 < p < \infty$, $0 < q \leqslant \infty$ and $0 < s < s_1$.

1. Introduction

In the present paper we deal with the mapping properties of the nonlinear composition operator $T_f : g \mapsto f \circ g$, associated with a function $f : \mathbb{R} \to \mathbb{R}$, on Besov spaces $B^s_{p,q}(\mathbb{R}^n)$ with $n > 1$. We are interested in the two following properties

$$T_f(B^s_{p,q}(\mathbb{R}^n)) \subseteq B^s_{p,q}(\mathbb{R}^n) \quad \text{(Full acting property)},$$

$$T_f(B^s_{p,q}(\mathbb{R}^n) \cap L_\infty(\mathbb{R}^n)) \subseteq B^s_{p,q}(\mathbb{R}^n) \quad \text{(Restricted acting property)}.$$

Based on the different previous works, e.g., [8], [9], [10] in cases $n = 1$ and $n \geqslant 2$, we believe on the following conjecture.

CONJECTURE 1. *If $1 \leqslant p < \infty$, $0 < q \leqslant \infty$ and $s > 1 + 1/p$, then the restricted acting property holds if, and only if, $f(0) = 0$ and $f \in B^{s,\ell oc}_{p,q}(\mathbb{R})$.*

In the context of this conjecture we recall:

(i) If $T_f(B^s_{p,q}(\mathbb{R}^n) \cap L_\infty(\mathbb{R}^n)) \subseteq B^s_{p,\infty}(\mathbb{R}^n)$, then f is locally Lipschitz continuous, i.e., $f \in W^{1,\ell oc}_\infty(\mathbb{R})$, see [3] or [19].

(ii) If $T_f(\mathscr{S}(\mathbb{R}^n)) \subseteq B^s_{p,q}(\mathbb{R}^n)$, then $f \in B^{s,\ell oc}_{p,q}(\mathbb{R})$, see [19, 5.3.1/thm. 2]

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With this respect, the conditions $f(0)=0$ (obtained by testing f on the zero function) and $f\in W^{1,\ell oc}_{\infty}(\mathbb{R})$ or $f\in B^{s,\ell oc}_{p,q}(\mathbb{R})$, are necessary for a function f, such that T_f maps $B^s_{p,q}(\mathbb{R}^n)$ to itself. We refer to [19, 5.3] for some sufficient conditions, and to [11] for motivations of this conjecture. In the case $n=1$, $1<p<\infty$ and $0<q\leqslant\infty$ Conjecture 1 has been established, see [10], the case $p=1$ and $s\in\mathbb{N}$ is an open question. On the contrary, for $n>1$ only partial results are known. The restricted acting property holds in the following two cases:

(A1) If $1<p<\infty$, $q=p$, $s>1+n/p$, $f(0)=0$ and $f\in B^{s,\ell oc}_{p,p}(\mathbb{R})$. See [10, rem. 1].

(A2) If $1<p<\infty$, $1\leqslant q\leqslant\infty$, $1+1/p<s<2$, $f(0)=0$ and $f\in B^{s+\varepsilon,\ell oc}_{p,q}(\mathbb{R})$ for some $\varepsilon>0$, see [5] and Remark 4 below.

In all known cases, if the acting property holds, then T_f is a bounded operator on $B^s_{p,q}(\mathbb{R}^n)$. For instance, the acting property in case $n=1$ and $s>1+1/p$, is reflected by the following estimation

$$\|f\circ g\|_{B^s_{p,q}(\mathbb{R})}\leqslant c\|f'\|_{B^{s-1}_{p,q}(\mathbb{R})}\|g\|_{B^s_{p,q}(\mathbb{R})}(1+\|g\|_{B^s_{p,q}(\mathbb{R})})^{s-1-1/p}, \tag{1}$$

(see our previous works [8], [9] and [17]). In (1), a replacement of the factor $(1+\|g\|_{B^s_{p,q}(\mathbb{R})})^{s-1-1/p}$ either by $(1+\|g\|_\infty)^{s-1-1/p}$ or by $(1+\|g'\|_\infty)^{s-1-1/p}$ can be used for an extension to the n-dimensional case, at least partly. In this sense we shall establish an estimate of the form

$$\|f\circ g\|_{B^s_{p,q}(\mathbb{R}^n)}\leqslant c(f)\|g\|_{B^s_{p,q}(\mathbb{R}^n)}$$

improving (A1)–(A2) on that way. Our method consists in studying the composition operator on intersections. Recall that the study of T_f on intersections has a certain history, e.g., in case of Sobolev spaces and/or Lizorkin-Triebel spaces by Adams and Frazier [1], [2], Brezis and Mironescu [12], Maz'ya and Shaposnikova [15], Bourdaud [6], Runst and Sickel [19, 5.3.7] as well as [16].

NOTATION. As usual, $\mathbb{N}$ denotes the natural numbers, $\mathbb{N}_0:=\mathbb{N}\cup\{0\}$, $\mathbb{Z}$ the integers, $\mathbb{R}$ the real numbers and if s is a real number then $[s]$ denotes its integer part. We work in Euclidean spaces $\mathbb{R}^n$. All functions are assumed to be real-valued. For $1\leqslant p\leqslant\infty$ we denote by $\|\cdot\|_p$ the L_p-norm, and by $W^m_p(\mathbb{R}^n)$ ($m\in\mathbb{N}$) the usual Sobolev space with $\dot W^m_p(\mathbb{R}^n)$ its the homogeneous counterpart (i.e., $\dot W^m_p(\mathbb{R}^n)$ is the set of $f\in\mathscr{S}'(\mathbb{R}^n)$ such that $f^{(\alpha)}\in L_p(\mathbb{R}^n)$ for $|\alpha|=m$ and is endowed with the seminorm $\|f\|_{\dot W^m_p(\mathbb{R}^n)}:=\sum_{|\alpha|=m}\|f^{(\alpha)}\|_p$). We denote by $C_b(\mathbb{R}^n)$ and $C_{ub}(\mathbb{R}^n)$ the Banach space of bounded and continuous functions on $\mathbb{R}^n$, and the Banach space of bounded and uniformly continuous functions on $\mathbb{R}^n$, respectively, endowed with the supremum. For a space E of functions defined on $\mathbb{R}^n$ the associated local space $E^{\ell oc}$ is the set of $f\in\mathscr{S}'(\mathbb{R}^n)$ such that $\varphi f\in E$ for all $\varphi\in\mathscr{D}(\mathbb{R}^n)$. We define the difference operators for an arbitrary function f by $\Delta_h f:=f(\cdot+h)-f$, $\Delta^1_h f:=\Delta_h f$ and $\Delta^{m+1}_h f:=\Delta_h(\Delta^m_h f)$, $h\in\mathbb{R}^n$, $m\in\mathbb{N}$. We denote by $\mathscr{S}_\infty(\mathbb{R}^n)$ the set of all $u\in\mathscr{S}(\mathbb{R}^n)$ such that $\langle x^\alpha,u\rangle=0$ for all $\alpha\in\mathbb{N}^n_0$, and $\mathscr{S}'_\infty(\mathbb{R}^n)$ denotes its topological dual. For all $f\in\mathscr{S}'(\mathbb{R}^n)$, we denote by $[f]_\infty$ the equivalence class of f modulo polynomials on $\mathbb{R}^n$. The mapping which takes any $[f]_\infty$ to the restriction of f to $\mathscr{S}_\infty(\mathbb{R}^n)$ turns out to be an isomorphism from the "$\mathscr{S}'(\mathbb{R}^n)$ modulo polynomials" onto $\mathscr{S}'_\infty(\mathbb{R}^n)$. Finally, constants $c, c_1, \ldots$ are strictly positive and depend only on the fixed parameters n, s, p and q, their values may vary from line to line.

Plan of the paper. In Section 2 we formulate the main results of the paper. In Section 3 we collect the needed properties of homogeneous and inhomogeneous Besov spaces. Section 4 is devoted to the proofs. In the last section, we discuss open questions.

2. The main results

The main result is the following.

THEOREM 1. *Let $1 < p < \infty$ and $0 < q \leqslant \infty$. Let s, s_1 be real numbers satisfying*

$$s_1 > 1 + \frac{1}{p} \quad \textit{and} \quad 0 < s < s_1.$$

If the function f belongs to $B^{s_1;\ell oc}_{p,\infty}(\mathbb{R})$ and $f(0) = 0$, then the composition operator T_f takes $B^s_{p,q}(\mathbb{R}^n) \cap W^1_\infty(\mathbb{R}^n)$ to $B^s_{p,q}(\mathbb{R}^n)$.

The proof of Theorem 1 relies upon the following proposition, which has its own interest. We need a standard cut-off function. More precisely, we consider once and for all a function ρ, that is a radial C^∞ on $\mathbb{R}$ satisfying $0 \leqslant \rho \leqslant 1$, $\rho(\xi) = 1$ for $|\xi| \leqslant 1$ and $\rho(\xi) = 0$ for $|\xi| \geqslant 3/2$. We put $\rho_t(x) := \rho(t^{-1}x)$ for all $x \in \mathbb{R}$ and all $t > 0$.

PROPOSITION 1. *Let s, s_1, p and q be real numbers as in Theorem 1. Then there exists a constant $c > 0$ such that*

$$\|f \circ g\|_{B^s_{p,q}(\mathbb{R}^n)} \leqslant c \|f \rho_{\|g\|_\infty}\|_{B^{s_1}_{p,\infty}(\mathbb{R})} \|g\|_{B^s_{p,q}(\mathbb{R}^n)} (1 + \|\nabla g\|_\infty)^{\frac{s}{s_1}(s_1 - 1 - 1/p)} \tag{2}$$

holds, for all $f \in B^{s_1;\ell oc}_{p,\infty}(\mathbb{R})$ such that $f(0) = 0$ and all $g \in B^s_{p,q}(\mathbb{R}^n) \cap W^1_\infty(\mathbb{R}^n)$.

REMARK 1. By definition of s_1, we notice that $\lim_{s_1 \downarrow s} \frac{s}{s_1}(s_1 - 1 - 1/p) = s - 1 - 1/p$.

REMARK 2. Concerning the sharpness of the estimate (2), we can observe the following: by Proposition 1, we obtain a class of functions f for which there exists a family of constants $c(f,t)$, $t > 0$, such that the inequality

$$\|f \circ g\|_{B^s_{p,q}(\mathbb{R}^n)} \leqslant c(f,t) \|g\|_{B^s_{p,q}(\mathbb{R}^n)}$$

holds, for all $g \in B^s_{p,q}(\mathbb{R}^n) \cap W^1_\infty(\mathbb{R}^n)$ satisfying $\|\nabla g\|_\infty \leqslant t$. Such an estimation cannot be improved, since it becomes an equality in case $f(x) := x$.

Now, using the homogeneous Besov space $\dot{B}^s_{p,q}(\mathbb{R})$, we introduce the space $\mathscr{B}^s_{p,q}(\mathbb{R})$ of essentially bounded functions f such that $[f]_\infty \in \dot{B}^s_{p,q}(\mathbb{R})$, and endow it with the natural quasi-norm

$$\|f\|_{\mathscr{B}^s_{p,q}(\mathbb{R})} := \|f\|_\infty + \|[f]_\infty\|_{\dot{B}^s_{p,q}(\mathbb{R})}. \tag{3}$$

In [17, sec. 3.4] we find some properties of $\mathscr{B}^s_{p,q}(\mathbb{R})$ as embeddings, equivalent quasi-norms, characterization by differences, etc., see also Section 3.3 below. Then, as mentioned before and in the sense of (1), we have the following result.

THEOREM 2. *Let s, s_1, p and q be real numbers as in Theorem 1. Then there exists a constant $c > 0$ such that*

$$\|f \circ g\|_{B^s_{p,q}(\mathbb{R}^n)} \leqslant c\|f'\|_{\mathscr{B}^{s_1-1}_{p,\infty}(\mathbb{R})} \|g\|_{B^s_{p,q}(\mathbb{R}^n)} (1 + \|\nabla g\|_\infty)^{\frac{s}{s_1}(s_1-1-1/p)} \tag{4}$$

holds, for all f such that $f(0) = 0$, $f' \in \mathscr{B}^{s_1-1}_{p,\infty}(\mathbb{R})$ and all $g \in B^s_{p,q}(\mathbb{R}^n) \cap \dot{W}^1_\infty(\mathbb{R}^n)$.

REMARK 3. Under the assumptions of Theorem 2, the inequality (4) holds also for functions f such that $f' \in B^{s_1-1}_{p,\infty}(\mathbb{R})$, since the embedding $B^{s_1}_{p,\infty}(\mathbb{R}) \hookrightarrow \mathscr{B}^{s_1}_{p,\infty}(\mathbb{R})$ is satisfied, see Proposition 11(i) below.

The following result concerns the particular case $0 < s < 2$.

THEOREM 3. *Let $1 < p < \infty$ and $0 < q \leqslant \infty$. Let s, s_1 be real numbers satisfying*

$$s_1 > 1 + \frac{1}{p} \quad \text{and} \quad 0 < s < \min(s_1, 2).$$

If the function f belongs to $B^{s_1,\ell oc}_{p,\infty}(\mathbb{R})$ and $f(0) = 0$, then the composition operator T_f takes $B^s_{p,q}(\mathbb{R}^n) \cap L_\infty(\mathbb{R}^n)$ to $B^s_{p,q}(\mathbb{R}^n)$.

REMARK 4. Theorem 3 has been formulated in [5, thm. 3.1] without the restriction $s < 2$. However, the proof given there contains a gap. Indeed, the argument in the proof of the induction is based on the algebra property of $B^{s-1}_{p,q}(\mathbb{R})$, $s > 1 + 1/p$, (see, e.g., [4, thm. 1.1/proof], [8, p. 951] or [10, 3.1.2]), which is no longer true in higher dimensions. We can extend Theorem 3 to some values $s \geqslant 2$ depending on the algebra property (see Theorem 5 below).

Similar to Theorem 1, the proof of Theorem 3 follows from an inequality of type (4), i.e., we prove the following assertion.

THEOREM 4. *Let $1 < p < \infty$ and $0 < q \leqslant \infty$. Let s, s_1 be real numbers satisfying*

$$1 + \frac{1}{p} < s_1 < 2 \quad \text{and} \quad 0 < s < s_1. \tag{5}$$

Then there exists a constant $c > 0$ such that

$$\|f \circ g\|_{B^s_{p,q}(\mathbb{R}^n)} \leqslant c\|f'\|_{\mathscr{B}^{s_1-1}_{p,\infty}(\mathbb{R})} \|g\|_{B^s_{p,q}(\mathbb{R}^n)} (1 + \|g\|_\infty)^{\frac{s}{s_1}(s_1-1-1/p)}$$

holds, for all f such that $f(0) = 0$, $f' \in \mathscr{B}^{s_1-1}_{p,\infty}(\mathbb{R})$ and all $g \in B^s_{p,q}(\mathbb{R}^n) \cap L_\infty(\mathbb{R}^n)$.

REMARK 5. (i) As in Remark 3, we can replace $\mathscr{B}_{p,\infty}^{s_1-1}(\mathbb{R})$ by $B_{p,\infty}^{s_1-1}(\mathbb{R})$ in Theorem 4, then in that case the condition (5) becomes $s_1 > 1+1/p$ and $0 < s < \min(s_1,2)$, i.e., we do not need the assumption $s_1 < 2$ since Besov spaces are monotone in the sense $B_{p,\infty}^{s_2}(\mathbb{R}) \hookrightarrow B_{p,\infty}^{s_1}(\mathbb{R})$ for $s_2 > s_1$.

(ii) As in Proposition 1, Theorem 4 also holds by taking both $f \in B_{p,\infty}^{s_1,\ell oc}(\mathbb{R})$ and $\|f\rho_{\|g\|_\infty}\|_{B_{p,\infty}^{s_1}(\mathbb{R})}$ instead of $f' \in \mathscr{B}_{p,\infty}^{s_1-1}(\mathbb{R})$ and $\|f'\|_{\mathscr{B}_{p,\infty}^{s_1-1}(\mathbb{R})}$, respectively. We also have the counterpart of Remark 2.

Now, we turn to the end of Remark 4 by considering the case of Besov algebra spaces. Then Theorem 3 becomes true for some values $s \geqslant 2$ and Conjecture 1 holds at least with $B_{p,p}^{s}(\mathbb{R}^n)$.

THEOREM 5. *Let* $0 < q \leqslant \infty$. *Let either*

$$\max\left(1,\frac{n}{2}\right) < p < \infty, \quad s_1 > 1+\frac{1}{p} \quad \textit{and} \quad \frac{n}{p} < s < \min(s_1,2)$$

or

$$1 < p < \infty,\ [s] \geqslant 2 \quad \textit{and} \quad \frac{n}{p}+1 < s < s_1$$

be satisfied. If $f \in B_{p,\infty}^{s_1,\ell oc}(\mathbb{R})$ *and* $f(0) = 0$, *then the composition operator* T_f *takes* $B_{p,q}^{s}(\mathbb{R}^n)$ *to itself.*

THEOREM 6. *Let* $\max(1,n/2) < p < \infty$. *Let* s *be a real number such that*

$$\max\left(\frac{n}{p},1+\frac{1}{p}\right) < s < 2.$$

Then the composition operator T_f *takes* $B_{p,p}^{s}(\mathbb{R}^n)$ *to itself if, and only if,* $f(0) = 0$ *and* $f \in B_{p,p}^{s,\ell oc}(\mathbb{R})$.

REMARK 6. Theorem 6 was proved in [10, rem. 1] under the assumption $s > 1+n/p$.

3. The Besov spaces

For Besov spaces we do not go into details, we recall some properties and refer the reader to [18], [14] [20], [21], [19], [7] and [17]. Let γ_1 be a fixed radial C^∞ function on $\mathbb{R}^n$ such that $0 \leqslant \gamma_1 \leqslant 1$, $\gamma_1(\xi) = 1$ if $|\xi| \leqslant 1$ and $\gamma_1(\xi) = 0$ if $|\xi| \geqslant 3/2$. We put $\gamma(\xi) := \gamma_1(\xi) - \gamma_1(2\xi)$ for all $\xi \in \mathbb{R}^n$. Then γ is supported by the compact annulus $1/2 \leqslant |\xi| \leqslant 3/2$, and the following identities hold:

$$\sum_{j\in\mathbb{Z}} \gamma(2^j\xi) = 1 \quad (\forall \xi \in \mathbb{R}^n \setminus \{0\}),$$

$$\gamma_1(\xi) + \sum_{j\geqslant 1} \gamma(2^{-j}\xi) = 1 \qquad (\forall \xi \in \mathbb{R}^n).$$

For any $j \in \mathbb{Z}$, we introduce the pseudodifferential operators $S_j := \gamma_1(2^{-j}D)$ and $Q_j := \gamma(2^{-j}D)$. It is clear that S_j is defined on $\mathscr{S}'(\mathbb{R}^n)$, and that Q_j is defined on $\mathscr{S}'_\infty(\mathbb{R}^n)$ since $Q_j f = 0$ $(\forall j \in \mathbb{Z})$ if, and only if, f is a polynomial. We also notice that according to Young inequality in $L_p(\mathbb{R}^n)$ the families of operators $(S_j)_{j\in\mathbb{Z}}$ and $(Q_j)_{j\in\mathbb{Z}}$ constitute bounded subsets of the normed space $\mathscr{L}(L_p(\mathbb{R}^n))$ for any $p \in [1, +\infty]$. On the other hand, the Littlewood-Paley decompositions of a tempered distribution are described in the following well known statement.

PROPOSITION 2. (i) *For every* $f \in \mathscr{S}_\infty(\mathbb{R}^n)$ (*resp.* $\mathscr{S}'_\infty(\mathbb{R}^n)$), *it holds* $f = \sum_{j\in\mathbb{Z}} Q_j f$, *in* $\mathscr{S}_\infty(\mathbb{R}^n)$ (*resp.* $\mathscr{S}'_\infty(\mathbb{R}^n)$).

(ii) *For every* $f \in \mathscr{S}(\mathbb{R}^n)$ (*resp.* $\mathscr{S}'(\mathbb{R}^n)$) *and every* $k \in \mathbb{Z}$, *it holds* $f = S_k f + \sum_{j>k} Q_j f$, *in* $\mathscr{S}(\mathbb{R}^n)$ (*resp.* $\mathscr{S}'(\mathbb{R}^n)$).

3.1. The homogeneous Besov spaces

DEFINITION 1. The homogeneous Besov space $\dot{B}^s_{p,q}(\mathbb{R}^n)$ is the set of $f \in \mathscr{S}'_\infty(\mathbb{R}^n)$ such that

$$\|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)} := \Big(\sum_{j\in\mathbb{Z}} (2^{sj}\|Q_j f\|_p)^q\Big)^{1/q} < \infty.$$

$\dot{B}^s_{p,q}(\mathbb{R}^n)$ is a quasi-Banach space for the above defined quasi-norm, contains $\mathscr{S}_\infty(\mathbb{R}^n)$ and is continuously embedded in $\mathscr{S}'_\infty(\mathbb{R}^n)$. It has the remarkable property

$$c_1\|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)} \leqslant \lambda^{n/p-s}\|f(\lambda(\cdot))\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)} \leqslant c_2\|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)} \quad (\forall \lambda > 0).$$

Another equivalent quasi-norm is given by the following assertion (cf., [7]).

PROPOSITION 3. *A member* f *of* $\mathscr{S}'_\infty(\mathbb{R}^n)$ *belongs to* $\dot{B}^s_{p,q}(\mathbb{R}^n)$ *if, and only if, its first order derivatives* $\partial_j f$ *belongs to* $\dot{B}^{s-1}_{p,q}(\mathbb{R}^n)$ *for* $j = 1, \dots, n$.
Moreover $\sum_{j=1}^n \|\partial_j f\|_{\dot{B}^{s-1}_{p,q}(\mathbb{R}^n)}$ *is an equivalent quasi-norm in* $\dot{B}^s_{p,q}(\mathbb{R}^n)$.

The following property is useful for us, see [14, thm. 2.1].

PROPOSITION 4. *The continuous embedding* $\dot{B}^{s_1}_{p_1,q}(\mathbb{R}^n) \hookrightarrow \dot{B}^{s_2}_{p_2,q}(\mathbb{R}^n)$ *holds for all parameters* s, s_1, p, p_1 *satisfying* $p_1 < p_2$ *and* $s_1 - n/p_1 = s_2 - n/p_2$.

3.2. The inhomogeneous Besov spaces

By using Proposition 2(ii), we obtain the (ordinary) Besov spaces.

DEFINITION 2. The Besov space $B^s_{p,q}(\mathbb{R}^n)$ is the set of $f \in \mathscr{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{B^s_{p,q}(\mathbb{R}^n)} := \|S_0 f\|_p + \Big(\sum_{j\geqslant 1} (2^{sj}\|Q_j f\|_p)^q\Big)^{1/q} < \infty.$$

In connection with the homogeneous Besov space, we dispose the following assertion proved in, e.g., [21, thm. 2.3.3, p. 98].

PROPOSITION 5. *Let* $s>0$, $1\leqslant p\leqslant\infty$ *and* $0<q\leqslant\infty$. *Then* $B^s_{p,q}(\mathbb{R}^n)$ *is the set of all* $f\in L_p(\mathbb{R}^n)$ *such that* $[f]_\infty\in\dot{B}^s_{p,q}(\mathbb{R}^n)$. *Moreover the expression* $\|f\|_p+\|[f]_\infty\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)}$ *defines an equivalent quasi-norm in* $B^s_{p,q}(\mathbb{R}^n)$.

Also, according to Triebel [21, thm. 2.6.1, p. 140] and [20, thm. 2.3.8, pp. 58–59], we have the following characterization and property of $B^s_{p,q}(\mathbb{R}^n)$.

PROPOSITION 6. *Let* $1\leqslant p\leqslant\infty$, $0<q\leqslant\infty$, $m\in\mathbb{N}$ *and* $0<s<m$. *Let* $\{e_1,\ldots,e_n\}$ *denotes the canonical basis of* $\mathbb{R}^n$. *Then the expression*

$$\|f\|_p+\sum_{j=1}^n\Big(\int_0^1 t^{-sq}\|\Delta^m_{te_j}f\|_p^q\,\frac{\mathrm{d}t}{t}\Big)^{1/q}$$

defines an equivalent quasi-norm in $B^s_{p,q}(\mathbb{R}^n)$.

PROPOSITION 7. *Let* $s\in\mathbb{R}$, $1\leqslant p\leqslant\infty$ *and* $0<q\leqslant\infty$. *There exists* $c>0$ *such that the estimate* $\|\partial_j f\|_{B^{s-1}_{p,q}(\mathbb{R}^n)}\leqslant c\|f\|_{B^s_{p,q}(\mathbb{R}^n)}$ $(j=1,\ldots,n)$ *holds, for all* $f\in B^s_{p,q}(\mathbb{R}^n)$.

We also have a characterization by the derivatives, which is a variant of the Triebel's theorem [20, thm. 2.3.8].

PROPOSITION 8. *If* $s>0$, $1\leqslant p\leqslant\infty$ *and* $0<q\leqslant\infty$, *then the expression* $\|f\|_p+\sum_{j=1}^n\|\partial_j f\|_{B^{s-1}_{p,q}(\mathbb{R}^n)}$ *is an equivalent quasi-norm in* $B^s_{p,q}(\mathbb{R}^n)$.

Proof. It suffices to take $n=1$. Let $f\in B^s_{p,q}(\mathbb{R})$. By Proposition 7 and the embedding $B^s_{p,q}(\mathbb{R})\hookrightarrow L_p(\mathbb{R})$ for $s>0$, it holds $\|f\|_p+\|f'\|_{B^{s-1}_{p,q}(\mathbb{R})}\leqslant c\|f\|_{B^s_{p,q}(\mathbb{R})}$. Conversely, let $f\in L_p(\mathbb{R})$ be such that $f'\in B^{s-1}_{p,q}(\mathbb{R})$. Applying Propositions 5 and 3 we obtain

$$\|f\|_{B^s_{p,q}(\mathbb{R})}\leqslant c_1(\|f\|_p+\|[f']_\infty\|_{\dot{B}^{s-1}_{p,q}(\mathbb{R})})\leqslant c_2(\|f\|_p+\|f'\|_{B^{s-1}_{p,q}(\mathbb{R})}).$$

The proof is complete. □

The Besov spaces have the Fatou property in the following sense (see, e.g., [13, thm. 2.6/1]).

PROPOSITION 9. *Let* $f\in\mathscr{S}'(\mathbb{R}^n)$. *If there exists a bounded sequence* $(u_k)_{k\geqslant 0}$ *in* $B^s_{p,q}(\mathbb{R}^n)$ *such that* $\lim_{k\to\infty}u_k=f$ *in* $\mathscr{S}'(\mathbb{R}^n)$, *then* $f\in B^s_{p,q}(\mathbb{R}^n)$ *and* $\|f\|_{B^s_{p,q}(\mathbb{R}^n)}\leqslant c\liminf_{k\to\infty}\|u_k\|_{B^s_{p,q}(\mathbb{R}^n)}$.

An interpolation argument of Besov spaces will be used in the proofs of the above results, for this reason we give the following statement.

PROPOSITION 10. *Let* $0<\theta<1$ *and* $0<q,q_i\leqslant\infty$ $(i=0,1,2,3)$. *Let* $s_0,s_1\in\mathbb{R}$ *be such that* $s_0<s_1$. *We put* $s:=(1-\theta)s_0+\theta s_1$. *Let* T *be an operator satisfying*

- $T(0)=0$,
- $\|T(g_1)-T(g_2)\|_{B^{s_0}_{p,q_2}(\mathbb{R}^n)}\leqslant h_0(\|g_1\|_{\dot W^1_\infty(\mathbb{R}^n)},\|g_2\|_{\dot W^1_\infty(\mathbb{R}^n)})\|g_1-g_2\|_{B^{s_0}_{p,q_0}(\mathbb{R}^n)}$ *for all* $g_1,g_2\in B^{s_0}_{p,q_0}(\mathbb{R}^n)\cap\dot W^1_\infty(\mathbb{R}^n)$,
- $\|T(g)\|_{B^{s_1}_{p,q_3}(\mathbb{R}^n)}\leqslant h_1(\|g\|_{\dot W^1_\infty(\mathbb{R}^n)})\|g\|_{B^{s_1}_{p,q_1}(\mathbb{R}^n)}$ *for all* $g\in B^{s_1}_{p,q_1}(\mathbb{R}^n)\cap\dot W^1_\infty(\mathbb{R}^n)$,

where h_0,h_1 *are nonnegative and nondecreasing functions. Then the operator* T *takes* $B^s_{p,q}(\mathbb{R}^n)\cap\dot W^1_\infty(\mathbb{R}^n)$ *to* $B^s_{p,q}(\mathbb{R}^n)$. *Moreover, there exist two positive constants* c_1,c_2 *such that*

$$\|T(g)\|_{B^s_{p,q}(\mathbb{R}^n)}\leqslant c_1h_0^{1-\theta}(c_2\|g\|_{\dot W^1_\infty(\mathbb{R}^n)},c_2\|g\|_{\dot W^1_\infty(\mathbb{R}^n)})h_1^{\theta}(c_2\|g\|_{\dot W^1_\infty(\mathbb{R}^n)})\|g\|_{B^s_{p,q}(\mathbb{R}^n)}$$

holds, for all $g\in B^s_{p,q}(\mathbb{R}^n)\cap\dot W^1_\infty(\mathbb{R}^n)$.

Proof. If we replace $L_\infty(\mathbb{R}^n)$ by $\dot W^1_\infty(\mathbb{R}^n)$ in all places of occurrence in the proof of Proposition 2.5.4/2 in [19], the claim follows. □

3.3. The modified Besov spaces $\mathscr{B}^s_{p,q}(\mathbb{R})$

In this section we restrict ourselves to the one-dimensional case. In case $s>0$, $\mathscr{B}^s_{p,q}(\mathbb{R})$ (see the formula (3) for its quasi-norm) is a quasi-Banach algebra for the pointwise product, see [17, thm. 3.26]. We also have the following assertion, which is proved in [17, prop. 3.21].

PROPOSITION 11. (i) *If* $s>1/p$, *we have*

$$B^s_{p,q}(\mathbb{R})\hookrightarrow\mathscr{B}^s_{p,q}(\mathbb{R})=\{f\in C_{ub}(\mathbb{R}):\ [f]_\infty\in\dot B^s_{p,q}(\mathbb{R})\}.$$

In case $p=\infty$, *the above embedding becomes an equality.*

(ii) *If* $s>1+1/p$, *then* $\mathscr{B}^s_{p,q}(\mathbb{R})$ *is the set of functions* $f\in L_\infty(\mathbb{R})$ *such that* $f'\in\mathscr{B}^{s-1}_{p,q}(\mathbb{R})$, *and the expression* $\|f\|_\infty+\|f'\|_{\mathscr{B}^{s-1}_{p,q}(\mathbb{R})}$ *defines an equivalent quasi-norm in* $\mathscr{B}^s_{p,q}(\mathbb{R})$.

4. Proof of the main results

We split the proof into three parts: the first and the second correspond to the proof of Theorems 1–2 and 3–4 respectively, and the third part concerns the Besov algebra spaces.

4.1. Proof of Theorems 1–2

Once Proposition 1 and Theorem 2 are proved, Theorem 1 follows easily. We also need the following assertion proved in [10, thm. 2].

THEOREM 7. ([10, thm. 2]) *Let $1 < p < \infty$ and $s > 1+1/p$. Then there exists a constant $c > 0$ such that the inequality*

$$\|(f\circ g)'\|_{\mathscr{B}^{s-1}_{p,\infty}(\mathbb{R})} \leqslant c\|f'\|_{\mathscr{B}^{s-1}_{p,\infty}(\mathbb{R})}\|g\|_{B^s_{p,\infty}(\mathbb{R})}(1+\|g'\|_\infty)^{s-1-1/p} \tag{6}$$

holds, for all functions f such that $f' \in \mathscr{B}^{s-1}_{p,\infty}(\mathbb{R})$ and all $g \in B^s_{p,\infty}(\mathbb{R})$.

The estimate (6) can be easily extended to n-dimensional case. We have the following generalization of Theorem 7.

LEMMA 1. *Let $1 < p < \infty$ and $s > 1+1/p$. Then there exists a constant $c > 0$ such that the inequality*

$$\|f\circ g\|_{B^s_{p,\infty}(\mathbb{R}^n)} \leqslant c\|f'\|_{\mathscr{B}^{s-1}_{p,\infty}(\mathbb{R})}\|g\|_{B^s_{p,p}(\mathbb{R}^n)}(1+\|\nabla g\|_\infty)^{s-1-1/p} \tag{7}$$

holds, for all f such that $f(0)=0$, $f' \in \mathscr{B}^{s-1}_{p,\infty}(\mathbb{R})$ and all $g \in B^s_{p,p}(\mathbb{R}^n)\cap \dot{W}^1_\infty(\mathbb{R}^n)$.

REMARK 7. Lemma 1 is a variant of [10, thm. 4], indeed the result in [10] is proved for functions f such that $f' \in \mathscr{B}^{s-1}_{p,p}(\mathbb{R})$, for this reason we prefer to give its proof.

Proof of Lemma 1. For $x\in\mathbb{R}^n$, we put $x'_1 := (x_2,\dots,x_n)$, $x'_2 := (x_1,x_3,\dots,x_n),\dots,$ $x'_j := (x_1,\dots,x_{j-1},x_{j+1},\dots,x_n),\dots$ and

$$g_{x'_j}(y) := g(x_1,\dots,x_{j-1},y,x_{j+1},\dots,x_n), \qquad \forall y\in\mathbb{R}. \tag{8}$$

In $L_p(\mathbb{R}^{n-1})$ it holds

$$\Big(\int_{\mathbb{R}^{n-1}}\|g_{x'_j}\|^p_{B^s_{p,p}(\mathbb{R})}\mathrm{d}x'_j\Big)^{1/p} \leqslant \|g\|_p + \Big(\int_0^1 t^{-sp}\|\Delta^m_{te_j}g\|^p_p\frac{\mathrm{d}t}{t}\Big)^{1/p} \leqslant \|g\|_{B^s_{p,p}(\mathbb{R}^n)}, \tag{9}$$

which implies that $\|g_{x'_j}\|_{B^s_{p,p}(\mathbb{R})} < \infty$, then by the embedding $B^s_{p,p}(\mathbb{R}) \hookrightarrow B^s_{p,\infty}(\mathbb{R})$, we can apply Theorem 7 to the function $g_{x'_j}$. The assumption $f(0)=0$ yields $\|f\circ g_{x'_j}\|_p \leqslant \|f'\|_\infty\|g_{x'_j}\|_p$. Since $\|g'_{x'_j}\|_\infty \leqslant \|\nabla g\|_\infty$ and by Proposition 3 it holds

$$\begin{aligned}\|f\circ g_{x'_j}\|_{B^s_{p,\infty}(\mathbb{R})} &\leqslant \|f'\|_\infty\|g_{x'_j}\|_p + \|(f\circ g_{x'_j})'\|_{\dot{B}^{s-1}_{p,\infty}(\mathbb{R})} \\ &\leqslant \|f'\|_\infty\|g_{x'_j}\|_p + c\|f'\|_{\mathscr{B}^{s-1}_{p,\infty}(\mathbb{R})}\|g_{x'_j}\|_{B^s_{p,p}(\mathbb{R})}(1+\|\nabla g\|_\infty)^{s-1-1/p},\end{aligned} \tag{10}$$

where c is defined by Theorem 7, it depends only on s and p. Let $m\in\mathbb{N}$ be such that $s<m$. Clearly, we have

$$\begin{aligned}\|f\circ g\|_{B^s_{p,\infty}(\mathbb{R}^n)} &\leqslant c_1\Big\{\|f'\|_\infty\|g\|_p+\sum_{j=1}^n\sup_{0\leqslant t\leqslant 1}t^{-s}\Big(\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}}|\Delta_t^m f\circ g_{x'_j}(y)|^p\mathrm{d}y\mathrm{d}x'_j\Big)^{1/p}\Big\}\\ &\leqslant c_2\Big\{\|f'\|_\infty\|g\|_p+\sum_{j=1}^n\Big(\int_{\mathbb{R}^{n-1}}\|f\circ g_{x'_j}\|^p_{B^s_{p,\infty}(\mathbb{R})}\mathrm{d}x'_j\Big)^{1/p}\Big\}.\end{aligned}\tag{11}$$

Then it suffices to combine (10), (11) and (9), the estimate (7) follows. □

Proof of Proposition 1. We have $f\circ g=(f\rho_t)\circ g$ for $t:=\|g\|_\infty$. By the embedding $B^{s_1-1}_{p,\infty}(\mathbb{R})\hookrightarrow\mathscr{B}^{s_1-1}_{p,\infty}(\mathbb{R})$ (see Proposition 11(i)), by Proposition 7 and by assumption on f, it holds

$$\|(f\rho_t)'\|_{\mathscr{B}^{s_1-1}_{p,\infty}(\mathbb{R})}\leqslant c_1\|(f\rho_t)'\|_{B^{s_1-1}_{p,\infty}(\mathbb{R})}\leqslant c_2\|f\rho_t\|_{B^{s_1}_{p,\infty}(\mathbb{R})}.$$

Since $(f\rho_t)(0)=0$, the desired estimate (2) follows by Theorem 2. Here we assumed that Theorem 2 is already proved. □

Proof of Theorem 2. Let f be a function such that $f(0)=0$ and $f'\in\mathscr{B}^{s_1-1}_{p,\infty}(\mathbb{R})$. Applying Lemma 1 with s replaced by s_1, we obtain

$$\|T_f(g)\|_{B^{s_1}_{p,\infty}(\mathbb{R}^n)}\leqslant c\|f'\|_{\mathscr{B}^{s_1-1}_{p,\infty}(\mathbb{R})}(1+\|\nabla g\|_\infty)^{s_1-1-1/p}\|g\|_{B^{s_1}_{p,p}(\mathbb{R}^n)},\tag{12}$$

for all $g\in B^{s_1}_{p,p}(\mathbb{R}^n)\cap\dot W^1_\infty(\mathbb{R}^n)$, where the constant c is independent of f and g. We also have

$$\|T_f(g_1)-T_f(g_2)\|_p\leqslant\|f'\|_\infty\|g_1-g_2\|_p\quad(\forall g_1,g_2\in L_p(\mathbb{R}^n)),$$

which implies (by using the embeddings $B^0_{p,1}(\mathbb{R}^n)\hookrightarrow L_p(\mathbb{R}^n)\hookrightarrow B^0_{p,\infty}(\mathbb{R}^n)$) that

$$\|T_f(g_1)-T_f(g_2)\|_{B^0_{p,\infty}(\mathbb{R}^n)}\leqslant c\|f'\|_\infty\|g_1-g_2\|_{B^0_{p,1}(\mathbb{R}^n)},\tag{13}$$

for all $g_1,g_2\in B^0_{p,1}(\mathbb{R}^n)\cap\dot W^1_\infty(\mathbb{R}^n)$, where the constant c is independent of f, g_1 and g_2 (recall that (13) is satisfied for all $g_1,g_2\in B^0_{p,1}(\mathbb{R}^n)$). By (12), (13) and the assumption $T_f(0)=0$, we can apply Proposition 10 to the operator T_f. Then for $0<q\leqslant\infty$ and $0<\theta<1$ it holds

$$\begin{aligned}\|T_f(g)\|_{B^s_{p,q}(\mathbb{R}^n)} &\leqslant c_1\|f'\|_\infty^{1-\theta}\Big(\|f'\|_{\mathscr{B}^{s_1-1}_{p,\infty}(\mathbb{R})}(1+\|\nabla g\|_\infty)^{s_1-1-1/p}\Big)^\theta\|g\|_{B^s_{p,q}(\mathbb{R}^n)}\\ &\leqslant c_2\|f'\|_{\mathscr{B}^{s_1-1}_{p,\infty}(\mathbb{R})}(1+\|\nabla g\|_\infty)^{\theta(s_1-1-1/p)}\|g\|_{B^s_{p,q}(\mathbb{R}^n)}\end{aligned}$$

for all $g\in B^s_{p,q}(\mathbb{R}^n)\cap\dot W^1_\infty(\mathbb{R}^n)$, where the constants c_1,c_2 are independent of f and g, and with $s:=\theta s_1$. Now, the inequality (4) follows and the desired result is proved. □

4.2. Proof of Theorems 3–4

We only prove Theorem 4, since for Theorem 3 we argue so as in the proof of Proposition 1. Indeed, by using the function ρ_t with $t := \|g\|_\infty$, we choose a number r such that $s < r < \min(2, s_1)$, and the fact that $f \in B^{r,\ell oc}_{p,\infty}(\mathbb{R})$ implies $(f\rho_t)' \in \mathscr{B}^{r-1}_{p,\infty}(\mathbb{R})$, then we apply Theorem 4 with r instead of s_1. Now, we turn to the proof of Theorem 4 and begin by some preparations.

LEMMA 2. *Let $1 < p < \infty$, $p \leqslant q \leqslant \infty$ and $1 + 1/p < s < 2$. We put $r := sp - 1$. Then there exists a constant $c > 0$ such that the inequality*

$$\|f \circ g\|_{B^s_{p,q}(\mathbb{R})} \leqslant c\|f'\|_{\mathscr{B}^{s-1}_{p,q}(\mathbb{R})}\Big(\|g\|_{B^s_{p,q}(\mathbb{R})} + \|g\|^{s-1/p}_{B^{1+1/r}_{r,1}(\mathbb{R})}\Big) \tag{14}$$

holds, for all f such that $f(0) = 0$, $f' \in \mathscr{B}^{s-1}_{p,q}(\mathbb{R})$ and all $g \in B^s_{p,q}(\mathbb{R})$.

Proof. This statement is essentially proved in [17], but we will give the following explanation. First, in [17, (4.15)-(4.16)] we have proved the estimate (14) under the following conditions:

- f is of class C^1, $f(0) = 0$ and $f' \in \mathscr{B}^{s-1}_{p,q}(\mathbb{R})$,
- g is real analytic and $g \in B^s_{p,q}(\mathbb{R})$.

Second, as in [17, p. 254] the general case can be obtained by applying, in $B^s_{p,q}(\mathbb{R})$, the Fatou property to the sequence of functions $(f_j \circ g_j)_{j\geqslant 0}$, where $f_j := \rho(2^{-j}D)f - \rho(2^{-j}D)f(0)\rho$ and $g_j := \rho(2^{-j}D)g$, with ρ is defined in the beginning of Section 2 and $\rho(2^{-j}D)f$ is defined by $\mathscr{F}\big(\rho(2^{-j}D)f\big)(\xi) = \rho(2^{-j}\xi)\mathscr{F}f(\xi)$ $(\forall \xi \in \mathbb{R})$ where $\mathscr{F}f$ is the Fourier transform of f. □

LEMMA 3. *Let $1 < p < \infty$ and $1 + 1/p < s < 2$. We put $\theta := (s - 1/p)^{-1}$. Then there exists a constant $c > 0$ such that the inequality*

$$\|(f \circ g)'\|_{\mathscr{B}^{s-1}_{p,\infty}(\mathbb{R})} \leqslant c\|f'\|_{\mathscr{B}^{s-1}_{p,\infty}(\mathbb{R})}\|g\|_{B^s_{p,\theta}(\mathbb{R})}(1 + \|g\|_\infty)^{s-1-1/p} \tag{15}$$

holds, for all functions f such that $f' \in \mathscr{B}^{s-1}_{p,\infty}(\mathbb{R})$ and all $g \in B^s_{p,\theta}(\mathbb{R})$.

Proof. We first put $f_1(x) := f(x) - f(0)$ for all $x \in \mathbb{R}$. Then we have

$$(f_1 \circ g)' = (f \circ g)'.$$

By the embedding $B^{s-1}_{p,\infty}(\mathbb{R}) \hookrightarrow C_b(\mathbb{R})$, Propositions 5 and 3, we get

$$\begin{aligned}
\|(f \circ g)'\|_{\mathscr{B}^{s-1}_{p,\infty}(\mathbb{R})} &= \|(f \circ g)'\|_\infty + \Big\|\big[(f_1 \circ g)'\big]_\infty\Big\|_{\dot{B}^{s-1}_{p,\infty}(\mathbb{R})} \\
&\leqslant \|f'\|_\infty\|g'\|_\infty + c_1\big\|f_1 \circ g\big\|_{B^s_{p,\infty}(\mathbb{R})} \\
&\leqslant c_2\Big(\|f'\|_{\mathscr{B}^{s-1}_{p,\infty}(\mathbb{R})}\|g\|_{B^s_{p,\infty}(\mathbb{R})} + \big\|f_1 \circ g\big\|_{B^s_{p,\infty}(\mathbb{R})}\Big).
\end{aligned} \tag{16}$$

Now, by applying Lemma 2 with the function f_1 (since $f_1(0)=0$) and the exponent $r := sp-1$, it holds

$$\big\|f_1\circ g\big\|_{B^s_{p,\infty}(\mathbb{R})} \leqslant c\|f'\|_{\mathscr{B}^{s-1}_{p,\infty}(\mathbb{R})}\Big(\|g\|_{B^s_{p,\infty}(\mathbb{R})}+\|g\|^{s-1/p}_{B^{1+1/r}_{r,1}(\mathbb{R})}\Big). \tag{17}$$

On the other hand, by the definition of θ, it holds that $\theta\in]0,1[$, $p/\theta = r$ and $\theta s = 1+1/r$. Using the Gagliardo-Nirenberg type inequality we obtain

$$\|g\|_{B^{\theta s}_{p/\theta,1}(\mathbb{R})} \leqslant c\|g\|^{\theta}_{B^s_{p,\theta}(\mathbb{R})}\|g\|^{1-\theta}_\infty \qquad (\forall g\in B^s_{p,\theta}(\mathbb{R})),$$

see, e.g., [19, thm. 2.2.5, p. 38]. Now inserting this inequality and (17) into (16), applying the embedding $B^s_{p,\theta}(\mathbb{R})\hookrightarrow B^s_{p,\infty}(\mathbb{R})$, we obtain (15). □

LEMMA 4. *Let $1<p<\infty$ and $1+1/p<s<2$. We put $\theta := (s-1/p)^{-1}$. Then there exists a constant $c>0$ such that the inequality*

$$\|f\circ g\|_{B^s_{p,\infty}(\mathbb{R}^n)} \leqslant c\|f'\|_{\mathscr{B}^{s-1}_{p,\infty}(\mathbb{R})}\|g\|_{B^s_{p,\theta}(\mathbb{R}^n)}(1+\|g\|_\infty)^{s-1-1/p}$$

holds, for all f such that $f(0)=0$, $f'\in\mathscr{B}^{s-1}_{p,\infty}(\mathbb{R})$ and all $g\in B^s_{p,\theta}(\mathbb{R}^n)\cap L_\infty(\mathbb{R}^n)$.

Proof. As in (9), by applying the inequality of Minkowski with respect to the space $L_{p/\theta}(\mathbb{R}^{n-1})$, we obtain

$$\begin{aligned}\Big(\int_{\mathbb{R}^{n-1}}\|g_{x'_j}\|^p_{B^s_{p,\theta}(\mathbb{R})}\mathrm{d}x'_j\Big)^{1/p} &\leqslant \|g\|_p+\Big(\int_0^1 t^{-s\theta}\Big(\int_{\mathbb{R}^n}|\Delta^2_{te_j}g(x)|^p\mathrm{d}x\Big)^{\theta/p}\frac{\mathrm{d}t}{t}\Big)^{1/\theta}\\ &\leqslant c\|g\|_{B^s_{p,\theta}(\mathbb{R}^n)} \qquad (j=1,\ldots,n),\end{aligned}$$

(see (8) for the definition of $g_{x'_j}$) where the constant c is independent of g. Then we apply Lemma 3, and the rest of the proof is similar to that given for Lemma 1. □

Proof of Theorem 4. The proof is similar to that of Theorem 2, where by using Lemma 4 with s replaced by s_1, the result follows by applying [19, prop. 2.5.4/2, p. 88] to the operator T_f. We omit the details. □

4.3. Proof of Theorems 5–6

Proof of Theorem 5. Step 1: the case $[s]=1$. By Theorem 4, Remark 5(i) and the embedding $B^s_{p,q}(\mathbb{R}^n)\hookrightarrow L_\infty(\mathbb{R}^n)$ it holds

$$\|f\circ g\|_{B^s_{p,q}(\mathbb{R}^n)} \leqslant c\|f'\|_{B^{s_1-1}_{p,\infty}(\mathbb{R})}\|g\|_{B^s_{p,q}(\mathbb{R}^n)}(1+\|g\|_{B^s_{p,q}(\mathbb{R}^n)})^{\frac{s}{s_1}(s_1-1-1/p)}$$

for all f such that $f(0)=0$, $f'\in B^{s_1-1}_{p,\infty}(\mathbb{R})$ and all $g\in B^s_{p,q}(\mathbb{R}^n)$.
Now, for a function $f\in B^{s_1,\ell oc}_{p,\infty}(\mathbb{R})$ which satisfies $f(0)=0$, we have both, the inequality

$$\|(f\rho_{\|g\|_\infty})'\|_{B^{s_1-1}_{p,\infty}(\mathbb{R})} \leqslant c\|f\rho_{\|g\|_\infty}\|_{B^{s_1}_{p,\infty}(\mathbb{R})}$$

and the equality $f\circ g = f\rho_{\|g\|_\infty}\circ g$ give the desired result.

Step 2: *the case* $[s]\geqslant 2$. The result follows by both Theorem 1 and the fact that $B^s_{p,q}(\mathbb{R}^n)\cap W^1_\infty(\mathbb{R}^n) = B^s_{p,q}(\mathbb{R}^n)$ since $s>1+n/p$. $\square$

Proof of Theorem 6. The necessity part is well known, cf., e.g., [19, 5.3.1]. For the sufficiency part we will split the proof in two parts.

Step 1. Assume first that $f'\in\mathscr{B}^{s-1}_{p,p}(\mathbb{R})$. We argue as in the proof of Lemma 1 and use its notations. We first have (for $s<m$ with $m\in\mathbb{N}$)

$$\|f\circ g\|_{B^s_{p,p}(\mathbb{R}^n)} \leqslant c_1\Big\{\|f'\|_\infty\|g\|_p + \sum_{j=1}^n\Big(\int_{0\leqslant t\leqslant 1} t^{-sp}\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}}|\Delta_t^m f\circ g_{x'_j}(y)|^p \mathrm{d}y\mathrm{d}x'_j\frac{\mathrm{d}t}{t}\Big)^{1/p}\Big\}$$
$$\leqslant c_2\Big\{\|f'\|_\infty\|g\|_p + \sum_{j=1}^n\Big(\int_{\mathbb{R}^{n-1}}\|f\circ g_{x'_j}\|^p_{B^s_{p,p}(\mathbb{R})}\mathrm{d}x'_j\Big)^{1/p}\Big\}. \tag{18}$$

On the other hand, because $r:=sp-1>p>1$, we have

$$\Big(\int_{\mathbb{R}^{n-1}}\|g_{x'_j}\|^r_{B^{1+1/r}_{r,1}(\mathbb{R})}\mathrm{d}x'_j\Big)^{1/r}\leqslant c\|g\|_{B^{1+1/r}_{r,1}(\mathbb{R}^n)} \qquad (\forall g\in B^{1+1/r}_{r,1}(\mathbb{R}^n)), \tag{19}$$

which implies $\|g_{x'_j}\|_{B^{1+1/r}_{r,1}(\mathbb{R})}<\infty$, then by Lemma 2 (with $p=q$) we obtain

$$\|f\circ g_{x'_j}\|_{B^s_{p,p}(\mathbb{R})}\leqslant c\|f'\|_{\mathscr{B}^{s-1}_{p,p}(\mathbb{R})}\Big(\|g_{x'_j}\|_{B^s_{p,p}(\mathbb{R})}+\|g_{x'_j}\|^{s-1/p}_{B^{1+1/r}_{r,1}(\mathbb{R})}\Big),$$

the constant c depends only on s and p (see again Lemma 2). Now, inserting the last inequality into (18) and using both (9) and (19), then by the embedding $B^s_{p,p}(\mathbb{R}^n)\hookrightarrow B^{1+1/r}_{r,1}(\mathbb{R}^n)$ it holds

$$\|f\circ g\|_{B^s_{p,p}(\mathbb{R}^n)}\leqslant c\|f'\|_{\mathscr{B}^{s-1}_{p,p}(\mathbb{R})}\Big(\|g\|_{B^s_{p,p}(\mathbb{R}^n)}+\|g\|^{s-1/p}_{B^s_{p,p}(\mathbb{R}^n)}\Big).$$

Step 2. The general case can be done as in the proof of Proposition 1. $\square$

5. Open question

As underlined in the introduction, the main problem is to obtain a class of nonlinear functions f, including for instance Schwartz functions vanishing at 0, for which the following estimate holds

$$\|f\circ g\|_{B^s_{p,q}(\mathbb{R}^n)}\leqslant c(f)\|g\|_{B^s_{p,q}(\mathbb{R}^n)}(1+\|g\|_\infty)^{s-1-1/p}.$$

It would be also interesting to extend the validity of Theorem 4 to any $s_1>1+1/p$.

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REFERENCES

[1] D. R. ADAMS AND M. FRAZIER, *BMO and smooth truncation in Sobolev spaces*, Studia Math. **89** (1988), 241–260.
[2] D. R. ADAMS AND M. FRAZIER, *Composition operators on potential spaces*, Proc. Amer. Math. Soc. **114** (1992), 155–165.
[3] G. BOURDAUD, *Fonctions qui opèrent sur les espaces de Besov et de Triebel*, Ann. I. H. Poincaré – AN **10** (1993), 413–422.
[4] G. BOURDAUD, *Une propriété de composition dans l'espace H^s*, C. R. Acad. Sci. Paris **340**, Série I (2005), 221–224.
[5] G. BOURDAUD, *Une propriété de composition dans l'espace H^s (II)*, C. R. Acad. Sci. Paris **342**, Série I (2006), 243–246.
[6] G. BOURDAUD, *Superposition in homogeneous and vector valued Sobolev spaces*, Trans. Amer. Math. Soc. **362** (2010), 6105–6130.
[7] G. BOURDAUD AND Y. MEYER, *Le calcul fonctionnel sous-linéaire dans les espaces de Besov homogènes*, Rev. Mat. Iberoamer. **22** (2006), 725–746.
[8] G. BOURDAUD, M. MOUSSAI AND W. SICKEL, *An optimal symbolic calculus on Besov algebras*, Ann. I. H. Poincaré – AN **23** (2006), 949–956.
[9] G. BOURDAUD, M. MOUSSAI AND W. SICKEL, *Towards sharp superposition theorems in Besov and Lizorkin-Triebel spaces*, Nonlinear Anal. Theory Methods Appl. **68** (2008), 2889–2912.
[10] G. BOURDAUD, M. MOUSSAI AND W. SICKEL, *Composition operators acting on Besov spaces on the real line*, Annali Mat. P. A. **193** (2014), 1519–1554.
[11] G. BOURDAUD AND W. SICKEL, *Composition operators on function spaces with fractional order of smoothness*, RIMS Kokyuroku Bessatsu **B26** (2011), 93–132.
[12] H. BREZIS AND P. MIRONESCU, *Gagliardo-Nirenberg, compositions and products in fractional Sobolev spaces*, J. Evol. Equ. **1** (2001), 387–404.
[13] J. FRANKE, *On the spaces F^s_{pq} of Triebel-Lizorkin type: Pointwise multipliers and spaces on domains*, Math. Nachr. **125** (1986), 29–68.
[14] B. JAWERTH, *Some observations on Besov and Lizorkin-Triebel spaces*, Math. Scand. **40** (1977), 94–104.
[15] V. MAZ'YA AND T. SHAPOSNIKOVA, *An elementary proof of the Brezis and Mironescu theorem on the composition operator in fractional Sobolev spaces*, J. Evol. Equ. **2** (2002), 113–125.
[16] M. MOUSSAI, *The composition in multidimensional Triebel-Lizorkin spaces*, Math. Nachr. **284** (2011), 317–331.
[17] M. MOUSSAI, *Composition operators on Besov algebras*, Rev. Mat. Iberoamer. **28** (2012), 239–272.
[18] J. PEETRE, *New Thoughts on Besov Spaces*, Duke Univ. Math. Series I, Durham, N. C., 1976.
[19] T. RUNST AND W. SICKEL, *Sobolev Spaces of Fractional Order, Nemytskij Operators, and Nonlinear Partial Differential Equations*, De Gruyter, Berlin, 1996.
[20] H. TRIEBEL, *Theory of Function Spaces*, Birkhäuser, Basel, 1983.
[21] H. TRIEBEL, *Theory of Function Spaces II*, Birkhäuser, Basel, 1992.

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