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ON THE TRIEBEL–LIZORKIN SPACE BOUNDEDNESS OF MARCINKIEWICZ INTEGRALS ALONG COMPOUND SURFACES

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Abstract. In this paper the author present the boundedness of Marcinkiewicz integral operators associated to compound surfaces with rough kernels given by $h \in \Delta_\gamma(\mathbb{R}_+)$ and $\Omega \in L(\log^+ L)^{1/2}(S^{n-1}) \cup (\cup_{1<r<\infty} B_r^{0,-1/2}(S^{n-1}))$ on Triebel-Lizorkin spaces and Besov spaces. The main results of this paper represent improvements as well as natural extensions of many previously known results.

1. Introduction

Let $\mathbb{R}^n$ $(n \geqslant 2)$ be the n-dimensional Euclidean space and S^{n-1} denote the unit sphere in $\mathbb{R}^n$ equipped with the induced Lebesgue measure $d\sigma$. Let $\Omega \in L^1(S^{n-1})$ be a homogeneous function of degree zero and satisfy

$$\int_{S^{n-1}} \Omega(u)d\sigma(u) = 0. \tag{1}$$

For a complex number $\rho = \varsigma + i\tau$ ($\varsigma, \tau \in \mathbb{R}$ with $\varsigma > 0$) and a suitable mapping $\Gamma : \mathbb{R}^n \to \mathbb{R}^d$ with $d \geqslant 1$, we consider the parametric Marcinkiewicz integral operator $\mathscr{M}_{h,\Omega,\Gamma,\rho}$ on $\mathbb{R}^d$ by

$$\mathscr{M}_{h,\Omega,\Gamma,\rho}(f)(x) = \left(\int_0^\infty \left| \frac{1}{t^\rho} \int_{|y|\leqslant t} f(x-\Gamma(y)) \frac{h(|y|)\Omega(y)}{|y|^{n-\rho}} dy \right|^2 \frac{dt}{t} \right)^{1/2}, \tag{2}$$

where $f \in \mathscr{S}(\mathbb{R}^d)$ (the Schwartz class on $\mathbb{R}^d$) and $h \in \Delta_1(\mathbb{R}_+)$. Here $\mathbb{R}_+ := (0,\infty)$ and $\Delta_\gamma(\mathbb{R}_+)$ $(\gamma \geqslant 1)$ denotes the set of all measurable functions h defined on $\mathbb{R}_+$ satisfying the condition

$$\|h\|_{\Delta_\gamma(\mathbb{R}_+)} := \sup_{R>0} \left(R^{-1} \int_0^R |h(t)|^\gamma dt \right)^{1/\gamma} < \infty.$$

Clearly, $L^\infty(\mathbb{R}_+) = \Delta_\infty(\mathbb{R}_+) \subsetneq \Delta_{\gamma_2}(\mathbb{R}_+) \subsetneq \Delta_{\gamma_1}(\mathbb{R}_+)$ if $1 \leqslant \gamma_1 < \gamma_2 < \infty$.

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When $n = d$ and $\Gamma(y) = y$, we denote $\mathscr{M}_{h,\Omega,\Gamma,\rho}$ by $\mathscr{M}_{h,\Omega,\rho}$. For $\rho = h = 1$, $\mathscr{M}_{h,\Omega,\rho}$ reduces to the classical Marcinkiewicz integral operator denoted by $\mathscr{M}_\Omega$, which was introduced by Stein [18] and investigated by many authors. For example, see [4, 5] for the case $\Omega \in H^1(S^{n-1})$ (the Hardy space on S^{n-1}), [2, 3] for the case $\Omega \in L(\log^+ L)^{1/2}(S^{n-1})$, [1, 3, 8] for the case $\Omega \in B_r^{(0,-1/2)}(S^{n-1})$ (the block space generated by q-blocks). For $h = 1$, the operator $\mathscr{M}_{h,\Omega,\rho}$ becomes the classical parametric Marcinkiewicz integral operator denoted by $\mathscr{M}_{\Omega,\rho}$. Hörmander [11] (resp., Sakamoto and Yabuta [17]) first studied the L^p boundedness of $\mathscr{M}_{\Omega,\rho}$ with real (resp., complex) number ρ. For further research on $\mathscr{M}_{h,\Omega,\rho}$, we refer the readers to consult [6, 8], among others. It is well known that $L(\log^+ L)^\alpha(S^{n-1}) \subsetneq H^1(S^{n-1})$ for $\alpha \geqslant 1$ and

$$L(\log^+ L)^{1/2}(S^{n-1}) \nsubseteq H^1(S^{n-1}) \nsubseteq L(\log^+ L)^{1/2}(S^{n-1}); \tag{3}$$

$$B_r^{(0,v)}(S^{n-1}) \subset H^1(S^{n-1}) + L(\log^+ L)^{1+v}(S^{n-1}),\ \forall r > 1 \text{ and } v > -1. \tag{4}$$

The main purpose of this paper is to investigate the boundedness of $\mathscr{M}_{h,\Omega,\Gamma,\rho}$ on Triebel-Lizorkin spaces and Besov spaces. As is well known, the Triebel-Lizorkin spaces and Besov spaces contain many important function spaces, such as Lebesgue spaces, Hardy spaces, Sobolev spaces and Lipschitz spaces. The homogeneous Triebel-Lizorkin spaces $\dot{F}_\alpha^{p,q}(\mathbb{R}^d)$ and homogeneous Besov spaces $\dot{B}_\alpha^{p,q}(\mathbb{R}^d)$ are defined, respectively, by

$$\dot{F}_\alpha^{p,q}(\mathbb{R}^d) := \Big\{ f \in \mathscr{S}'(\mathbb{R}^d) :\ \|f\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^d)} := \Big\| \Big(\sum_{i\in\mathbb{Z}} 2^{-i\alpha q} |\Theta_i * f|^q \Big)^{1/q} \Big\|_{L^p(\mathbb{R}^d)} < \infty \Big\}, \tag{5}$$

$$\dot{B}_\alpha^{p,q}(\mathbb{R}^d) := \Big\{ f \in \mathscr{S}'(\mathbb{R}^d) :\ \|f\|_{\dot{B}_\alpha^{p,q}(\mathbb{R}^d)} := \Big(\sum_{i\in\mathbb{Z}} 2^{-i\alpha q} \|\Theta_i * f\|_{L^p(\mathbb{R}^d)}^q \Big)^{1/q} < \infty \Big\}, \tag{6}$$

where $0 < p, q \leqslant \infty$ $(p \neq \infty)$, $\alpha \in \mathbb{R}$, $\mathscr{S}'(\mathbb{R}^d)$ denotes the tempered distribution class on $\mathbb{R}^d$, $\widehat{\Theta}_i(\xi) = \phi(2^i\xi)$ for $i \in \mathbb{Z}$ and $\phi \in \mathscr{C}_c^\infty(\mathbb{R}^d)$ satisfies the conditions: $0 \leqslant \phi(x) \leqslant 1$; $\mathrm{supp}(\phi) \subset \{x : 1/2 \leqslant |x| \leqslant 2\}$; $\phi(x) > c > 0$ if $3/5 \leqslant |x| \leqslant 5/3$. The inhomogeneous versions of Triebel-Lizorkin spaces and Besov spaces, which are denoted by $F_\alpha^{p,q}(\mathbb{R}^d)$ and $B_\alpha^{p,q}(\mathbb{R}^d)$, respectively, are obtained by adding the term $\|\Phi * f\|_{L^p(\mathbb{R}^d)}$ to the right hand side of (5) or (6) with $\sum_{i\in\mathbb{Z}}$ replaced by $\sum_{i\geqslant 1}$, where $\Phi \in \mathscr{S}(\mathbb{R}^d)$, $\mathrm{supp}(\hat{\Phi}) \subset \{\xi \in \mathbb{R}^d : |\xi| \leqslant 2\}$, $\hat{\Phi}(x) > c > 0$ if $|x| \leqslant 5/3$. The following properties are well known (see [9, 19], for example):

$$\dot{F}_0^{p,2}(\mathbb{R}^d) = L^p(\mathbb{R}^d)\ \ \forall 1 < p < \infty; \tag{7}$$

$$\dot{F}_\alpha^{p,p}(\mathbb{R}^d) = \dot{B}_\alpha^{p,p}(\mathbb{R}^d)\ \ \forall 1 < p < \infty \text{ and } \alpha \in \mathbb{R}; \tag{8}$$

$$F_\alpha^{p,q}(\mathbb{R}^d) \sim \dot{F}_\alpha^{p,q}(\mathbb{R}^d) \cap L^p(\mathbb{R}^d) \text{ and } \|f\|_{F_\alpha^{p,q}(\mathbb{R}^d)} \sim \|f\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^d)} + \|f\|_{L^p(\mathbb{R}^d)}\ \ \forall \alpha > 0; \tag{9}$$

$$B_\alpha^{p,q}(\mathbb{R}^d) \sim \dot{B}_\alpha^{p,q}(\mathbb{R}^d) \cap L^p(\mathbb{R}^d) \text{ and } \|f\|_{B_\alpha^{p,q}(\mathbb{R}^d)} \sim \|f\|_{\dot{B}_\alpha^{p,q}(\mathbb{R}^d)} + \|f\|_{L^p(\mathbb{R}^d)}\ \ \forall \alpha > 0. \tag{10}$$

In recent years, the investigation of boundedness of parametric Marcinkiewicz integral operators on Triebel-Lizorkin spaces has also attracted the attention of many authors. In 2009, Zhang and Chen [21] proved that $\mathscr{M}_\Omega$ is bounded on $F_\alpha^{p,q}(\mathbb{R}^n)$ for $0<\alpha<1$ and $1<p,q<\infty$ under the condition $\Omega\in H^1(S^{n-1})$. Later on, they [22] showed that $\mathscr{M}_{h,\Omega,\rho}$ with $\rho=1$ is bounded on $F_\alpha^{p,q}(\mathbb{R}^n)$ for $0<\alpha<1$ and $1+\frac{n+1}{n+2-1/r}<p,q<2+\frac{1-1/r}{n+1}$ if $\Omega\in L^r(S^{n-1})$ for some $r>1$ and $h\in L^\infty(\mathbb{R}_+)$. Very recently, Yabuta [20] obtained the following result.

THEOREM A. ([20]) *Let $n=d$, $\rho>0$ and $\Gamma(y)=\varphi(|y|)y'$ with $\varphi\in\mathfrak{F}$, where $\mathfrak{F}$ is the set of all functions ϕ satisfying*

(a) *ϕ is a positive increasing $\mathscr{C}^1(\mathbb{R}_+)$ function;*

(b) *there exist $C_\phi, c_\phi>0$ such that $t\phi'(t)\geqslant C_\phi\phi(t)$ and $\phi(2t)\leqslant c_\phi\phi(t)$ for all $t>0$.*

Suppose that $\Omega\in L(\log^+L)^{1/2}(S^{n-1})\cup(\cup_{r>1}B_r^{(0,-1/2)}(S^{n-1}))$ satisfies (1) and $h\in\Delta_\gamma(\mathbb{R}_+)$ for some $\gamma>1$. Then

(i) *$\mathscr{M}_{h,\Omega,\Gamma,\rho}$ is bounded on $\dot F_\alpha^{p,q}(\mathbb{R}^d)$ for $\alpha\in(0,1)$ and $(1/p,1/q)\in\mathscr{R}_\gamma$, where $\mathscr{R}_\gamma$ is the set of all interiors of the convex hull of three squares $(\frac12,\frac12+\frac{1}{\max\{2,\gamma'\}})^2$, $(\frac12-\frac{1}{\max\{2,\gamma'\}},\frac12)^2$ and $(\frac{1}{2\gamma},1-\frac{1}{2\gamma})^2$;*

(ii) *$\mathscr{M}_{h,\Omega,\Gamma,\rho}$ is bounded on $\dot B_\alpha^{p,q}(\mathbb{R}^d)$ for $\alpha\in(0,1)$, $|1/p-1/2|<\min\{1/2,1/\gamma'\}$ and $1<q<\infty$.*

REMARK 1. It should be pointed out that the questions concerning the $\dot F_\alpha^{p,q}(\mathbb{R}^d)$ bounds and $\dot B_\alpha^{p,q}(\mathbb{R}^d)$ bounds for $\mathscr{M}_{h,\Omega,\Gamma,\rho}$ with Γ being as in Theorem A and $\Omega\in H^1(S^{n-1})$ have been answered by Yabuta in [20]. There are some model examples for the class $\mathfrak{F}$, such as $t^\alpha\,(\alpha>0)$, $t^\beta\ln(1+t)\,(\beta\geqslant1)$, $t\ln\ln(e+t)$, real-valued polynomials P on $\mathbb{R}$ with positive coefficients and $P(0)=0$ and so on. Note that there exists $B_\varphi>1$ such that $\varphi(2t)\geqslant B_\varphi\varphi(t)$ for any $\varphi\in\mathfrak{F}$ (see [13]). On the other hand, we remark that $\mathscr{R}_{\gamma_1}\subsetneq\mathscr{R}_{\gamma_2}$ for any $1<\gamma_1<\gamma_2\leqslant\infty$. Specially, $\mathscr{R}_\infty=(0,1)\times(0,1)$.

In this paper we shall present several new results for the boundedness of parametric Marcinkiewicz integral operators along certain compound surfaces on Triebel-Lizorkin spaces and Besov spaces. We now briefly describe each of our main results.

The first type of our operators we consider is the parametric Marcinkiewicz integral operators supported by polynomial mappings. When $\Gamma(y)=\mathscr{P}(y)=(P_1(y),P_2(y),\dots,P_d(y))$ with P_j being real valued polynomials on $\mathbb{R}^n$, we denote $\mathscr{M}_{h,\Omega,\Gamma,\rho}$ by $\mathscr{M}_{h,\Omega,\mathscr{P},\rho}$. The L^p bounds of $\mathscr{M}_{h,\Omega,\mathscr{P},\rho}$ has been studied by many authors (see [3, 5, 7, 14] etc.). In particular, Al-Qassem and Pan [3] proved that $\mathscr{M}_{h,\Omega,\mathscr{P},\rho}$ is bounded on $L^p(\mathbb{R}^d)$ for $|1/p-1/2|<\min\{1/2,1/\gamma'\}$ if $h\in\Delta_\gamma(\mathbb{R}_+)$ for some $\gamma>1$ and $\Omega\in L(\log^+L)^{1/2}(S^{n-1})\cup(\cup_{r>1}B_r^{(0,-1/2)}(S^{n-1}))$.

Based on the above result and (7), it is natural to ask whether the condition $\Omega\in L(\log^+L)^{1/2}(S^{n-1})\cup(\cup_{r>1}B_r^{(0,-1/2)}(S^{n-1}))$ implies the $\dot F_\alpha^{p,q}(\mathbb{R}^d)$ bounds of $\mathscr{M}_{h,\Omega,\mathscr{P},\rho}$ for some $\alpha\neq0$ or $q\neq2$. Our investigation will not only address this problem, but also deal with a more general class of operators. More precisely, we shall establish the following

THEOREM 1. *Let $\varphi\in\mathfrak{F}$ and $\Gamma(y)=\mathscr{P}(\varphi(|y|)y')$, where $\mathscr{P}=(P_1,P_2,\dots,P_d)$*

with P_j *being real valued polynomials on* $\mathbb{R}^n$. *Suppose that* $h \in \Delta_\gamma(\mathbb{R}_+)$ *for some* $\gamma > 1$ *and* $\Omega \in L(\log^+ L)^{1/2}(S^{n-1}) \cup (\cup_{r>1} B_r^{(0,-1/2)}(S^{n-1}))$ *satisfies* (1). *Then*

(i) $\mathscr{M}_{h,\Omega,\Gamma,\rho}$ *is bounded on* $\dot{F}_\alpha^{p,q}(\mathbb{R}^d)$ *for* $\alpha \in (0,1)$ *and* $(1/p, 1/q) \in \mathscr{R}_\gamma$*;*

(ii) $\mathscr{M}_{h,\Omega,\Gamma,\rho}$ *is bounded on* $\dot{B}_\alpha^{p,q}(\mathbb{R}^d)$ *for* $\alpha \in (0,1)$, $|1/p - 1/2| < \min\{1/2, 1/\gamma'\}$ *and* $1 < q < \infty$.

The bounds are independent of the coefficients of P_j *for* $1 \leqslant j \leqslant d$, *but depend on* n, d, φ *and* $\deg(P_j)$ *for* $1 \leqslant j \leqslant d$.

REMARK 2. Theorem 1 extends Theorem A, which corresponds to the case $\rho > 0, n = d$ and $\mathscr{P}(y) = y$.

The second type of our operators we consider are the parametric Marcinkiewicz integral operators along polynomial compound curves. In this paper we shall establish the following

THEOREM 2. *Let* $n = d$ *and* $\Gamma(y) = (P_1(\varphi(|y|))y_1', \dots, P_n(\varphi(|y|))y_n')$ *with* P_j *being real valued polynomials on* $\mathbb{R}$ *with satisfying* $P_j(0) = 0$ *and* $\varphi \in \mathfrak{F}$. *Suppose that* $\Omega \in L(\log^+ L)^{1/2}(S^{n-1}) \cup (\cup_{r>1} B_r^{(0,-1/2)}(S^{n-1}))$ *satisfies* (1) *and* $h \in \Delta_\gamma(\mathbb{R}_+)$ *for some* $\gamma > 1$. *Then*

(i) $\mathscr{M}_{h,\Omega,\Gamma,\rho}$ *is bounded on* $\dot{F}_\alpha^{p,q}(\mathbb{R}^n)$ *for* $\alpha \in (0,1)$ *and* $(1/p, 1/q) \in \mathscr{R}_\gamma$*;*

(ii) $\mathscr{M}_{h,\Omega,\Gamma,\rho}$ *is bounded on* $\dot{B}_\alpha^{p,q}(\mathbb{R}^n)$ *for* $\alpha \in (0,1)$, $|1/p - 1/2| < \min\{1/2, 1/\gamma'\}$ *and* $1 < q < \infty$.

The bounds are independent of the coefficients of P_j *for* $1 \leqslant j \leqslant n$, *but depend on* n, φ *and* $\deg(P_j)$ *for* $1 \leqslant j \leqslant n$.

REMARK 3. Theorem 2 extends Theorem A, which corresponds to the case $\rho > 0$ and $P_1(t) = P_2(t) = \cdots = P_n(t) = t$.

REMARK 4. By employing the method in the proof of [3, Theorem 2.3] and applying some estimates about Fourier transforms of measures appeared in [14, 15], one can easily obtain that $\mathscr{M}_{h,\Omega,\Gamma,\rho}$ is bounded on $L^p(\mathbb{R}^d)$ for $|1/p - 1/2| < \min\{1/2, 1/\gamma'\}$ if h, Ω are given as in Theorem 1 and Γ is given as in Theorem 1 or 2.

As a direct consequence of Theorems 1 and 2, we have the following

COROLLARY 1. *Let* $h \in \Delta_\infty(\mathbb{R}_+)$ *and* Ω, Γ *be given as in Theorem* 1 *or* 2. *Then* $\mathscr{M}_{h,\Omega,\Gamma,\rho}$ *is bounded on* $\dot{F}_\alpha^{p,q}(\mathbb{R}^d)$ *and* $\dot{B}_\alpha^{p,q}(\mathbb{R}^d)$ *for* $\alpha \in (0,1)$ *and* $1 < p, q < \infty$.

To obtain Theorems 1 and 2, we need to establish the following delicate sharp $\dot{F}_\alpha^{p,q}(\mathbb{R}^d)$ bounds for $\mathscr{M}_{h,\Omega,\Gamma,\rho}$.

THEOREM 3. *Let* h, Γ *be given as in Theorem* 1. *Suppose that* $\Omega \in L^s(S^{n-1})$ *for some* $s \in (1,2]$ *satisfying* (1). *Then for* $\alpha \in (0,1)$ *and* $(1/p, 1/q) \in \mathscr{R}_\gamma$, *there exists* $C > 0$ *such that*

$$\|\mathscr{M}_{h,\Omega,\Gamma,\rho}(f)\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^d)} \leqslant C(s-1)^{-1/2} \|\Omega\|_{L^s(S^{n-1})} \|f\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^d)},$$

where $C = C_{\rho,\alpha,p,q,n,d,\varphi,\deg(\mathscr{P})}$ *is independent of* s, Ω *and the coefficients of* P_j *for* $1 \leqslant j \leqslant d$.

THEOREM 4. *Let* h, Γ *be given as in Theorem* 2*. Suppose that* $\Omega \in L^s(S^{n-1})$ *for some* $s \in (1,2]$ *satisfying* (1)*. Then for* $\alpha \in (0,1)$ *and* $(1/p, 1/q) \in \mathscr{R}_\gamma$*, there exists* $C > 0$ *such that*

$$\|\mathscr{M}_{h,\Omega,\Gamma,\rho}(f)\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^n)} \leqslant C(s-1)^{-1/2}\|\Omega\|_{L^s(S^{n-1})}\|f\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^n)},$$

where $C = C_{\rho,\alpha,p,q,n,\varphi,\max_{1\leqslant j\leqslant n}\deg(P_j)}$ *is independent of* s, Ω *and the coefficients of* P_j *for* $1 \leqslant j \leqslant n$.

Applying Remark 4, the properties (9)–(10) and Theorems 1 and 2, we can get the following conclusion immediately.

COROLLARY 2. *Under the same conditions of Theorems* 1 *and* 2*, the operator* $\mathscr{M}_{h,\Omega,\Gamma,\rho}$ *is bounded on* $F_\alpha^{p,q}(\mathbb{R}^d)$ *and* $B_\alpha^{p,q}(\mathbb{R}^d)$*, respectively.*

REMARK 5. When $\gamma = \infty$, the range of (p,q) in Corollary 2 becomes $(1,\infty)^2$. Therefore, Corollary 2 extends and improves greatly the result of [22], even in the special case $\rho = 1$, $n = d$ and $\Gamma(y) = y$. It should be pointed out that all of our main results are new, even in the special case: $\rho = 1$, $n = d$, $h(t) \equiv 1$ and $\varphi(t) = t$.

The rest of this paper is organized as follows. After presenting some technical lemmas in Section 2, we shall give the proofs of our main results in Section 3. We would like to remark that some ideas in our proofs are taken from [3, 10, 12, 20] and the main novelty in this paper is to give a standard approach on the bounds for Marcinkiewicz integral operators in Triebel-Lizorkin spaces and Besov spaces. The proofs of Theorems 3–4 are based on two important lemmas (see Lemmas 2.4 and 2.6). The proofs of Theorems 1-2 follows from Theorems 3-4 and an extrapolation method followed from [3].

Throughout this paper, we let p' denote the dual exponent to p defined $1/p + 1/p' = 1$. The letter C, sometimes with additional parameters, will stand for positive constants, not necessarily the same one at each occurrence but is independent of the essential variables. We shall use $\delta_{\mathbb{R}^n}$ to denote the Dirac delta function on $\mathbb{R}^n$ and $\pi_n^m (m \geqslant n)$ to denote a projection operator from $\mathbb{R}^m$ to $\mathbb{R}^n$. We also denote by D^{-1} the inverse transform of D and D^t the transpose of D for any linear transformation D. In what follows, we set $\mathfrak{R}_d = \{\zeta \in \mathbb{R}^d;\ 1/2 < |\zeta| \leqslant 1\}$. We also use the conventions $\sum_{j\in\emptyset} a_j = 0$ and $\prod_{j\in\emptyset} a_j = 1$.

2. Preliminary Lemmas

In this section, we shall present some necessary lemmas, which will play key roles in our proofs. Let us begin with some useful characterizations of Triebel-Lizorkin spaces and Besov spaces, which are followed from [20].

LEMMA 1. ([20]) *Let* $0<\alpha<\infty$ *and* l *be an integer satisfying* $l>\alpha$. *We denote by* $\triangle_{\zeta}^{l}(f)$ *the* l*-th difference of* f *for an arbitrary function* f *defined on* $\mathbb{R}^d$ *and* $\zeta\in\mathfrak{R}_d$.

(i) *If* $1<p<\infty,\ 1<q\leqslant\infty$ *and* $1\leqslant r<\min\{p,q\}$. *Then*

$$\|f\|_{\dot{F}_{\alpha}^{p,q}(\mathbb{R}^d)}\approx\Big\|\Big(\sum_{k\in\mathbb{Z}}2^{kq\alpha}\Big(\int_{\mathfrak{R}_d}|\triangle_{2^{-k}\zeta}^{l}(f)|^r d\zeta\Big)^{q/r}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)};$$

(ii) *If* $1\leqslant p<\infty,\ 1\leqslant q\leqslant\infty$ *and* $1\leqslant r\leqslant p$. *Then*

$$\|f\|_{\dot{B}_{\alpha}^{p,q}(\mathbb{R}^d)}\approx\Big(\sum_{k\in\mathbb{Z}}2^{kq\alpha}\Big\|\Big(\int_{\mathfrak{R}_d}|\triangle_{2^{-k}\zeta}^{l}(f)|^r d\zeta\Big)^{1/r}\Big\|_{L^p(\mathbb{R}^d)}^{q}\Big)^{1/q}.$$

The following lemma can be used in the estimates about Fourier transformations of some measures appeared in the proof of Theorem 4.

LEMMA 2. *([16]). Let* $\lambda\neq 0$. *Suppose* $\Phi(t)=t^{\alpha_1}+\mu_2 t^{\alpha_2}+\cdots+\mu_n t^{\alpha_n}$ *and* $\Psi\in C^1[a,b]$, *where* $\mu_2,\ldots,\mu_n$ *are real parameters, and* $\alpha_1,\ldots,\alpha_n$ *are distinct positive (not necessarily integer) exponents. Then*

$$\Big|\int_a^b\exp(i\lambda\Phi(t))\Psi(t)dt\Big|\leqslant C|\lambda|^{-\varepsilon}\Big\{\sup_{a\leqslant t\leqslant b}|\Psi(t)|+\int_a^b|\Psi'(t)|dt\Big\},$$

with $\varepsilon=\min\{1/\alpha_1,1/n\}$ *and* C *does not depend on* $\mu_2,\ldots,\mu_n$ *as long as* $0\leqslant a<b\leqslant 1$.

The following results obtained by Liu in [12] are two vector-valued norm inequalities of the Hardy-Littlewood maximal operator.

LEMMA 3. ([12]) (i) *Let* $M_{(d)}$ *be the Hardy-Littlewood maximal operator on* $\mathbb{R}^d$. *Then*

$$\begin{aligned}&\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big\|\Big(\sum_{k\in\mathbb{Z}}|M_{(d)}g_{j,\zeta,k}|^2\Big)^{1/2}\Big\|_{L^r(\mathfrak{R}_d)}^{q}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)}\\&\leqslant C\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big\|\Big(\sum_{k\in\mathbb{Z}}|g_{j,\zeta,k}|^2\Big)^{1/2}\Big\|_{L^r(\mathfrak{R}_d)}^{q}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)}\end{aligned}$$

for any $1<p,q,r<\infty$, *where* $C>0$ *is independent of functions* $\{g_{j,\zeta,k}\}_{j,\zeta,k}$ *on* $\mathbb{R}^d$ *parametrized by* $\zeta\in\mathfrak{R}_d$ *and* $j,k\in\mathbb{Z}$;

(ii) *Let* $\mathscr{M}_{\mathscr{P}}$ *be the Hardy-Littlewood maximal operator supported by polynomial mappings defined by*

$$\mathscr{M}_{\mathscr{P}}f(x)=\sup_{r>0}\frac{1}{r^n}\int_{|y|\leqslant r}|f(x-\mathscr{P}(y))|dy,$$

where $\mathscr{P}=(P_1,\ldots,P_d)$ *with* P_j *being real-valued polynomials on* $\mathbb{R}^n$. *Then*

$$\Big\|\Big(\sum_{j\in\mathbb{Z}}\|\mathscr{M}_{\mathscr{P}}f_{j,\zeta}\|_{L^r(\mathfrak{R}_d)}^{q}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)}\leqslant C\Big\|\Big(\sum_{j\in\mathbb{Z}}\|f_{j,\zeta}\|_{L^r(\mathfrak{R}_d)}^{q}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)},$$

for any $1 < p, q, r < \infty$*, where* $C > 0$ *is independent of the coefficients of* P_j *for* $1 \leqslant j \leqslant d$.

Let h, Ω, Γ, ρ be goven as in (2), we define the family of measures $\{\sigma_{h,\Omega,\Gamma,t}\}$ and the related maximal operators $\sigma^*_{h,\Omega,\Gamma}$ on $\mathbb{R}^d$ by

$$\int_{\mathbb{R}^d} f(x)d\sigma_{h,\Omega,\Gamma,t}(x) = \frac{1}{t^\rho}\int_{t/2<|x|\leqslant t} f(\Gamma(x))\frac{h(|x|)\Omega(x)}{|x|^{n-\rho}}dx,$$

$$\sigma^*_{h,\Omega,\Gamma}(f)(x) = \sup_{t>0}\big||\sigma_{h,\Omega,\Gamma,t}| * f(x)\big|,$$

where $|\sigma_{h,\Omega,\Gamma,t}|$ is defined in the same way as $\sigma_{h,\Omega,\Gamma,t}$, but with Ω and h replaced by $|\Omega|$ and $|h|$, respectively.

LEMMA 4. *Let* $v > 1$ *and* $\Gamma(y) = (P_1(\varphi(|y|))a_1(y), \ldots, P_d(\varphi(|y|))a_d(y))$*, where* $\varphi \in \mathfrak{F}$*,* $P_1, \ldots, P_d$ *are real-valued polynomials on* $\mathbb{R}_+$ *and* $a_1(y), \ldots, a_d(y)$ *are arbitrary functions independent of* $|y|$*. Suppose that* $h \in \Delta_\gamma(\mathbb{R}_+)$ *for some* $\gamma > 1$ *and* $\Omega \in L^1(S^{n-1})$ *satisfies* (1)*. If* $(1/p, 1/q, 1/r)$ *belongs to the interior of the convex hull of three cubes* $(\frac{1}{2}, \frac{1}{2} + \frac{1}{\max\{2,\gamma'\}})^3$*,* $(\frac{1}{2} - \frac{1}{\max\{2,\gamma'\}}, \frac{1}{2})^3$ *and* $(\frac{1}{2\gamma}, 1 - \frac{1}{2\gamma})^3$*, then for arbitrary functions* $\{g_{j,\zeta,k}\}_{j,\zeta,k} \in L^p(\ell^q(L^r(\ell^2)), \mathbb{R}^d)$*, there exists* $C > 0$ *such that*

$$\begin{aligned}&\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big(\int_{\mathfrak{R}_d}\Big(\sum_{k\in\mathbb{Z}}\int_{2^{kv}}^{2^{(k+1)v}}\big||\sigma_{h,\Omega,\Gamma,t}| * g_{j,\zeta,k}\big|^2\frac{dt}{t}\Big)^{1/2}d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)}\\ &\leqslant Cv^{1/2}\|\Omega\|_{L^1(S^{n-1})}\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big\|\Big(\sum_{k\in\mathbb{Z}}|g_{j,\zeta,k}|^2\Big)^{1/2}\Big\|^q_{L^r(\mathfrak{R}_d)}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)}.\end{aligned} \tag{11}$$

The constant $C > 0$ *is independent of* v*,* Ω *and the coefficients of* P_j *for* $1 \leqslant j \leqslant d$.

Proof. By Hölder's inequality we have

$$\begin{aligned}&\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big(\int_{\mathfrak{R}_d}\Big(\sum_{k\in\mathbb{Z}}\int_{2^{kv}}^{2^{(k+1)v}}\big||\sigma_{h,\Omega,\Gamma,t}| * g_{j,\zeta,k}\big|^2\frac{dt}{t}\Big)^{1/2}d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)}\\ &\leqslant \Big\|\Big(\sum_{j\in\mathbb{Z}}\Big\|\Big(\sum_{k\in\mathbb{Z}}\int_{2^{kv}}^{2^{(k+1)v}}\big||\sigma_{h,\Omega,\Gamma,t}| * g_{j,\zeta,k}\big|^2\frac{dt}{t}\Big)^{1/2}\Big\|^q_{L^r(\mathfrak{R}_d)}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)}\end{aligned}$$

for any $1 < p, q, r < \infty$. Thus, to prove (11), it suffices to show that

$$\begin{aligned}&\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big\|\Big(\sum_{k\in\mathbb{Z}}\int_{2^{kv}}^{2^{(k+1)v}}\big||\sigma_{h,\Omega,\Gamma,t}| * g_{j,\zeta,k}\big|^2\frac{dt}{t}\Big)^{1/2}\Big\|^q_{L^r(\mathfrak{R}_d)}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)}\\ &\leqslant Cv^{1/2}\|\Omega\|_{L^1(S^{n-1})}\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big\|\Big(\sum_{k\in\mathbb{Z}}|g_{j,\zeta,k}|^2\Big)^{1/2}\Big\|^q_{L^r(\mathfrak{R}_d)}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)}\end{aligned} \tag{12}$$

for $(1/p, 1/q, 1/r)$ belonging to the interior of the convex hull of three cubes $(\frac{1}{2}, \frac{1}{2} + \frac{1}{\max\{2,\gamma'\}})^3$, $(\frac{1}{2} - \frac{1}{\max\{2,\gamma'\}}, \frac{1}{2})^3$ and $(\frac{1}{2\gamma}, 1 - \frac{1}{2\gamma})^3$, and arbitrary functions $\{g_{j,\zeta,k}\}_{j,\zeta,k} \in$

$L^p(\ell^q(L^r(\ell^2,\mathfrak{R}_d)),\mathbb{R}^d)$, where $C>0$ is independent of v, Ω and the coefficients of P_j for $1\leqslant j\leqslant d$.

Below we shall prove (12). We first conclude that

$$\Big\|\Big(\sum_{j\in\mathbb{Z}}\|\sigma^*_{h,\Omega,\Gamma}(f_{j,\zeta})\|^q_{L^r(\mathfrak{R}_d)}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)}\leqslant C\|\Omega\|_{L^1(S^{n-1})}\Big\|\Big(\sum_{j\in\mathbb{Z}}\|f_{j,\zeta}\|^q_{L^r(\mathfrak{R}_d)}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)} \tag{13}$$

for any $\gamma'<p,q,r<\infty$ and arbitrary functions $\{f_{j,\zeta}\}\in L^p(\ell^q(L^r(\mathfrak{R}_d)),\mathbb{R}^d)$. By the change of variables and Hölder's inequality,

$$\begin{aligned}
&\sigma^*_{h,\Omega,\Gamma}(f)(x)\\
&\leqslant \sup_{t>0}\int_{t/2<|y|\leqslant t}|f(x-\Gamma(y))|\frac{|h(|y|)\Omega(y)|}{|y|^n}dy\\
&=\sup_{t>0}\int_{t/2}^{t}\int_{S^{n-1}}|f(x-\Gamma(r\theta))||\Omega(\theta)|d\sigma(\theta)|h(r)|\frac{dr}{r}\\
&\leqslant 2\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|\Omega\|^{1/\gamma}_{L^1(S^{n-1})}\Big(\int_{S^{n-1}}\sup_{t>0}\int_{t/2}^{t}|f(x-\Gamma(r\theta))|^{\gamma'}\frac{dr}{r}|\Omega(\theta)|d\sigma(\theta)\Big)^{1/\gamma'}\\
&\leqslant C\|\Omega\|^{1/\gamma}_{L^1(S^{n-1})}\\
&\quad\times\Big(\int_{S^{n-1}}\sup_{t>0}\int_{\varphi(t/2)}^{\varphi(t)}|f(x-\Gamma(\varphi^{-1}(s)\theta))|^{\gamma'}\frac{ds}{\varphi^{-1}(s)\varphi'(\varphi^{-1}(s))}|\Omega(\theta)|d\sigma(\theta)\Big)^{1/\gamma'}\\
&\leqslant C(\varphi)\|\Omega\|^{1/\gamma}_{L^1(S^{n-1})}\Big(\int_{S^{n-1}}\sup_{t>0}\frac{1}{t}\int_{|s|\leqslant t}|f(x-\Gamma(\varphi^{-1}(s)\theta))|^{\gamma'}ds|\Omega(\theta)|d\sigma(\theta)\Big)^{1/\gamma'},
\end{aligned}$$

which combining (ii) of Lemma 3 with Minkowski's inequality yields (13).

We now discuss the following three cases:

Case 1 $(1<\gamma\leqslant\infty)$. Note that

$$\sup_{k\in\mathbb{Z}}\sup_{t\in[2^{kv},2^{(k+1)v}]}\big||\sigma_{h,\Omega,\Gamma,t}|*g_{j,\zeta,k}\big|\leqslant\sigma^*_{h,\Omega,\Gamma}\Big(\sup_{k\in\mathbb{Z}}|g_{j,\zeta,k}|\Big),$$

which together with (13) leads to

$$\begin{aligned}
&\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big\|\sup_{k\in\mathbb{Z}}\sup_{t\in[2^{kv},2^{(k+1)v}]}\big||\sigma_{h,\Omega,\Gamma,t}|*g_{j,\zeta,k}\big|\Big\|^q_{L^r(\mathfrak{R}_d)}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)}\\
&\leqslant C\|\Omega\|_{L^1(S^{n-1})}\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big\|\sup_{k\in\mathbb{Z}}|g_{j,\zeta,k}|\Big\|^q_{L^r(\mathfrak{R}_d)}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)}
\end{aligned} \tag{14}$$

for any $\gamma'<p,q,r<\infty$. On the other hand, by the duality, Hölder's inequality, Fubini's theorem and (13) we have that for any $1<p,q,r<\gamma$, there exists a sequence of functions $\{f_{j,\zeta}\}_{j,\zeta}$ with $\|\{f_{j,\zeta}\}\|_{L^{p'}(\ell^{q'}(L^{r'}(\mathfrak{R}_d)),\mathbb{R}^d)} = 1$ such that

$$\begin{aligned}
&\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big\|\sum_{k\in\mathbb{Z}}\int_{2^{kv}}^{2^{(k+1)v}}\big||\sigma_{h,\Omega,\Gamma,t}|*g_{j,\zeta,k}\big|\frac{dt}{t}\Big\|^q_{L^r(\mathfrak{R}_d)}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)}\\
&=\sum_{j\in\mathbb{Z}}\int_{\mathbb{R}^d}\int_{\mathfrak{R}_d}\sum_{k\in\mathbb{Z}}\int_{2^{kv}}^{2^{(k+1)v}}\big||\sigma_{h,\Omega,\Gamma,t}|*g_{j,\zeta,k}(x)\big|\frac{dt}{t}|f_{j,\zeta}(x)|d\zeta dx\\
&\leqslant\sum_{j\in\mathbb{Z}}\int_{\mathbb{R}^d}\int_{\mathfrak{R}_d}\sum_{k\in\mathbb{Z}}|g_{j,\zeta,k}(x)|\int_{2^{kv}}^{2^{(k+1)v}}|\sigma_{h,\Omega,\Gamma,t}|*\widetilde{|f_{j,\zeta}|}(-x)|\frac{dt}{t}d\zeta dx\\
&\leqslant v\sum_{j\in\mathbb{Z}}\int_{\mathbb{R}^d}\int_{\mathfrak{R}_d}\sum_{k\in\mathbb{Z}}|g_{j,\zeta,k}(x)|\sigma^*_{h,\Omega,\Gamma}(\widetilde{|f_{j,\zeta}|})(-x)d\zeta dx\\
&\leqslant v\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big\|\sum_{k\in\mathbb{Z}}|g_{j,\zeta,k}|\Big\|^q_{L^r(\mathfrak{R}_d)}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)}\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big\|\sigma^*_{h,\Omega,\Gamma}(\widetilde{|f_{j,\zeta}|})\Big\|^{q'}_{L^{r'}(\mathfrak{R}_d)}\Big)^{1/q'}\Big\|_{L^{p'}(\mathbb{R}^d)}\\
&\leqslant Cv\|\Omega\|_{L^1(S^{n-1})}\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big\|\sum_{k\in\mathbb{Z}}|g_{j,\zeta,k}|\Big\|^q_{L^r(\mathfrak{R}_d)}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)},
\end{aligned}\tag{15}$$

where $\widetilde{f_{j,\zeta}}(x) = f_{j,\zeta}(-x)$. Interpolating between (14) and (15) yields that (12) holds for $(1/p,1/q,1/r)$ belonging to the interior of the cube $(\frac{1}{2\gamma},1-\frac{1}{2\gamma})^3$.

Case 2 $(1<\gamma\leqslant 2)$. By Hölder's inequality we have

$$\begin{aligned}
\big||\sigma_{h,\Omega,\Gamma,t}|*g_{j,\zeta,k}(x)\big| &\leqslant \int_{t/2<|y|\leqslant t}|g_{j,\zeta,k}(x-\Gamma(y))|\frac{|h(y)\Omega(y)|}{|y|^n}dy\\
&\leqslant\Big(\int_{t/2<|y|\leqslant t}|g_{j,\zeta,k}(x-\Gamma(y))|^2\frac{|h(y)|^{2-\gamma}|\Omega(y)|}{|y|^n}dy\Big)^{1/2}\\
&\quad\times\Big(\int_{t/2<|y|\leqslant t}\frac{|h(y)|^{\gamma}|\Omega(y)|}{|y|^n}dy\Big)^{1/2}\\
&\leqslant C\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\|\Omega\|^{1/2}_{L^1(S^{n-1})}\Big(|\sigma_{|h|^{2-\gamma},\Omega,\Gamma,t}|*|g_{j,\zeta,k}|^2(x)\Big)^{1/2}.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big\|\Big(\sum_{k\in\mathbb{Z}}\int_{2^{kv}}^{2^{(k+1)v}}\big||\sigma_{h,\Omega,\Gamma,t}|*g_{j,\zeta,k}\big|^2\frac{dt}{t}\Big)^{1/2}\Big\|^q_{L^r(\mathfrak{R}_d)}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)}\\
&\leqslant C\|\Omega\|^{1/2}_{L^1(S^{n-1})}\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big\|\sum_{k\in\mathbb{Z}}\int_{2^{kv}}^{2^{(k+1)v}}|\sigma_{|h|^{2-\gamma},\Omega,\Gamma,t}|*|g_{j,\zeta,k}|^2\frac{dt}{t}\Big\|^q_{L^r(\mathfrak{R}_d)}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)}
\end{aligned}\tag{16}$$

Note that $|h|^{2-\gamma}\in\Delta_{\frac{\gamma}{2-\gamma}}(\mathbb{R}_+)$. By (16) and (15) with γ, p, q, r replacing by $\frac{\gamma}{2-\gamma}$, $\frac{p}{2}$, $\frac{q}{2}$, $\frac{r}{2}$, respectively we have (12) for $(1/p,1/q,1/r)$ belonging to the interior of the cube $(\frac{1}{2}-\frac{1}{\gamma'},\frac{1}{2})^3$. By duality we see that (12) holds for $(1/p,1/q,1/r)$ belonging to the interior of the cube $(\frac{1}{2},\frac{1}{2}+\frac{1}{\gamma'})^3$. Interpolating these two cases, we see that (12) holds for $(1/p,1/q,1/r)$ belonging to the interior of the convex hull of two cubes $(\frac{1}{2}-\frac{1}{\gamma'},\frac{1}{2})^3$ and $(\frac{1}{2},\frac{1}{2}+\frac{1}{\gamma'})^3$. Note that in this case the interior of the cubes $(\frac{1}{2\gamma},1-\frac{1}{2\gamma})^3$ contains in the interior of the convex hull of two cubes $(\frac{1}{2}-\frac{1}{\gamma'},\frac{1}{2})^3$ and $(\frac{1}{2},\frac{1}{2}+\frac{1}{\gamma'})^3$.

Case 3 $(\gamma \geqslant 2)$. Note that $\Delta_\gamma(\mathbb{R}_+) \subset \Delta_2(\mathbb{R}_+)$ for $\gamma \geqslant 2$. Interpolating between cases 1 and 2 we obtain (12) for $(1/p, 1/q, 1/r)$ belonging to the interior of the convex hull of three cubes $(\frac{1}{2\gamma}, 1-\frac{1}{2\gamma})^3$, $(0,\frac{1}{2})^3$ and $(\frac{1}{2},1)^3$. This finishes the proof of Lemma 4. □

Let $\eta_0 \in \mathscr{C}^\infty(\mathbb{R})$ be an even function satisfying $0 \leqslant \eta_0(t) \leqslant 1$, $\eta_0(0) = 1$ and $\eta_0(t) = 0$ for $|t| \geqslant 1$. Set $\eta(\xi) = 1$ for $|\xi| \leqslant 1$, $\eta(\xi) = \eta_0(\frac{|\xi|-1}{a-1})$, where $a > 1$. Then, η satisfies $\chi_{|\xi|\leqslant 1}(\xi) \leqslant \eta(\xi) \leqslant \chi_{|\xi|\leqslant a}(\xi)$ and $|\partial^\alpha \eta(\xi)| \leqslant c_\alpha (a-1)^{-|\alpha|}$ for $\xi \in \mathbb{R}^d$ and $\alpha \in \mathbb{N}^d$, where c_α is independent of a. Let $\{a_k\}$ be a lacunary sequence satisfying $\inf_{k\in\mathbb{Z}} \frac{a_{k+1}}{a_k} \geqslant a > 1$. We define functions $\{\psi_k\}_k$ on $\mathbb{R}^d$ by

$$\psi_k(\xi) = \eta(a_{k+1}^{-1}\xi) - \eta(a_k^{-1}\xi), \ \ \xi \in \mathbb{R}^d. \tag{17}$$

Then observe that

$$\operatorname{supp}(\psi_k) \subset \{a_k \leqslant |\xi| \leqslant a a_{k+1}\}; \ \ \operatorname{supp}(\psi_k) \cap \operatorname{supp}(\psi_j) = \emptyset \ \text{ for } \ |j-k| \geqslant 2;$$

$$\sum_{k\in\mathbb{Z}} \psi_k(\xi) = 1 \ \ \forall \xi \in \mathbb{R}^d \setminus \{0\}.$$

Let $m \leqslant d$. Since ψ_k is radial, we shall use the convention $\psi_k(|\zeta|) = \psi_k(\xi)$ for $\zeta \in \mathbb{R}^m$ satisfying $|\zeta| = |\xi|$ with $\xi \in \mathbb{R}^d$. We have the following lemma.

LEMMA 5. *Let $m \leqslant d$, $H : \mathbb{R}^m \to \mathbb{R}^m$ and $G : \mathbb{R}^d \to \mathbb{R}^d$ be two nonsingular linear transformations. Let ψ_k be given as in* (17)*. Define the multiplier operator S_k on $\mathbb{R}^d$ by $\widehat{S_k f}(\xi) = \psi_k(|H\pi_m^d G\xi|)\hat{f}(\xi)$. Then*

$$\begin{aligned} &\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big\|\Big(\sum_{k\in\mathbb{Z}}|S_k f_{j,\zeta}|^2\Big)^{1/2}\Big\|_{L^r(\mathfrak{R}_d)}^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)} \\ &\leqslant C\Big(\frac{a}{a-1}\Big)^{d+2}\Big\|\Big(\sum_{j\in\mathbb{Z}}\|f_{j,\zeta}\|_{L^r(\mathfrak{R}_d)}^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)}. \end{aligned}$$

for $1 < p, q, r < \infty$, where $C > 0$ is independent of a and $\{f_{j,\zeta}\}$.

Proof. Define the function Ψ_k by $\widehat{\Psi_k}(\xi) = \psi_k(\xi)$. By [20, Lemma 2.5] we have

$$\begin{aligned} &\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big\|\Big(\sum_{k\in\mathbb{Z}}|\Psi_k * f_{j,\zeta}|^2\Big)^{1/2}\Big\|_{L^r(\mathfrak{R}_d)}^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)} \\ &\leqslant C\Big(\frac{a}{a-1}\Big)^{d+2}\Big\|\Big(\sum_{j\in\mathbb{Z}}\|f_{j,\zeta}\|_{L^r(\mathfrak{R}_d)}^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)}. \end{aligned} \tag{18}$$

Define J by $J = G^{-1}(H^{-1} \otimes \delta_{\mathbb{R}^{d-m}})$. Obviously, J is a nonsingular linear transformation on $\mathbb{R}^d$. Let $y = (y^1, y^2)$ with $y^1 = (y_1, y_2, \ldots, y_m)$ and $y^2 = (y_{m+1}, y_{m+2}, \ldots, y_d)$. One can easily check that

$$S_k f(x) = |J|\Psi_k \otimes \delta_{\mathbb{R}^{d-m}} * f^J(J^t x), \tag{19}$$

where $f^J(\xi)=|J|^{-1}f((J^t)^{-1}\xi)$. We get from (18) and (19) that

$$\begin{aligned}
&\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big\|\Big(\sum_{k\in\mathbb{Z}}|S_kf_{j,\zeta}|^2\Big)^{1/2}\Big\|^q_{L^r(\mathfrak{R}_d)}\Big)^{1/q}\Big\|^p_{L^p(\mathbb{R}^d)}\\
&\leqslant\int_{\mathbb{R}^d}\Big(\sum_{j\in\mathbb{Z}}\Big\|\Big(\sum_{k\in\mathbb{Z}}||J|\Psi_k\otimes\delta_{\mathbb{R}^{d-m}}*f^J_{j,\zeta}(J^tx)|^2\Big)^{1/2}\Big\|^q_{L^r(\mathfrak{R}_d)}\Big)^{p/q}dx\\
&=|J|^{p-1}\int_{\mathbb{R}^d}\Big(\sum_{j\in\mathbb{Z}}\Big\|\Big(\sum_{k\in\mathbb{Z}}|\Psi_k\otimes\delta_{\mathbb{R}^{d-m}}*f^J_{j,\zeta}(y)|^2\Big)^{1/2}\Big\|^q_{L^r(\mathfrak{R}_d)}\Big)^{p/q}dy\\
&=|J|^{p-1}\int_{\mathbb{R}^{d-m}}\int_{\mathbb{R}^m}\Big(\sum_{j\in\mathbb{Z}}\Big\|\Big(\sum_{k\in\mathbb{Z}}|[\Psi_k*f^J_{j,\zeta}(\cdot,y^2)](y^1)|^2\Big)^{1/2}\Big\|^q_{L^r(\mathfrak{R}_d)}\Big)^{p/q}dy^1dy^2\\
&\leqslant C|J|^{p-1}\Big(\frac{a}{a-1}\Big)^{p(d+2)}\int_{\mathbb{R}^d}\Big(\sum_{j\in\mathbb{Z}}\|f^J_{j,\zeta}(y)\|^q_{L^r(\mathfrak{R}_d)}\Big)^{p/q}dy\\
&\leqslant C\Big(\frac{a}{a-1}\Big)^{p(d+2)}\Big\|\Big(\sum_{j\in\mathbb{Z}}\|f_{j,\zeta}\|^q_{L^r(\mathfrak{R}_d)}\Big)^{1/q}\Big\|^p_{L^p(\mathbb{R}^d)},
\end{aligned}$$

which is just the conclusion of Lemma 5. □

We end this section by presenting the following key lemma, which is the heart of our proofs.

LEMMA 6. *Let $v>1$, $\Lambda\in\mathbb{N}\setminus\{0\}$ and $\{\sigma_{s,t}:t\in\mathbb{R}_+,\ 1\leqslant s\leqslant\Lambda\}$ be a family of Borel measures on $\mathbb{R}^d$ with $\sigma_{0,t}=0$ for all $t\in\mathbb{R}_+$. We also denote by $|\sigma_{s,t}|$ the total variation of $\sigma_{s,t}$. For $1\leqslant s\leqslant\Lambda$, let $\gamma_s,\beta_s>0$, $M_s\in\mathbb{N}\setminus\{0\}$ and $L_s:\mathbb{R}^d\to\mathbb{R}^{M_s}$ be linear transformations. Suppose that $\varphi\in\mathfrak{F}$ and there exist $p_0,q_0>1$, $1<r_0<\min\{p_0,q_0\}$ and $C,A>0$ independent of v such that the following conditions are satisfied for $1\leqslant s\leqslant\Lambda$, $t\in\mathbb{R}_+$, $\xi\in\mathbb{R}^d$ and $\{g_{l,\zeta,k}\}\in L^{p_0}(\ell^{q_0}(L^{r_0}(\ell^2,\mathfrak{R}_d)),\mathbb{R}^d)$:*

(i) $|\widehat{\sigma_{s,t}}(\xi)-\widehat{\sigma_{s-1,t}}(\xi)|\leqslant CA\varphi(t)^{\gamma_s}|L_s(\xi)|$;

(ii) $|\widehat{\sigma_{s,t}}(\xi)|\leqslant CA\min\{1,(\varphi(t)^{\gamma_s}|L_s(\xi)|)^{-\beta_s/v}\}$;

(iii)

$$\begin{aligned}
&\Big\|\Big(\sum_{l\in\mathbb{Z}}\Big(\int_{\mathfrak{R}_d}\Big(\sum_{k\in\mathbb{Z}}\int_{2^{kv}}^{2^{(k+1)v}}||\sigma_{s,t}|*g_{l,\zeta,k}|^2\frac{dt}{t}\Big)^{1/2}d\zeta\Big)^{q_0}\Big)^{1/q_0}\Big\|_{L^{p_0}(\mathbb{R}^d)}\\
&\leqslant CAv^{1/2}\Big\|\Big(\sum_{l\in\mathbb{Z}}\Big\|\Big(\sum_{k\in\mathbb{Z}}|g_{l,\zeta,k}|^2\Big)^{1/2}\Big\|^{q_0}_{L^{r_0}(\mathfrak{R}_d)}\Big)^{1/q_0}\Big\|_{L^{p_0}(\mathbb{R}^d)}
\end{aligned}$$

Then for $\alpha\in(0,1)$ and $(1/p,1/q)\in P_1P_2\setminus\{(\frac{1}{p_0},\frac{1}{q_0})\}$, there exists $C>0$ independent of A and v such that

$$\Big\|\Big(\sum_{l\in\mathbb{Z}}2^{lq\alpha}\Big(\int_{\mathfrak{R}_d}\Big(\int_0^\infty|\sigma_{\Lambda,t}*\triangle_{2^{-l}\zeta}(f)|^2\frac{dt}{t}\Big)^{1/2}d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)}\leqslant CAv^{1/2}\|f\|_{\dot F^{p,q}_\alpha(\mathbb{R}^d)},\tag{20}$$

where $\triangle_{2^{-l}\zeta}(f)(x)=f(x+2^{-l}\zeta)-f(x)$ and P_1P_2 is the line segment from P_1 to P_2 with $P_1=(\frac{1}{2},\frac{1}{2})$ and $P_2=(\frac{1}{p_0},\frac{1}{q_0})$.

Proof. For any $1 \leqslant s \leqslant \Lambda$, let $l_s = \operatorname{rank}(L_s) \leqslant \min\{d, M_s\}$. By [10, Lemma 6.1], there are two nonsingular linear transformations $\mathscr{H}_s : \mathbb{R}^{l_s} \to \mathbb{R}^{l_s}$ and $\mathscr{G}_s : \mathbb{R}^d \to \mathbb{R}^d$ such that

$$|\mathscr{H}_s \pi_{l_s}^d \mathscr{G}_s \xi| \leqslant |L_s(\xi)| \leqslant M_s |\mathscr{H}_s \pi_{l_s}^d \mathscr{G}_s \xi|, \;\; \forall\, \xi \in \mathbb{R}^d. \tag{21}$$

Now we can choose a function $\psi \in \mathscr{C}_0^\infty(\mathbb{R})$ such that $\psi(t) \equiv 1$ for $|t| \leqslant 1/2$ and $\psi(t) \equiv 0$ for $|t| > 1$. For $t > 0$ and $1 \leqslant s \leqslant \Lambda$, we define the family of measures $\{\tau_{s,t}\}$ by

$$\widehat{\tau_{s,t}}(\xi) = \widehat{\sigma_{s,t}}(\xi) \prod_{j=s+1}^{\Lambda} \psi(|\varphi(t)^{\gamma_j} \mathscr{H}_j \pi_{l_j}^d \mathscr{G}_j \xi|) - \widehat{\sigma_{s-1,t}}(\xi) \prod_{j=s}^{\Lambda} \psi(|\varphi(t)^{\gamma_j} \mathscr{H}_j \pi_{l_j}^d \mathscr{G}_j \xi|). \tag{22}$$

By our assumption $\sigma_{0,t} = 0$ and (22) we get

$$\sigma_{\Lambda,t} = \sum_{s=1}^{\Lambda} \tau_{s,t}. \tag{23}$$

It follows that

$$\begin{aligned} &\Big\| \Big(\sum_{l \in \mathbb{Z}} 2^{lq\alpha} \Big(\int_{\mathfrak{R}_d} \Big(\int_0^\infty |\sigma_{\Lambda,t} * \triangle_{2^{-l}\zeta}(f)|^2 \frac{dt}{t} \Big)^{1/2} d\zeta \Big)^q \Big)^{1/q} \Big\|_{L^p(\mathbb{R}^d)} \\ &\leqslant \sum_{s=1}^{\Lambda} \Big\| \Big(\sum_{l \in \mathbb{Z}} 2^{lq\alpha} \Big(\int_{\mathfrak{R}_d} \Big(\int_0^\infty |\tau_{s,t} * \triangle_{2^{-l}\zeta}(f)|^2 \frac{dt}{t} \Big)^{1/2} d\zeta \Big)^q \Big)^{1/q} \Big\|_{L^p(\mathbb{R}^d)}. \end{aligned} \tag{24}$$

Therefore, to prove (20), it suffices to show that

$$\Big\| \Big(\sum_{l \in \mathbb{Z}} 2^{lq\alpha} \Big(\int_{\mathfrak{R}_d} \Big(\int_0^\infty |\tau_{s,t} * \triangle_{2^{-l}\zeta}(f)|^2 \frac{dt}{t} \Big)^{1/2} d\zeta \Big)^q \Big)^{1/q} \Big\|_{L^p(\mathbb{R}^d)} \leqslant C A v^{1/2} \|f\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^d)} \tag{25}$$

for any $1 \leqslant s \leqslant \Lambda$, $\alpha \in (0,1)$ and $(1/p, 1/q) \in P_1P_2 \setminus \{(\frac{1}{p_0}, \frac{1}{q_0})\}$, where $C > 0$ is independent of A, v.

Next we prove (25). Fix $0 < \alpha < 1$. By straightforward calculations, conditions (i)–(ii) and (21)–(22) we obtain that for any $1 \leqslant s \leqslant \Lambda$,

$$|\widehat{\tau_{s,t}}(\xi)| \leqslant CA \min\{1, (\varphi(t)^{\gamma_s} |L_s(\xi)|)^{1/v}\}; \tag{26}$$

$$|\widehat{\tau_{s,t}}(\xi)| \leqslant CA (\varphi(t)^{\gamma_s} |L_s(\xi)|)^{-\beta_s/v}, \;\text{ if }\; \varphi(t)^{\gamma_s} |\mathscr{H}_s \pi_{l_s}^d \mathscr{G}_s \xi| \geqslant 1. \tag{27}$$

For any fixed $s \in \{1, 2, \ldots, \Lambda\}$. Let ψ_k be given as in (17) with $a_k = \varphi(2^{-kv})^{-\gamma_s}$ and $a = B_\varphi^{v\gamma_s}$, where B_φ is given as in Remark 1. Define the multiplier operator $S_{k,s}$ on $\mathbb{R}^d$ by

$$\widehat{S_{k,s} f}(\xi) = \psi_k(|\mathscr{H}_s \pi_{l_s}^d \mathscr{G}_s \xi|) \hat{f}(\xi).$$

By Minkowski's inequality we have

$$\begin{aligned}
&\Big\|\Big(\sum_{l\in\mathbb{Z}}2^{lq\alpha}\Big(\int_{\mathfrak{R}_d}\Big(\int_0^{\infty}|\tau_{s,t}*\triangle_{2^{-l}\zeta}(f)|^2\frac{dt}{t}\Big)^{1/2}d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)}\\
&=\Big\|\Big(\sum_{l\in\mathbb{Z}}2^{lq\alpha}\Big(\int_{\mathfrak{R}_d}\Big(\sum_{k\in\mathbb{Z}}\int_{2^{kv}}^{2^{(k+1)v}}|\tau_{s,t}*\triangle_{2^{-l}\zeta}(f)|^2\frac{dt}{t}\Big)^{1/2}d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)}\\
&=\Big\|\Big(\sum_{l\in\mathbb{Z}}2^{lq\alpha}\Big(\int_{\mathfrak{R}_d}\Big(\sum_{k\in\mathbb{Z}}\int_{2^{kv}}^{2^{(k+1)v}}|\tau_{s,t}*\sum_{j\in\mathbb{Z}}S_{j-k,s}\triangle_{2^{-l}\zeta}(f)|^2\frac{dt}{t}\Big)^{1/2}d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)}\\
&\leqslant\sum_{j\in\mathbb{Z}}\Big\|\Big(\sum_{l\in\mathbb{Z}}2^{lq\alpha}\Big(\int_{\mathfrak{R}_d}\Big(\sum_{k\in\mathbb{Z}}\int_{2^{kv}}^{2^{(k+1)v}}|\tau_{s,t}*S_{j-k,s}\triangle_{2^{-l}\zeta}(f)|^2\frac{dt}{t}\Big)^{1/2}d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)}.
\end{aligned}\tag{28}$$

Define the mixed norm $\|\cdot\|_{E_\alpha^{p,q}}$ for measurable functions on $\mathbb{R}^d\times\mathfrak{R}_d\times\mathbb{Z}\times\mathbb{Z}\times\mathbb{R}_+$ by

$$\|g\|_{E_\alpha^{p,q}}:=\Big\|\Big(\sum_{l\in\mathbb{Z}}2^{lq\alpha}\Big(\int_{\mathfrak{R}_d}\Big(\sum_{k\in\mathbb{Z}}\int_0^{\infty}|g(x,\zeta,l,k,t)|^2\frac{dt}{t}\Big)^{1/2}d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)}.$$

For any $j\in\mathbb{Z}$, let

$$V_{j,s}(f)(x,\zeta,l,k,t):=\tau_{s,t}*S_{j-k,s}\triangle_{2^{-l}\zeta}(f)(x)\chi_{[2^{kv},2^{(k+1)v})}(t).$$

Thus we have

$$\begin{aligned}
&\Big\|\Big(\sum_{l\in\mathbb{Z}}2^{lq\alpha}\Big(\int_{\mathfrak{R}_d}\Big(\sum_{k\in\mathbb{Z}}\int_{2^{kv}}^{2^{(k+1)v}}|\tau_{s,t}*\triangle_{2^{-l}\zeta}(f)|^2\frac{dt}{t}\Big)^{1/2}d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)}\\
&\leqslant\sum_{j\in\mathbb{Z}}\|V_{j,s}(f)\|_{E_\alpha^{p,q}}.
\end{aligned}\tag{29}$$

By (26)–(27), Hölder's inequality, Minkowski's inequality, Fubini's theorem and Plancherel's theorem we have

$$\begin{aligned}
&\|V_{j,s}(f)\|^2_{E_\alpha^{2,2}}\\
&=\Big\|\Big(\sum_{l\in\mathbb{Z}}2^{2l\alpha}\Big(\int_{\mathfrak{R}_d}\Big(\sum_{k\in\mathbb{Z}}\int_{2^{kv}}^{2^{(k+1)v}}|\tau_{s,t}*S_{j-k,s}\triangle_{2^{-l}\zeta}(f)|^2\frac{dt}{t}\Big)^{1/2}d\zeta\Big)^2\Big)^{1/2}\Big\|^2_{L^2(\mathbb{R}^d)}\\
&=\int_{\mathbb{R}^d}\sum_{l\in\mathbb{Z}}2^{2l\alpha}\Big(\int_{\mathfrak{R}_d}\Big(\sum_{k\in\mathbb{Z}}\int_{2^{kv}}^{2^{(k+1)v}}|\tau_{s,t}*S_{j-k,s}\triangle_{2^{-l}\zeta}(f)(x)|^2\frac{dt}{t}\Big)^{1/2}d\zeta\Big)^2dx\\
&\leqslant C\sum_{l\in\mathbb{Z}}2^{2l\alpha}\int_{\mathfrak{R}_d}\sum_{k\in\mathbb{Z}}\int_{2^{kv}}^{2^{(k+1)v}}\int_{\mathbb{R}^d}|\tau_{s,t}*S_{j-k,s}\triangle_{2^{-l}\zeta}(f)(x)|^2dx\frac{dt}{t}d\zeta\\
&\leqslant C\sum_{l\in\mathbb{Z}}2^{2l\alpha}\int_{\mathfrak{R}_d}\sum_{k\in\mathbb{Z}}\int_{E_{j-k,s}}\int_{2^{kv}}^{2^{(k+1)v}}|\widehat{\tau_{s,t}}(x)|^2\frac{dt}{t}|\widehat{\triangle_{2^{-l}\zeta}(f)}(x)|^2dxd\zeta\\
&\leqslant CA^2vB_\varphi^{-2c|j|}\|f\|^2_{\dot{B}_\alpha^{2,2}(\mathbb{R}^d)},
\end{aligned}\tag{30}$$

where $C,c>0$ are independent of v and

$$E_{j-k,s}=\{x\in\mathbb{R}^d:\ \varphi(2^{(k-j)v})^{-\gamma_s}\leqslant|\mathscr{H}_s\pi_{l_s}^d\mathscr{G}_s\xi|\leqslant B_\varphi^{v\gamma_s}\varphi(2^{(k-j-1)v})^{-\gamma_s}\}.$$

Combining (30) with (8) yields

$$\|V_{j,s}(f)\|_{E_\alpha^{2,2}} \leqslant CAv^{1/2}B_\varphi^{-c|j|}\|f\|_{\dot F_\alpha^{2,2}(\mathbb{R}^d)}, \tag{31}$$

where $C>0$ is independent of v. Following that, we will prove that there exists $C>0$ which is independent of v such that

$$\|V_{j,s}(f)\|_{E_\alpha^{p_0,q_0}} \leqslant CAv^{1/2}\|f\|_{\dot F_\alpha^{p_0,q_0}(\mathbb{R}^d)}. \tag{32}$$

In fact, interpolation between (31) and (32) implies that for $\alpha\in(0,1)$ and $(1/p,1/q)\in P_1P_2\setminus\{(\frac{1}{p_0},\frac{1}{q_0})\}$, there exists $\theta\in(0,1)$ such that

$$\|V_{j,s}(f)\|_{E_\alpha^{p,q}} \leqslant CAv^{1/2}B_\varphi^{-c\theta|j|}\|f\|_{\dot F_\alpha^{p,q}(\mathbb{R}^d)}, \tag{33}$$

where C is independent of v. (33) together with (29) yields (25).

It remains to show (32). For $1\leqslant s\leqslant\Lambda$, let Φ^s be a radial function in $\mathscr{S}(\mathbb{R}^{l_s})$ defined by $\widehat{\Phi^s}(x)=\psi(|x|)$, where $x\in\mathbb{R}^{l_s}$ and ψ is given as in (22). Define J_s and X_s by

$$J_sf(x):=f(\mathscr{G}_s^t(\mathscr{H}_s^t\otimes id_{\mathbb{R}^{d-l_s}})x)$$

and

$$X_sf(x)=\sup_{k\in\mathbb{Z}}\sup_{t\in[2^{kv},2^{(k+1)v}]}|X_{k,t;s}f(x)|,$$

where

$$X_{k,t;s}f(x)=J_s^{-1}((\Phi_{k,t;s}\otimes\delta_{\mathbb{R}^{d-l_s}})*J_sf)(x),$$

and

$$\Phi_{k,t;s}(x^0)=(\varphi(t)^{\gamma_s})^{-l_s}\Phi^s(\varphi(t)^{-\gamma_s}x^0),$$

where $x^0\in\mathbb{R}^{l_s}$. One can easily check that

$$|X_sf(x)|\leqslant C_s[J_s^{-1}\circ(M_{(l_s)}\otimes id_{\mathbb{R}^{d-l_s}})\circ J_s](f)(x), \tag{34}$$

where $x=(x^0,x^1)\in\mathbb{R}^{l_s}\times\mathbb{R}^{d-l_s}$. This combining with (i) of Lemma 3 yields

$$\begin{aligned}
&\Big\|\Big(\sum_{l\in\mathbb{Z}}\Big\|\Big(\sum_{k\in\mathbb{Z}}|X_sg_{l,\zeta,k}|^2\Big)^{1/2}\Big\|^q_{L^r(\mathfrak{R}_d)}\Big)^{1/q}\Big\|^p_{L^p(\mathbb{R}^d)}\\
&\leqslant C\Big\|\Big(\sum_{l\in\mathbb{Z}}\Big\|\Big(\sum_{k\in\mathbb{Z}}|[J_s^{-1}\circ(M_{(l_s)}\otimes id_{\mathbb{R}^{d-l_s}})\circ J_s](g_{l,\zeta,k})|^2\Big)^{1/2}\Big\|^q_{L^r(\mathfrak{R}_d)}\Big)^{1/q}\Big\|^p_{L^p(\mathbb{R}^d)}\\
&\leqslant C|J_s|\int_{\mathbb{R}^{d-l_s}}\int_{\mathbb{R}^{l_s}}\Big(\sum_{l\in\mathbb{Z}}\Big\|\Big(\sum_{k\in\mathbb{Z}}|(M_{(l_s)}[(J_sg_{l,\zeta,k}(\cdot,x^1))](x^0)^2\Big)^{1/2}\Big\|^q_{L^r(\mathfrak{R}_d)}\Big)^{p/q}dx^0dx^1\\
&\leqslant C\Big\|\Big(\sum_{l\in\mathbb{Z}}\Big\|\Big(\sum_{k\in\mathbb{Z}}|g_{l,\zeta,k}|^2\Big)^{1/2}\Big\|^q_{L^r(\mathfrak{R}_d)}\Big)^{1/q}\Big\|^p_{L^p(\mathbb{R}^d)}
\end{aligned} \tag{35}$$

for any $1 \leqslant s \leqslant \Lambda$ and $1 < p, q, r < \infty$. Define $X^s f = X_s \circ X_{s+1} \circ \cdots \circ X_\Lambda f$ for $1 \leqslant s \leqslant \Lambda$. We get from (35) that

$$\begin{aligned} &\Big\|\Big(\sum_{l\in\mathbb{Z}}\Big\|\Big(\sum_{k\in\mathbb{Z}}|X^s g_{l,\zeta,k}|^2\Big)^{1/2}\Big\|^q_{L^r(\mathfrak{R}_d)}\Big)^{1/q}\Big\|^p_{L^p(\mathbb{R}^d)} \\ &\leqslant C\Big\|\Big(\sum_{l\in\mathbb{Z}}\Big\|\Big(\sum_{k\in\mathbb{Z}}|g_{l,\zeta,k}|^2\Big)^{1/2}\Big\|^q_{L^r(\mathfrak{R}_d)}\Big)^{1/q}\Big\|^p_{L^p(\mathbb{R}^d)}. \end{aligned} \tag{36}$$

for any $1 \leqslant s \leqslant \Lambda$ and $1 < p, q, r < \infty$. On the other hand, by the definition of $X_{k,t;s}$ we have

$$\tau_{s,t} * f = \sigma_{s,t} * (X_{k,t;s+1} \circ X_{k,t;s+2} \circ \cdots \circ X_{k,t;\Lambda} f) - \sigma_{s-1,t} * (X_{k,t;s} \circ X_{k,t;s+1} \circ \cdots \circ X_{k,t;\Lambda} f).$$

It follows that

$$\int_{2^{kv}}^{2^{(k+1)v}} |\tau_{s,t} * f|^2 \frac{dt}{t} \leqslant 2\Big(\int_{2^{kv}}^{2^{(k+1)v}} \big||\sigma_{s,t}| * X^{s+1} f\big|^2 \frac{dt}{t} + \int_{2^{kv}}^{2^{(k+1)v}} \big||\sigma_{s-1,t}| * X^s f\big|^2 \frac{dt}{t}\Big). \tag{37}$$

From (36)–(37) and assumption (iii), one can get

$$\begin{aligned} &\Big\|\Big(\sum_{l\in\mathbb{Z}}\Big(\int_{\mathfrak{R}_d}\Big(\sum_{k\in\mathbb{Z}}\int_{2^{kv}}^{2^{(k+1)v}} |\tau_{s,t} * g_{l,\zeta,k}|^2 \frac{dt}{t}\Big)^{1/2} d\zeta\Big)^{q_0}\Big)^{1/q_0}\Big\|_{L^{p_0}(\mathbb{R}^d)} \\ &\leqslant CAv^{1/2}\Big\|\Big(\sum_{l\in\mathbb{Z}}\Big\|\Big(\sum_{k\in\mathbb{Z}}|g_{l,\zeta,k}|^2\Big)^{1/2}\Big\|^{q_0}_{L^{r_0}(\mathfrak{R}_d)}\Big)^{1/q_0}\Big\|_{L^{p_0}(\mathbb{R}^d)} \end{aligned} \tag{38}$$

for arbitrary functions $\{g_{l,\zeta,k}\} \in L^{p_0}(\ell^{q_0}(L^{r_0}(\ell^2, \mathfrak{R}_d)), \mathbb{R}^d)$ and $1 \leqslant s \leqslant \Lambda$. Then (38) together with (i) of Lemmas 1 and 5 leads to

$$\begin{aligned} \|V_{j,s}(f)\|_{E_\alpha^{p_0,q_0}} &\leqslant CAv^{1/2}\Big\|\Big(\sum_{l\in\mathbb{Z}} 2^{lq_0\alpha}\Big\|\Big(\sum_{k\in\mathbb{Z}}|S_{j-k,s}\triangle_{2^{-l}\zeta}(f)|^2\Big)^{1/2}\Big\|^{q_0}_{L^{r_0}(\mathfrak{R}_d)}\Big)^{1/q_0}\Big\|_{L^{p_0}(\mathbb{R}^d)} \\ &\leqslant CAv^{1/2}\Big(\frac{B_\varphi^{v\gamma_s}}{B_\varphi^{v\gamma_s}-1}\Big)^{d+2}\Big\|\Big(\sum_{l\in\mathbb{Z}} 2^{lq_0\alpha}\|\triangle_{2^{-l}\zeta}(f)\|^{q_0}_{L^{r_0}(\mathfrak{R}_d)}\Big)^{1/q_0}\Big\|_{L^{p_0}(\mathbb{R}^d)} \\ &\leqslant CAv^{1/2}\Big(\frac{B_\varphi^{\gamma_s}}{B_\varphi^{\gamma_s}-1}\Big)^{d+2}\|f\|_{\dot{F}_\alpha^{p_0,q_0}(\mathbb{R}^d)}. \end{aligned}$$

This yields (32) and completes the proof of Lemma 6. □

3. Proofs of Theorems 1–4

This section is devoted to presenting the proofs of main results. In what follows, let $\sigma_{h,\Omega,\Gamma,t}$ be defined as in Section 2 and set $\triangle_{2^{-l}\zeta}(f)(x) = f(x+2^{-l}\zeta) - f(x)$ for any $l \in \mathbb{Z}$, $\zeta \in \mathfrak{R}_d$ and $x \in \mathbb{R}^d$. Let us begin with the proof of Theorem 3.

Proof of Theorem 3. Following from [10], we first recall some notations. For $l \in \mathbb{N}\setminus\{0\}$, we denote $V_{n,l}$ as the space of real-valued homogeneous polynomials of degree l on $\mathbb{R}^n$ and $\mathscr{A}_n$ denotes the class of polynomials of n variables with real coefficients.

Let $\mathscr{P} = (P_1, \ldots, P_d)$ with $P_j \in \mathscr{A}_n$ for $1 \leqslant j \leqslant d$ and $\deg(\mathscr{P}) = \max_{1\leqslant j\leqslant d} \deg(P_j)$. There are integers $0 < l_1 < l_2 < \cdots < l_{\mathscr{N}} \leqslant \deg(\mathscr{P})$, and polynomials $Q_j^\eta \in V_{n,l_\eta} \subset \mathscr{A}_n$, $R_j \in \mathscr{A}_1$ with $\deg(R_j) \leqslant \deg(\mathscr{P})$ for $1 \leqslant \eta \leqslant \mathscr{N}$, $1 \leqslant j \leqslant d$ such that

$$\mathscr{P}(x) = \mathscr{R}(|x|) + \sum_{\eta=1}^{\mathscr{N}} \mathscr{Q}^\eta(x),$$

where $\mathscr{Q}^\eta = (Q_1^\eta, Q_2^\eta, \ldots, Q_d^\eta)$, $\mathscr{R} = (R_1, R_2, \ldots, R_d)$ and $Z_{l_\eta}(Q_j^\eta) = Q_j^\eta$ for $1 \leqslant \eta \leqslant \mathscr{N}$ and $1 \leqslant j \leqslant d$, where $Z_{l_\eta} : V_{n,l_\eta} \to V_{n,l_\eta}$ is a linear transformation defined as in (3.10) in [10]. Note that for each $1 \leqslant \eta \leqslant \mathscr{N}$, there is at least one $1 \leqslant j \leqslant d$ such that $Q_j^\eta \neq 0$. For $1 \leqslant j \leqslant d$ and $1 \leqslant \eta \leqslant \mathscr{N}$, write

$$Q_j^\eta(x) = \sum_{|\beta|=l_\eta} b_{\eta j\beta} x^\beta = \sum_{s=1}^{\ell(\eta)} b'_{\eta js} x^{\beta_{\eta,s}},$$

where $\ell(\eta) = \dim(V_{n,l_\eta})$ and $|\beta_{\eta,v}| = l_\eta$ for any $1 \leqslant v \leqslant \ell(\eta)$. For $1 \leqslant \eta \leqslant \mathscr{N}$, define the linear transformations $L_\eta : \mathbb{R}^d \to \mathbb{R}^{\ell(\eta)}$ by

$$L_\eta(\xi) = \Big(\sum_{j=1}^{d} b'_{\eta j1} \xi_j, \cdots, \sum_{j=1}^{d} b'_{\eta j\ell(\eta)} \xi_j \Big).$$

Define $\Phi_0, \ldots, \Phi_{\mathscr{N}}$ by

$$\Phi_\eta(x) = \mathscr{R}(|x|) + \sum_{u=1}^{\eta} \mathscr{Q}^u(x) \ \text{ for } 0 \leqslant \eta \leqslant \mathscr{N}.$$

For any $0 \leqslant \eta \leqslant \mathscr{N}$ and $t \in \mathbb{R}_+$, we denote $\sigma_{t,\eta}$ by $\sigma_{h,\Omega,\Gamma_\eta,t}$ with $\Gamma_\eta(y) = \Phi_\eta(\varphi(|y|)y')$. By the change of variables and Hölder's inequality we have

$$\begin{aligned}
&|\widehat{\sigma_{t,\eta}}(\xi)| \\
&= \Big| \frac{1}{t^\rho} \int_{t/2}^{t} \int_{S^{n-1}} \Omega(y') \exp(-2\pi i \xi \cdot \Phi_\eta(\varphi(r)y')) d\sigma(y') h(r) \frac{dr}{r^{1-\rho}} \Big| \\
&\leqslant C\|h\|_{\Delta_\gamma(\mathbb{R}_+)} \Big(\int_{t/2}^{t} \Big| \int_{S^{n-1}} \Omega(y') \exp(-2\pi i \xi \cdot \Phi_\eta(\varphi(r)y')) d\sigma(y') \Big|^{\gamma'} \frac{dr}{r} \Big)^{1/\gamma'} \\
&\leqslant C \Big(\int_{\varphi(t/2)}^{\varphi(t)} \Big| \int_{S^{n-1}} \Omega(y') \exp(-2\pi i \xi \cdot \Phi_\eta(ry')) d\sigma(y') \Big|^{\gamma'} \frac{dr}{\varphi'(\varphi^{-1}(r))\varphi^{-1}(r)} \Big)^{1/\gamma'} \\
&\leqslant C(\varphi) \Big(\int_{1}^{c_\varphi} \Big| \int_{S^{n-1}} \Omega(y') \exp(-2\pi i \xi \cdot \Phi_\eta(\varphi(t)ry')) d\sigma(y') \Big|^{\gamma'} \frac{dr}{r} \Big)^{1/\gamma'} \\
&\leqslant C(\varphi) \|\Omega\|_{L^s(S^{n-1})}^{\max\{1-2/\gamma',0\}} \\
&\quad \times \Big(\int_{1}^{c_\varphi} \Big| \int_{S^{n-1}} \Omega(y') \exp(-2\pi i \xi \cdot \Phi_\eta(\varphi(t)ry')) d\sigma(y') \Big|^{s'} \frac{dr}{r} \Big)^{2/(s' \max\{2,\gamma'\})}.
\end{aligned} \tag{39}$$

By the similar argument as in getting [10, Corollary 4.3] with $\varepsilon = (8l_\eta)^{-1}$ and careful analysis, there exists $C > 0$ independent of s such that

$$\begin{aligned}
&\Big(\int_{1}^{c_\varphi} \Big| \int_{S^{n-1}} \Omega(y') \exp(-2\pi i \xi \cdot \Phi_\eta(\varphi(t)ry')) d\sigma(y') \Big|^{s'} \frac{dr}{r} \Big)^{1/s'} \\
&\leqslant C|\varphi(t)^{l_\eta} L_\eta(\xi)|^{-1/(4l_\eta s')} \|\Omega\|_{L^s(S^{n-1})}.
\end{aligned}$$

Combining this inequality with (39) implies

$$|\widehat{\sigma_{t,\eta}}(\xi)| \leqslant C(\varphi)\|\Omega\|_{L^s(S^{n-1})}|\varphi(t)^{l_\eta}L_\eta(\xi)|^{-1/(2l_\eta s' \max\{2,\gamma'\})}. \tag{40}$$

It is easy to see that $\sigma_{t,0}=0$ for all $t\in\mathbb{R}_+$ and

$$|\widehat{\sigma_{t,\eta}}(\xi)| \leqslant C\|\Omega\|_{L^s(S^{n-1})}; \tag{41}$$

$$|\widehat{\sigma_{t,\eta}}(\xi)-\widehat{\sigma_{t,\eta-1}}(\xi)| \leqslant C\|\Omega\|_{L^s(S^{n-1})}\varphi(t)^{l_\eta}|L_\eta(\xi)|. \tag{42}$$

It follows from (40)–(42) that

$$|\widehat{\sigma_{t,\eta}}(\xi)-\widehat{\sigma_{t,\eta-1}}(\xi)| \leqslant C\|\Omega\|_{L^s(S^{n-1})}(\varphi(t)^{l_\eta}|L_\eta(\xi)|)^{1/s'}; \tag{43}$$

$$|\widehat{\sigma_{t,\eta}}(\xi)| \leqslant C(\varphi)\|\Omega\|_{L^s(S^{n-1})}\min\{1,|\varphi(t)^{l_\eta}L_\eta(\xi)|\}^{-1/(4l_\eta\gamma' s')}. \tag{44}$$

Let $\alpha\in(0,1)$ and $(1/p,1/q)\in\mathscr{R}_\gamma$, we can choose $1<r<\min\{p,q\}$ such that $(1/p,1/q,1/r)$ belongs to the interior of the convex hull of three cubes $(\frac{1}{2},\frac{1}{2}+\frac{1}{\max\{2,\gamma'\}})^3$, $(\frac{1}{2}-\frac{1}{\max\{2,\gamma'\}},\frac{1}{2})^3$, and $(\frac{1}{2\gamma},1-\frac{1}{2\gamma})^3$. Invoking Lemma 4 we have

$$\begin{aligned}&\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big(\int_{\mathfrak{R}_d}\Big(\sum_{k\in\mathbb{Z}}\int_{2^{ks'}}^{2^{(k+1)s'}}\big||\sigma_{t,\eta}|*g_{j,\zeta,k}\big|^2\frac{dt}{t}\Big)^{1/2}d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)}\\&\leqslant C\Big(\frac{s}{s-1}\Big)^{1/2}\|\Omega\|_{L^s(S^{n-1})}\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big\|\Big(\sum_{k\in\mathbb{Z}}|g_{j,\zeta,k}|^2\Big)^{1/2}\Big\|^q_{L^r(\mathfrak{R}_d)}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)}\end{aligned} \tag{45}$$

for arbitrary functions $\{g_{j,\zeta,k}\}_{j,\zeta,k}\in L^p(\ell^q(L^r(\ell^2)),\mathbb{R}^d)$, where $C>0$ is independent of s, Ω and the coefficients of P_j for $1\leqslant j\leqslant d$. Applying Lemma 6 and (43)–(45) we get

$$\begin{aligned}&\Big\|\Big(\sum_{l\in\mathbb{Z}}2^{lq\alpha}\Big(\int_{\mathfrak{R}_d}\Big(\int_0^\infty|\sigma_{t,\mathscr{N}}*\triangle_{2^{-l}\zeta}(f)|^2\frac{dt}{t}\Big)^{1/2}d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)}\\&\leqslant C(s-1)^{-1/2}\|\Omega\|_{L^s(S^{n-1})}\|f\|_{\dot F_\alpha^{p,q}(\mathbb{R}^d)},\end{aligned} \tag{46}$$

where $C>0$ is independent of s, Ω and the coefficients of P_j for $1\leqslant j\leqslant d$. By Minkowski's inequality it follows that

$$\begin{aligned}\mathscr{M}_{h,\Omega,\Gamma,\rho}(f)(x)&=\Big(\int_0^\infty\Big|\sum_{k=-\infty}^{0}2^{k\rho}\sigma_{2^kt,\mathscr{N}}*f(x)\Big|^2\frac{dt}{t}\Big)^{1/2}\\&\leqslant\sum_{k=-\infty}^{0}2^{k\varsigma}\Big(\int_0^\infty|\sigma_{2^kt,\mathscr{N}}*f(x)|^2\frac{dt}{t}\Big)^{1/2}\\&\leqslant\frac{1}{1-2^{-\varsigma}}\Big(\int_0^\infty|\sigma_{t,\mathscr{N}}*f(x)|^2\frac{dt}{t}\Big)^{1/2}.\end{aligned} \tag{47}$$

By (46)–(47) and (i) of Lemma 1 we get

$$
\begin{aligned}
&\|\mathscr{M}_{h,\Omega,\Gamma,\rho}(f)\|_{\dot{F}_{\alpha}^{p,q}(\mathbb{R}^d)}\\
&\leqslant C\Big\|\Big(\sum_{l\in\mathbb{Z}}2^{lq\alpha}\Big(\int_{\mathfrak{R}_d}|\mathscr{M}_{h,\Omega,\Gamma,\rho}(f)(\cdot+2^{-l}\zeta)-\mathscr{M}_{h,\Omega,\Gamma,\rho}(f)(\cdot)|d\zeta\Big)^{q}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)}\\
&\leqslant C\Big\|\Big(\sum_{l\in\mathbb{Z}}2^{lq\alpha}\Big(\int_{\mathfrak{R}_d}|\mathscr{M}_{h,\Omega,\Gamma,\rho}(\triangle_{2^{-l}\zeta}(f))|d\zeta\Big)^{q}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)}\\
&\leqslant C\Big\|\Big(\sum_{l\in\mathbb{Z}}2^{lq\alpha}\Big(\int_{\mathfrak{R}_d}\Big(\int_0^\infty|\sigma_{t,\mathscr{N}}*\triangle_{2^{-l}\zeta}(f)|^2\frac{dt}{t}\Big)^{1/2}d\zeta\Big)^{q}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)}\\
&\leqslant C(s-1)^{-1/2}\|\Omega\|_{L^s(S^{n-1})}\|f\|_{\dot{F}_{\alpha}^{p,q}(\mathbb{R}^d)}
\end{aligned}
\tag{48}
$$

for $\alpha\in(0,1)$ and $(1/p,1/q)\in\mathscr{R}_\gamma$. This completes the proof of Theorem 3. □

Proof of Theorem 4. Let $N=\max_{1\leqslant j\leqslant n}\deg(P_j)$. For $1\leqslant j\leqslant n$, we set $P_l(t)=\sum_{i=1}^N a_{i,j}t^i$. There are integers $0<l_1<l_2<\ldots<l_\Lambda\leqslant N$ such that $P_j(t)=\sum_{i=1}^\Lambda a_{l_i,j}t^{l_i}$ for any $1\leqslant j\leqslant n$ and $(a_{l_i,1},a_{l_i,2},\ldots,a_{l_i,n})\neq(0,0,\ldots,0)\in\mathbb{R}^n$ for all $1\leqslant i\leqslant\Lambda$. For $1\leqslant j\leqslant n$ and $1\leqslant\eta\leqslant\Lambda$, set $P_j^{(\eta)}(t)=\sum_{i=1}^\eta a_{l_i,j}t^{l_i}$. Define $\Phi_0,\Phi_1,\ldots,\Phi_\Lambda$ by

$$\Phi_\eta(y)=(P_1^{(\eta)}(|y|)y_1',\ldots,P_n^{(\eta)}(|y|)y_n'),\ \ 0\leqslant\eta\leqslant\Lambda.$$

Clearly,

$$\Phi_\eta(y)\cdot\xi=\sum_{j=1}^n P_j^{(\eta)}(|y|)y'\cdot\xi_j=\sum_{i=1}^\eta(L_i(\xi)\cdot y')|y|^{l_i},$$

for any $y,\xi\in\mathbb{R}^n$ and $1\leqslant\eta\leqslant\Lambda$, where $L_i:\mathbb{R}^n\to\mathbb{R}^n$ is the linear transformation given by

$$L_i(\xi)=(a_{l_i,1}\xi_1,a_{l_i,2}\xi_2,\ldots,a_{l_i,n}\xi_n).$$

For any $0\leqslant\eta\leqslant\Lambda$ and $t\in\mathbb{R}_+$, we denote $\sigma_{t,\eta}$ by $\sigma_{h,\Omega,\Gamma_\eta,t}$ with $\Gamma_\eta(y)=\Phi_\eta(\varphi(|y|)y')$. One can easily check that $\sigma_{t,0}=0$ for all $t\in\mathbb{R}_+$ and

$$|\widehat{\sigma_{t,\eta}}(\xi)|\leqslant C\|\Omega\|_{L^s(S^{n-1})};\tag{49}$$

$$|\widehat{\sigma_{t,\eta}}(\xi)-\widehat{\sigma_{t,\eta-1}}(\xi)|\leqslant C\|\Omega\|_{L^s(S^{n-1})}\varphi(t)^{l_\eta}|L_\eta(\xi)|.\tag{50}$$

By a change of variable and Hölder's inequality we have

$$
\begin{aligned}
|\widehat{\sigma_{t,\eta}}(\xi)|&=\Big|\frac{1}{t^\rho}\int_{t/2}^t\int_{S^{n-1}}\Omega(y')\exp(-2\pi i\xi\cdot\Phi_\eta(\varphi(r)y'))d\sigma(y')h(r)\frac{dr}{r^{1-\rho}}\Big|\\
&\leqslant C\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\Big(\int_{t/2}^t\Big|\int_{S^{n-1}}\Omega(y')\exp(-2\pi i\xi\cdot\Phi_\eta(\varphi(r)y'))d\sigma(y')\Big|^{\gamma'}\frac{dr}{r}\Big)^{1/\gamma'}\\
&\leqslant C\|\Omega\|_{L^1(S^{n-1})}^{\max\{1-2/\gamma',0\}}\\
&\quad\times\Big(\int_{t/2}^t\Big|\int_{S^{n-1}}\Omega(y')\exp(-2\pi i\xi\cdot\Phi_\eta(\varphi(r)y'))d\sigma(y')\Big|^2\frac{dr}{r}\Big)^{1/\max\{2,\gamma'\}}.
\end{aligned}
\tag{51}
$$

By Lemma 2 and Hölder's inequality,

$$
\begin{aligned}
&\int_{t/2}^{t}\Big|\int_{S^{n-1}}\Omega(y')\exp(-2\pi i\xi\cdot\Phi_\eta(\varphi(r)y'))d\sigma(y')\Big|^2\frac{dr}{r}\\
&=\int_{\varphi(t/2)}^{\varphi(t)}\Big|\int_{S^{n-1}}\Omega(y')\exp(-2\pi i\xi\cdot\Phi_\eta(ry'))d\sigma(y')\Big|^2\frac{dr}{\varphi^{-1}(r)\varphi'(\varphi^{-1}(r))}\\
&\leqslant C(\varphi)\int_{\varphi(t/2)}^{\varphi(t)}\Big|\int_{S^{n-1}}\Omega(y')\exp(-2\pi i\xi\cdot\Phi_\eta(ry'))d\sigma(y')\Big|^2\frac{dr}{r}\\
&\leqslant C(\varphi)\int_{c_\varphi^{-1}}^{1}\Big|\int_{S^{n-1}}\Omega(y')\exp\Big(-2\pi i\sum_{i=1}^{\eta}(L_i(\xi)\cdot y')\varphi(t)^{l_i}r^{l_i}\Big)d\sigma(y')\Big|^2\frac{dr}{r}\\
&\leqslant C(\varphi)\iint_{(S^{n-1})^2}|\Omega(y')\overline{\Omega(u')}|\\
&\quad\times\Big|\int_{c_\varphi^{-1}}^{1}\exp\Big(-2\pi i\sum_{i=1}^{\eta}(L_i(\xi)\cdot(y'-u'))\varphi(t)^{l_i}r^{l_i}\Big)\frac{dr}{r}\Big|d\sigma(y')d\sigma(u')\\
&\leqslant C(\varphi)\iint_{(S^{n-1})^2}|\Omega(y')\overline{\Omega(u')}|\\
&\quad\times\min\{\log c_\varphi,|\varphi(t)^{l_\eta}L_\eta(\xi)\cdot(y'-u')|^{-1/l_\eta}\}d\sigma(y')d\sigma(u')\\
&\leqslant C(\varphi)\|\Omega\|^2_{L^s(S^{n-1})}|\varphi(t)^{l_\eta}L_\eta(\xi)|^{-1/(l_\eta s')},
\end{aligned}
$$

which together with (51) yields

$$|\widehat{\sigma_{t,\eta}}(\xi)|\leqslant C\|\Omega\|_{L^s(S^{n-1})}|\varphi(t)^{l_\eta}L_\eta(\xi)|^{-1/(l_\eta s'\max\{2,\gamma'\})}. \tag{52}$$

It follows from (49)–(50) and (52) that

$$|\widehat{\sigma_{t,\eta}}(\xi)-\widehat{\sigma_{t,\eta-1}}(\xi)|\leqslant C\|\Omega\|_{L^s(S^{n-1})}(\varphi(t)^{l_\eta}|L_\eta(\xi)|)^{1/s'}; \tag{53}$$

$$|\widehat{\sigma_{t,\eta}}(\xi)|\leqslant C\|\Omega\|_{L^s(S^{n-1})}\min\{1,|\varphi(t)^{l_\eta}L_\eta(\xi)|\}^{-1/(2l_\eta\gamma's')}. \tag{54}$$

Let $\alpha\in(0,1)$ and $(1/p,1/q)\in\mathscr{R}_\gamma$. We can choose $1<r<\min\{p,q\}$ such that $(1/p,1/q,1/r)$ belongs to the interior of the convex hull of three cubes $(\frac12,\frac12+\frac{1}{\max\{2,\gamma'\}})^3$, $(\frac12-\frac{1}{\max\{2,\gamma'\}},\frac12)^3$ and $(\frac{1}{2\gamma},1-\frac{1}{2\gamma})^3$. Invoking Lemma 4 we obtain

$$
\begin{aligned}
&\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big(\int_{\mathfrak{R}_d}\Big(\sum_{k\in\mathbb{Z}}\int_{2^{ks'}}^{2^{(k+1)s'}}||\sigma_{t,\eta}|*g_{j,\zeta,k}|^2\frac{dt}{t}\Big)^{1/2}d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)}\\
&\leqslant C\Big(\frac{s}{s-1}\Big)^{1/2}\|\Omega\|_{L^s(S^{n-1})}\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big\|\Big(\sum_{k\in\mathbb{Z}}|g_{j,\zeta,k}|^2\Big)^{1/2}\Big\|^q_{L^r(\mathfrak{R}_d)}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)}.
\end{aligned} \tag{55}
$$

for any $1\leqslant\eta\leqslant\Lambda$. Applying Lemma 6, we get from (53)–(55) that

$$
\begin{aligned}
&\Big\|\Big(\sum_{l\in\mathbb{Z}}2^{lq\alpha}\Big(\int_{\mathfrak{R}_d}\Big(\int_0^\infty|\sigma_{t,\Lambda}*\triangle_{2^{-l}\zeta}(f)|^2\frac{dt}{t}\Big)^{1/2}d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)}\\
&\leqslant C(s-1)^{-1/2}\|\Omega\|_{L^s(S^{n-1})}\|f\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^d)},
\end{aligned} \tag{56}
$$

for $\alpha \in (0,1)$ and $(1/p,1/q) \in \mathscr{R}_\gamma$. On the other hand, by the similar argument as in getting (47) we have

$$\mathscr{M}_{h,\Omega,\Gamma,\rho}(f)(x) \leqslant \frac{1}{1-2^{-\varsigma}}\Big(\int_0^\infty |\sigma_{t,\Lambda} * f(x)|^2 \frac{dt}{t}\Big)^{1/2}. \tag{57}$$

for any $0<\alpha<1$ and $1<p,q<\infty$. Theorem 4 follows from (56)–(57) and the same argument as in getting (48). □

Proof of Theorem 1. By Theorem 3 and some extrapolation arguments (see the proof of [3, Theorem 2.3(a)]). One can easily get (i) of Theorem 1. Let $\alpha \in (0,1)$, $|1/p-1/2| < \min\{1/2, 1/\gamma'\}$ and $1<q<\infty$. By Remark 4, (ii) of Lemma 1 and Fubini's theorem we have

$$\begin{aligned}
&\|\mathscr{M}_{h,\Omega,\Gamma,\rho}(f)\|_{\dot{B}_\alpha^{p,q}(\mathbb{R}^d)} \\
&\leqslant C\Big(\sum_{l\in\mathbb{Z}} 2^{lq\alpha}\Big\|\Big(\int_{\mathfrak{R}_d} |\mathscr{M}_{h,\Omega,\Gamma,\rho}(f)(\cdot+2^{-l}\zeta) - \mathscr{M}_{h,\Omega,\Gamma,\rho}(f)(\cdot)|^p d\zeta\Big)^{1/p}\Big\|_{L^p(\mathbb{R}^d)}^q\Big)^{1/q} \\
&\leqslant C\Big(\sum_{l\in\mathbb{Z}} 2^{lq\alpha}\Big\|\Big(\int_{\mathfrak{R}_d} |\mathscr{M}_{h,\Omega,\Gamma,\rho}(\triangle_{2^{-l}\zeta}(f))|^p d\zeta\Big)^{1/p}\Big\|_{L^p(\mathbb{R}^d)}^q\Big)^{1/q} \\
&= C\Big(\sum_{l\in\mathbb{Z}} 2^{lq\alpha}\Big(\int_{\mathfrak{R}_d}\int_{\mathbb{R}^d} |\mathscr{M}_{h,\Omega,\Gamma,\rho}(\triangle_{2^{-l}\zeta}(f))(x)|^p dx d\zeta\Big)^{q/p}\Big)^{1/q} \\
&\leqslant C(1+\|\Omega\|_{L(\log^+ L)^{1/2}(S^{n-1})})\Big(\sum_{l\in\mathbb{Z}} 2^{lq\alpha}\Big(\int_{\mathbb{R}^d}\int_{\mathfrak{R}_d} |\triangle_{2^{-l}\zeta}(f)(x)|^p d\zeta dx\Big)^{q/p}\Big)^{1/q} \\
&= C(1+\|\Omega\|_{L(\log^+ L)^{1/2}(S^{n-1})})\|f\|_{\dot{B}_\alpha^{p,q}(\mathbb{R}^d)}.
\end{aligned}$$

This yields (ii) of Theorem 1. □

Proof of Theorem 2. By Theorem 4 and the same arguments as in the proof of [3, Theorem 2.3], we can get (i) of Theorem 2. (ii) of Theorem 2 follows form the same arguments as in getting (ii) of Theorem 1. We omit the details. □

REFERENCES

[1] H. M. AL-QASSEM AND A. J. AL-SALMAN, *A note on Marcinkiewicz integral operators*, J. Math. Anal. Appl. **282**, 2 (2003), 698–710.

[2] A. AL-SALMAN, H. AL-QASSEM, L. C. CHENG AND Y. PAN, *L^p bounds for the function of Marcinkiewicz*, Math. Res. Lett. **9**, 5 (2002), 697–700.

[3] H. AL-QASSEM AND Y. PAN, *On certain estimates for Marcinkiewicz integrals and extrapolation*, Collect. Math. **60**, 2 (2009), 123–145.

[4] Y. DING, D. FAN AND Y. PAN, *L^p-boundedness of Marcinkiewicz integrals with Hardy space function kernel*, Acta Math. Sin. (Engl. Ser.), **16**, 4 (2000), 593–600.

[5] Y. DING, D. FAN AND Y. PAN, *On the L^p boundedness of Marcinkiewicz integrals*, Michigan Math. J. **50**, (2002), 17–26.

[6] Y. DING, S. LU AND K. YABUTA, *A problem on rough parametric Marcinkiewicz functions*, J. Austr. Math. Soc. **72**, 1 (2002), 13–21.

[7] Y. DING AND Y. PAN, *L^p bounds for Marcinkiewicz integrals*, Proc. Edinb. Math. Soc. **46**, 3 (2003), 669–677.
[8] Y. DING, Q. XUE AND K. YABUTA, *Boundedness of the Marcinkiewicz integrals with rough kernel associated to surfaces*, Tohoku Math. J. **62**, 2 (2010), 233–262.
[9] M. FRAZIER, B. JAWERTH AND G. WEISS, *Littlewood-Paley theory and the study of function spaces*, CBMS Reg. Conf. Ser. vol. **79**, Amer. Math. Soc., Providence, RI 1991.
[10] D. FAN AND Y. PAN, *Singular integral operators with rough kernels supported by subvarieties*, Amer. J. Math. **119**, 4 (1997), 799–839.
[11] HÖRMANDER, *Estimates for translation invariant operators in L^p spaces*, Acta Math., **104**, 10 (1960), 93–104.
[12] F. LIU, *Integral operators of Marcinkiewicz type on Triebel-Lizorkin spaces*, Math. Nachr., (to appear) (2016). DOI: 10.1002/mana.201500374.
[13] F. LIU AND H. WU, *Multiple singular integrals and Marcinkiewicz integrals with mixed homogeneity along surfaces*, J. Inequ. Appl. **2012**, 189 (2012), 1–23.
[14] F. LIU AND H. WU, *On Marcinkiewicz integrals associated to compound mappings with rough kernels*, Acta Math. Sin. (Engl. Ser.), **30**, 7 (2014), 1210–1230.
[15] F. LIU AND D. ZHANG, *Parabolic Marcinkiewicz integrals associated to polynomials compound curves and extrapolation*, Bull. Korean Math. Soc. **52**, 3 (2015), 771–788.
[16] R. RICCI AND E. M. STEIN, *Harmonic analysis on nilpotent groups and singular integrals I: Oscillatory integrals*, J. Funct. Anal. **73**, 1 (1987), 179–184.
[17] M. SAKAMOTO AND K. YABUTA, *Boundedness of Marcinkiewicz functions*, Studia Math. **135**, (1999), 103–142.
[18] E. M. STEIN, *On the function of Littlewood-Paley, Lusin and Marcinkiewicz*, Trans. Amer. Math. Soc. **88**, 2 (1958), 430–466.
[19] H. TRIEBEL, *Theory of Function Spaces*, Monogr. Math. vol. **78**, Birkhäser Verlag, Basel, 1983.
[20] K. YABUTA, *Triebel-Lizorkin space bounddedness of Marcinkiewicz integrals associated to surfaces*, Appl. Math. J. Chinese Univ. **30**, 4 (2015), 418–446.
[21] C. ZHANG AND J. CHEN, *Boundedness of g-functions on Triebel-Lizorkin spaces*, Taiwanese J. Math. **13**, 3 (2009), 973–981.
[22] C. ZHANG AND J. CHEN, *Boundedness of Marcinkiewicz integral on Triebel-Lizorkin spaces*, Appl. Math. J. Chinese Univ. Ser. B, **25**, 25 (2010), 48–54.

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