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DETERMINANT INEQUALITIES FOR HADAMARD PRODUCT OF POSITIVE DEFINITE MATRICES

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Abstract. Let A_i, $i=1,\dots,m$, be $n\times n$ positive definite matrices whose diagonal blocks are n_j-square matrices $A_i^{(j)}$, $j=1,\dots,k$. Choi recently proved

$$\det\left(\sum_{i=1}^{m}A_i^{-1}\right)\geqslant\det\left(\sum_{i=1}^{m}(A_i^{(1)})^{-1}\right)\cdots\det\left(\sum_{i=1}^{m}(A_i^{(k)})^{-1}\right).$$

We first give a new proof of this inequality, and then present an analogous inequality involving the Hadamard product

$$\det\left(\prod_{i=1}^{m}\circ A_i^{-1}\right)\geqslant\det\left(\prod_{i=1}^{m}\circ(A_i^{(1)})^{-1}\right)\cdots\det\left(\prod_{i=1}^{m}\circ(A_i^{(k)})^{-1}\right).$$

1. Introduction

Let $\mathbb{M}_n$ be the space of $n\times n$ complex matrices. For two Hermitian matrices X and Y, we write $X\geqslant Y$ to mean that $X-Y$ is positive semidefinite, so $X\geqslant 0$ denotes that X is positive semidefinite. If X is positive definite, then we write $X>0$. Let I denote the identity matrix of a proper size. The Hadamard product (i.e., the entrywise product) of $A,\ B\in\mathbb{M}_n$ is denoted by $A\circ B$. If more matrices $A_1,\dots,A_m\in\mathbb{M}_n$ are involved, we then use $\prod_{i=1}^{m}\circ A_i$ to denote the Hadamard product of these matrices.

If $X=\begin{bmatrix}X_{11} & X_{12}\\ X_{21} & X_{22}\end{bmatrix}\in\mathbb{M}_n$ with X_{11} nonsingular, then the Schur complement of X_{11} in X is defined as

$$X/X_{11}=X_{22}-X_{21}X_{11}^{-1}X_{12}.$$

A well known property of the Schur complement is

$$\det X=\det X_{11}\det(X/X_{11}).$$

For more information on the Schur complement, we refer to the comprehensive survey (see [9]).

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Fischer's inequality [3, p. 506] states that for a positive semidefinite matrix $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ with A_{11} square, it holds

$$\det A \leqslant \det A_{11} \det A_{22}.$$

Let $A_i \in \mathbb{M}_n$, $i = 1,\dots,m$, be positive definite whose diagonal blocks are n_j-square matrices $A_i^{(j)}$, $j = 1,\dots,k$, (so $n_1 + \cdots + n_k = n$). Then

$$\det\left(\prod_{i=1}^{m} \circ A_i\right) \leqslant \det\left(\prod_{i=1}^{m} \circ A_i^{(1)}\right) \cdots \det\left(\prod_{i=1}^{m} \circ A_i^{(k)}\right)$$

follows directly from Fischer's inequality. Determinantal inequalities for positive definite matrices is the theme of a number of recent research papers, see for example [1, 4, 5, 7].

In [1], Choi proved the following result for positive definite matrices.

THEOREM 1. [1, Theorem 2] *Let $A_i \in \mathbb{M}_n$, $i = 1,\dots,m$, be positive definite whose diagonal blocks are n_j-square matrices $A_i^{(j)}$ for $j = 1,\dots,k$, (so $n_1 + \cdots + n_k = n$). Then*

$$\det\left(\sum_{i=1}^{m} A_i^{-1}\right) \geqslant \det\left(\sum_{i=1}^{m} (A_i^{(1)})^{-1}\right) \cdots \det\left(\sum_{i=1}^{m} (A_i^{(k)})^{-1}\right).$$

The main auxiliary result in Choi's proof is from [6]. In this paper, we first give an alternative proof of Theorem 1 using some properties on the Schur complement. This is done in section 2. In section 3, we prove the following analogue of Theorem 1 involving the Hadamard product.

THEOREM 2. *Let $A_i \in \mathbb{M}_n$, $i = 1,\dots,m$, be positive definite whose diagonal blocks are n_j-square matrices $A_i^{(j)}$ for $j = 1,\dots,k$ (so $n_1 + \cdots + n_k = n$). Then*

$$\det\left(\prod_{i=1}^{m} \circ A_i^{-1}\right) \geqslant \det\left(\prod_{i=1}^{m} \circ (A_i^{(1)})^{-1}\right) \cdots \det\left(\prod_{i=1}^{m} \circ (A_i^{(k)})^{-1}\right).$$

2. New proof of Theorem 1

We need some lemmas which are useful for our new proof of Theorem 1.

LEMMA 1. [3, Corollary 7.7.4] *If $A, B \in \mathbb{M}_n$ such that $0 < A \leqslant B$, then $B^{-1} \leqslant A^{-1}$ and $\det A \leqslant \det B$.*

LEMMA 2. [8, Theorem 7.13] *Let $A \in \mathbb{M}_n$ be positive definite. Partition A as $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ with A_{11} square. Let A^{-1} be conformally partitioned as A. Then*

1. *$(A_{ii})^{-1} \leqslant (A^{-1})_{ii}$, $i = 1,2$;*
2. *$A^{-1}/(A^{-1})_{11} = (A_{22})^{-1}$.*

LEMMA 3. [2, Theorem 2] *For $i=1,\dots,m$, let $A_i \in \mathbb{M}_n$ be positive definite and conformally partitioned. Then*

$$\left(\sum_{i=1}^m A_i\right) \Big/ \left(\sum_{i=1}^m A_i^{(1)}\right) \geqslant \sum_{i=1}^m \left(A_i \big/ A_i^{(1)}\right).$$

where $A_i^{(1)}$ is the $(1,1)$ block of A_i, $i=1,\dots,m$.

Now we are ready to present

Proof of Theorem 1. Since A_i are positive definite for all $i=1,\dots,m$, then A_i^{-1} are all positive definite. Let A_i^{-1} be conformally partitioned as A_i for $i=1,\dots,m$. Then the diagonal blocks of A_i^{-1} are n_j-square matrices $(A_i^{-1})^{(j)}$ for $j=1,\dots,k$. Using mathematical induction on k, we may assume $k=2$. Applying Lemma 3, we have

$$\left(\sum_{i=1}^m A_i^{-1}\right) \Big/ \left(\sum_{i=1}^m (A_i^{-1})^{(1)}\right) \geqslant \sum_{i=1}^m \left((A_i^{-1}) \big/ (A_i^{-1})^{(1)}\right).$$

Taking determinants on both sides, we get

$$\begin{aligned}
\det\left(\sum_{i=1}^m A_i^{-1}\right) &\geqslant \det\left(\sum_{i=1}^m (A_i^{-1})^{(1)}\right) \det\left(\sum_{i=1}^m \left((A_i^{-1}) \big/ (A_i^{-1})^{(1)}\right)\right)\\
&\geqslant \det\left(\sum_{i=1}^m \left(A_i^{(1)}\right)^{-1}\right) \det\left(\sum_{i=1}^m \left((A_i^{-1}) \big/ (A_i^{-1})^{(1)}\right)\right)\\
&= \det\left(\sum_{i=1}^m \left(A_i^{(1)}\right)^{-1}\right) \det\left(\sum_{i=1}^m \left(A_i^{(2)}\right)^{-1}\right),
\end{aligned}$$

where the first inequality is due to the property of the Schur complement, the second is a consequence of Lemma 1 and Lemma 2 (1), and the last equality follows from Lemma 2 (2). □

3. Proof of Theorem 2

In order to prove Theorem 2, we need to show a new result. which could be regarded as a complement of [6, Theorem 1]. We require the following basic result which is due to Schur.

LEMMA 4. [3, Theorem 7.5.3] *Let $A,B \in \mathbb{M}_n$. If $A \geqslant 0$ and $B \geqslant 0$, then $A \circ B \geqslant 0$.*

THEOREM 3. *Let* $T_k = \begin{bmatrix} X_k & Y_k \\ 0 & Z_k \end{bmatrix} \in \mathbb{M}_n$, $k = 1,\ldots,m$, *be* n*-square conformally partitioned matrices, where* X_k, Z_k *are* r*-square and* $(n-r)$*-square, respectively. Then*

$$\det\left(\prod_{k=1}^{m} \circ T_k^* T_k\right) \geqslant \det\left(\prod_{k=1}^{m} \circ X_k^* X_k\right) \cdot \det\left(\prod_{k=1}^{m} \circ Z_k^* Z_k\right).$$

Proof. We first assume that $X_k^* X_k$ is nonsingular for all $k = 1,\ldots,m,$ then

$$\begin{bmatrix} X_k & Y_k \end{bmatrix}^* \begin{bmatrix} X_k & Y_k \end{bmatrix} = \begin{bmatrix} X_k^* X_k & X_k^* Y_k \\ Y_k^* X_k & Y_k^* Y_k \end{bmatrix} \geqslant 0,$$

for $k = 1,\ldots,m,$ using Lemma 4, we have

$$\prod_{k=1}^{m} \circ \begin{bmatrix} X_k & Y_k \end{bmatrix}^* \begin{bmatrix} X_k & Y_k \end{bmatrix} = \begin{bmatrix} \prod_{k=1}^{m} \circ X_k^* X_k & \prod_{k=1}^{m} \circ X_k^* Y_k \\ \prod_{k=1}^{m} \circ Y_k^* X_k & \prod_{k=1}^{m} \circ Y_k^* Y_k \end{bmatrix} \geqslant 0.$$

Further

$$\prod_{k=1}^{m} \circ Y_k^* Y_k - \left(\prod_{k=1}^{m} \circ Y_k^* X_k\right)\left(\prod_{k=1}^{m} \circ X_k^* X_k\right)^{-1}\left(\prod_{k=1}^{m} \circ X_k^* Y_k\right) \geqslant 0.$$

On the other hand

$$T_k^* T_k = \begin{bmatrix} X_k^* X_k & X_k^* Y_k \\ Y_k^* X_k & Y_k^* Y_k + Z_k^* Z_k \end{bmatrix} \geqslant 0.$$

Hence, we get

$$\begin{aligned} \left(\prod_{k=1}^{m} \circ T_k^* T_k\right) \Big/ \left(\prod_{k=1}^{m} X_k^* X_k\right) &= \prod_{k=1}^{m} \circ (Y_k^* Y_k + Z_k^* Z_k) \\ &\quad - \left(\prod_{k=1}^{m} \circ Y_k^* X_k\right)\left(\prod_{k=1}^{m} \circ X_k^* X_k\right)^{-1}\left(\prod_{k=1}^{m} \circ X_k^* Y_k\right) \\ &\geqslant \prod_{k=1}^{m} \circ Z_k^* Z_k \geqslant 0. \end{aligned}$$

Applying the determinant on both sides, the desired inequality follows whenever $X_k^* X_k$ is nonsingular for all $k = 1,\ldots,m.$ By a continuity argument, the assertion also holds if $X_k^* X_k$ is singular for all $k = 1,\ldots,m.$ □

REMARK 1. By a simple induction, Theorem 3 can be extended to the $k \times k$ $(k \geqslant 2)$ block upper triangular case.

The following lemma is useful for the proof of Theorem 2.

LEMMA 5. [1, Lemma 2] *Let* $P = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \in \mathbb{M}_n$ *be positive definite. Then* P *can be factorized as* $P = T^* T$ *with* $T = \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix}$ *being conformally partitioned as* P.

Now we are in a position to present

Proof of Theorem 2. Mathematical induction allows us to prove Theorem 2 for $k=2$. By Lemma 5, for each $i=1,\ldots,m$, there exists a matrix $T_i=\begin{bmatrix} X_i & Y_i \\ 0 & Z_i \end{bmatrix}$ being conformally partitioned as A_i^{-1} such that $A_i^{-1}=T_i^*T_i$. Then

$$\det\left(\prod_{i=1}^{m}\circ A_i^{-1}\right)\geqslant \det\left(\prod_{i=1}^{m}\circ X_i^*X_i\right)\cdot\det\left(\prod_{i=1}^{m}\circ Z_i^*Z_i\right)$$

follows by Theorem 3. Now it is enough to show $(X_i^*X_i)^{-1}\leqslant A_i^{(1)}$ and $(Z_i^*Z_i)^{-1}\leqslant A_i^{(2)}$ for each i, since these relations and the inequality above imply

$$\det\left(\prod_{i=1}^{m}\circ A_i^{-1}\right)\geqslant \det\left(\prod_{i=1}^{m}\circ (A_i^{(1)})^{-1}\right)\det\left(\prod_{i=1}^{m}\circ (A_i^{(2)})^{-1}\right)$$

by Lemma 1. From

$$A_i^{-1}=T_i^*T_i=\begin{bmatrix} X_i^*X_i & X_i^*Y_i \\ Y_i^*X_i & Y_i^*Y_i+Z_i^*Z_i \end{bmatrix},$$

we have

$$A_i^{(1)}=(X_i^*X_i-X_i^*Y_i(Y_i^*Y_i+Z_i^*Z_i)^{-1}Y_i^*X_i)^{-1},$$

and thus $(X_i^*X_i)^{-1}\leqslant A_i^{(1)}$ by Lemma 1 again. Similarly,

$$\begin{aligned} A_i^{(2)} &= (Y_i^*Y_i+Z_i^*Z_i-Y_i^*X_i(X_i^*X_i)^{-1}X_i^*Y_i)^{-1} \\ &= (Y_i^*Y_i+Z_i^*Z_i-Y_i^*X_iX_i)^{-1}X_i^*)^{-1}X_i^*Y_i)^{-1} \\ &= (Z_i^*Z_i)^{-1}. \quad \square \end{aligned}$$

COROLLARY 1. *Let $A\in\mathbb{M}_n$ be positive definite whose diagonal blocks are n_i-square matrices A_i, for each $i=1,\ldots,k$. Then*

$$\det(I\circ A^{-1})\geqslant \det(I\circ A_1^{-1})\cdots\det(I\circ A_k^{-1}).$$

COROLLARY 2. *Let $C\in\mathbb{M}_n$ be positive definite whose diagonal blocks are n_i-square matrices C_i, for $i=1,\ldots,k$. Let $D_i\geqslant 0$ be n_i-square matrices, for each $i=1,\ldots,k$, and $D=diag(D_1,\ldots,D_k)$. Then*

$$\det(C^{-1}\circ D)\geqslant \det(C_1^{-1}\circ D_1)\cdots\det(C_k^{-1}\circ D_k).$$

Proof. We first assume that D is nonsingular, that is, D_i are all invertible for $i=1,\ldots,k$. Then, by Theorem 2

$$\begin{aligned} \det(C^{-1}\circ D) &= \det(C^{-1}\circ (D^{-1})^{-1}) \\ &\geqslant \det(C_1^{-1}\circ (D_1^{-1})^{-1})\cdots\det(C_k^{-1}\circ (D_k^{-1})^{-1}) \\ &= \det(C_1^{-1}\circ D_1)\cdots\det(C_k^{-1}\circ D_k). \end{aligned}$$

By a standard continuity argument, the statement is also true if D is singular. □

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REFERENCES

[1] D. CHOI, *Determinantal inequalities of positive definite matrices*, Math. Inequal. Appl. **19** (2016), 167–172.
[2] E. V. HAYNSWORTH, *Applications of an inequality for the Schur complement*, Amer. Math. Soc. **24** (1970), 512–516.
[3] R. A. HORN, C. R. JOHNSON, *Matrix Analysis*, Cambridge University Press, Cambridge, 1990.
[4] M. LIN, *A determinantal inequalities for positive definite matrices*, Electron. J. Linear Algebra. **27** (2014), 821–826.
[5] M. LIN, *An Oppenheim type inequalities for a block Hadamard product*, Linear Algebra Appl. **452** (2014), 1–6.
[6] M. LIN, *Determinantal inequalities for block triangular matrices*, Math. Inequal. Appl. **18** (2015), 1079–1086.
[7] I. MATIC, *Inequalities with determinannts of perturbed positive matrices*, Linear Algebra Appl. **449** (2014), 166–174.
[8] F. ZHANG, *Matrix Theory, Basic results and techniques*, Second Edition, Springer, 2011.
[9] F. ZHANG, *The Schur Complement and its Applications*, Springer, New York, 2005.

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