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BOUNDS FOR THE ZEROS OF POLYNOMIALS FROM NUMERICAL RADIUS INEQUALITIES

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Abstract. We apply matrix norms and recent numerical radius inequalities to a certain Frobenius companion matrix to derive several bounds for the zeros of polynomials. Our results are related to some classical and recent bounds and lead to improve these bounds.

1. Introduction

The problem in locating the zeros of complex polynomials has been frequently investigated. Matrix analysis methods have been used to obtain new proofs of classical bounds for the zeros of polynomials and to derive new bounds for these zeros. Over many decades a large number of research papers have been published. Matrix norms computations and numerical radii estimations used to obtain bounds for zeros of polynomials in terms of the entries of the first row of the Frobenius companion matrix and the first row of the square of the Frobenius companion matrix. See, e.g., [2], [6], [8], [9], [10], [11], and the references therein.

Let $p(z) = z^n + a_n z^{n-1} + \cdots + a_2 z + a_1$ be a complex monic polynomial with $a_1 \neq 0$. Let $z_1, z_2, \ldots, z_n$ be the zeros of p arranged in such a way that $|z_1| \geqslant |z_2| \geqslant \ldots \geqslant |z_n|$. Then $p(z)$ is the characteristic polynomial of the Frobenius companion matrix C_p of p, which is given by

$$C_p = \begin{bmatrix} -a_n & -a_{n-1} & \cdots & -a_2 & -a_1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

(see [4, p. 316]). So

$$C_p^2 = \begin{bmatrix} b_n & b_{n-1} & \cdots & b_3 & b_2 & b_1 \\ -a_n & -a_{n-1} & \cdots & -a_3 & -a_2 & -a_1 \\ 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 \end{bmatrix}$$

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where $b_j = a_n a_j - a_{j-1}$ for $j = 1, 2, \ldots, n$, with $a_0 = 0$.

Let $q(z) = (z - a_n)\, p(z) = z^{n+1} - b_n z^{n-1} - b_{n-1} z^{n-2} - \cdots - b_2 z - b_1$. So $z_1, z_2, \ldots, z_n$ and a_n are the zeros of q. The Frobenius companion matrix C_q of q, which is given by

$$C_q = \begin{bmatrix} 0 & b_n & b_{n-1} & \cdots & b_2 & b_1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

Since the spectral radius of a matrix A is dominated by its numerical radius, that is, $|\lambda_1(A)| \leqslant w(A)$, it follows that

$$\left|z_j\right| \leqslant w(C_q) \tag{1}$$

for $j = 1, 2, \ldots, n$.

Let $M_n(\mathbb{C})$ denote the algebra of all $n \times n$ complex matrices. For $A \in M_n(\mathbb{C})$, let $\|A\|$ denote the spectral norm of A. The numerical radius of $A \in M_n(\mathbb{C})$ is defined as

$$w(A) = \sup\{|\langle Ax, x\rangle| : x \in \mathbb{C}^n, \|x\| = 1\},$$

where $\langle ., . \rangle$ is the Euclidean inner product on $\mathbb{C}^n$. It is known that $w(.)$ is a norm on $M_n(\mathbb{C})$, which is equivalent to the spectral norm $\|.\|$.

In this paper, we apply several matrix inequalities to C_q to obtain bounds for the zeros of f in terms of the first row of C_p^2. In particular, we apply matrix norms and recent numerical radius inequalities to obtain new bounds. This is a continuation of the earlier work [10] and [11].

2. Bounds for the zeros of polynomials

To achieve our goal of obtaining new bounds for the zeros of polynomials we need the following lemmas, The first two lemmas are well-known and they can be found in [12] and [7, pp. 8–9], respectively.

LEMMA 1. *Let $A \in M_n(\mathbb{C})$. Then*

$$w(A) = \max_{\theta \in \mathbb{R}} \left\| \operatorname{Re}\left(e^{i\theta} A\right) \right\|.$$

LEMMA 2. *Let L_n be the $n \times n$ matrix given by*

$$L_n = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

Then $w(L_n) = \cos \frac{\pi}{n+1}$.

The third lemma contains two recent inequalities for numerical radius, see [3].

LEMMA 3. *Let* $A \in M_k(\mathbb{C}), B \in M_{k\times m}(\mathbb{C}), C \in M_{m\times k}(\mathbb{C})$, *and* $D \in M_m(\mathbb{C})$, *and let* $T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$. *Then*

$$w(T) \leqslant \frac{1}{2}\left(w(A)+w(D)+\sqrt{(w(A)-w(D))^2+4w^2(T_O)}\right) \tag{2}$$

$$\leqslant \frac{1}{2}\left(w(A)+w(D)+\sqrt{(w(A)-w(D))^2+(\|B\|+\|C\|)^2}\right), \tag{3}$$

where $M_{k\times m}(\mathbb{C})$ *is the space of all* $k\times m$ *complex matrices and* $T_O = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$.

Fujii and Kubo [5] have used the fact that the spectral radius of the companion matrix is dominated by any of its matrix norms to give proofs of some classical bounds, such as:

Carmichael and Mason's bound: for $j = 1,2,\ldots,n$, we have

$$|z_j| \leqslant \sqrt{1+\sum_{i=1}^{n}|a_i|^2}.$$

Cauchy's bound: for $j = 1,2,\ldots,n$, we have

$$|z_j| \leqslant \max(|a_1|, 1+|a_2|, \ldots, 1+|a_n|)$$
$$\leqslant 1+\max(|a_1|, |a_2|, \ldots, |a_n|).$$

Abdurakhmanov [1], Fujii and Kubo [6], Kittaneh [8], and others have given bounds for the zeros of polynomials from the matrix inequalities applied to the companion matrices, such as:

Abdurakhmanov's bound: for $j = 1,2,\ldots,n$, we have

$$|z_j| \leqslant \frac{1}{2}\left(|a_n|+\cos\frac{\pi}{n}+\sqrt{\left(|a_n|-\cos\frac{\pi}{n}\right)^2+\left(1+\sqrt{\sum_{i=1}^{n-1}|a_i|^2}\right)^2}\right).$$

Fujii and Kubo's bound: for $j = 1,2,\ldots,n$, we have

$$|z_j| \leqslant \cos\frac{\pi}{n+1}+\frac{1}{2}\left(|a_n|+\sqrt{\sum_{i=1}^{n}|a_i|^2}\right).$$

Kittaneh's bound: for $j = 1,2,\ldots,n$, we have

$$|z_j| \leqslant \frac{1}{2}\left(|a_n|+\cos\frac{\pi}{n}+\sqrt{\left(|a_n|-\cos\frac{\pi}{n}\right)^2+(1+|a_{n-1}|)^2+\sum_{i=1}^{n-2}|a_i|^2}\right).$$

Our first bound is related to Cauchy's bound.

THEOREM 1. *For* $j = 1, 2, \ldots, n$, *we have*

$$\left|z_j\right| \leqslant \max\left(|b_1|, 1+|b_2|, 1+|b_3|, \ldots, 1+|b_n|\right). \tag{4}$$

Proof. For a matrix $A = [a_{ij}]_{n\times n}$, the maximum column sum matrix norm $\|.\|_1$ is defined on $M_n(\mathbb{C})$ by (see [4, p. 294])

$$\|A\|_1 = \max_{1\leqslant j\leqslant n} \sum_{i=1}^{n} \left|a_{ij}\right|.$$

Applying this norm to C_q, we get

$$\left\|C_q\right\|_1 = \max\left(|b_1|, 1+|b_2|, 1+|b_3|, \ldots, 1+|b_n|\right).$$

Since the spectral radius of any matrix is dominated by any matrix norm, we get the required result. □

The second bound is related to Carmichael and Mason's bound.

THEOREM 2. *For* $j = 1, 2, \ldots, n$, *we have*

$$\left|z_j\right| \leqslant \sqrt{1 + \sum_{j=1}^{n} \left|b_j\right|^2}. \tag{5}$$

Proof. Let

$$R = \begin{bmatrix} 0 & b_n & b_{n-1} & \cdots & b_2 & b_1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

Then $C_q = R + S$ and $R^*S = S^*R = 0$. So, by the triangle inequality, we have

$$\left\|C_q\right\|^2 = \left\|C_q^* C_q\right\| = \|R^*R + S^*S\| \leqslant \|R^*R\| + \|S^*S\| = \|RR^*\| + 1 = 1 + \sum_{j=1}^{n} \left|b_j\right|^2.$$

Since the spectral radius of any matrix is dominated by operator norm, the desired result follows. □

Our third bound is related to Abdurakhmanov's bound.

THEOREM 3. *For* $j = 1, 2, \ldots, n$, *we have*

$$\left|z_j\right| \leqslant \frac{1}{2}\left(\cos\frac{\pi}{n+1} + \sqrt{\cos^2\frac{\pi}{n+1} + \left(1+\sqrt{\alpha}\right)^2}\right), \tag{6}$$

where $\alpha = \sum_{i=1}^{n} |b_i|^2$.

Proof. Let $v = \begin{bmatrix} b_n \ b_{n-1} \ \cdots \ b_2 \ b_1 \end{bmatrix}$, $e_1 = \begin{bmatrix} 1 \ 0 \ 0 \ \cdots \ 0 \end{bmatrix}^t$, and $T_n = \begin{bmatrix} 0 & 0 \\ I_{n-1} & 0 \end{bmatrix}_{n\times n}$, where I_{n-1} is the identity matrix of order $n-1$. Then $C_q = \begin{bmatrix} 0 & v \\ e_1 & T_n \end{bmatrix}$. Applying inequality (3), we get

$$\begin{aligned} w(C_q) &= w\left(\begin{bmatrix} 0 & v \\ e_1 & T_n \end{bmatrix}\right) \\ &\leqslant \frac{1}{2}\left(w(T_n) + \sqrt{w^2(T_n) + (\|v\| + \|e_1\|)^2}\right) \\ &= \frac{1}{2}\left(\cos\frac{\pi}{n+1} + \sqrt{\cos^2\frac{\pi}{n+1} + \left(1+\sqrt{\alpha}\right)^2}\right). \end{aligned}$$

Now the desired result follows from inequality (1). $\square$

The fourth bound is related to Fujii-Kubo's bound.

THEOREM 4. *For* $j = 1,2,\ldots,n$, *we have*

$$|z_j| \leqslant \cos\frac{\pi}{n+2} + \frac{1}{2}\sqrt{\alpha}, \tag{7}$$

where $\alpha = \sum_{i=1}^{n} |b_i|^2$.

Proof. Let $v = \begin{bmatrix} b_n \ b_{n-1} \ \cdots \ b_2 \ b_1 \end{bmatrix}$, and $T_{n+1} = \begin{bmatrix} 0 & 0 \\ I_n & 0 \end{bmatrix}_{(n+1)\times(n+1)}$, where I_n is the identity matrix of order n. Then $C_q = \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix} + T_{n+1}$. Using the triangle inequality and Lemma 2, we get

$$\begin{aligned} w(C_q) &= w\left(\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix} + T_{n+1}\right) \\ &\leqslant w\left(\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\right) + w(T_{n+1}) \\ &= \cos\frac{\pi}{n+2} + \frac{1}{2}\sqrt{\alpha}. \end{aligned}$$

The inequality (7) follows directly by recalling that $|z_j| \leqslant w(C_q)$. $\square$

Our final bound is related to Kittaneh's bound.

THEOREM 5. *For* $j = 1,2,\ldots,n$, *we have*

$$|z_j| \leqslant \frac{1}{2}\left(\cos\frac{\pi}{n+1} + \sqrt{\cos^2\frac{\pi}{n+1} + (1+|b_n|)^2 + \alpha}\right), \tag{8}$$

where $\alpha = \sum_{i=1}^{n-1} |b_i|^2$.

Proof. Let $v = \left[b_n \ b_{n-1} \ \cdots \ b_2 \ b_1 \right]$, $e_1 = \left[1 \ 0 \ 0 \cdots 0 \right]^t$, $T_n = \begin{bmatrix} 0 & 0 \\ I_{n-1} & 0 \end{bmatrix}_{n\times n}$, $C_D = \begin{bmatrix} 0 & 0 \\ 0 & T_n \end{bmatrix}$ and $C_O = \begin{bmatrix} 0 & v \\ e_1 & 0 \end{bmatrix}$, where I_{n-1} is the identity matrix of order $n-1$. Then $C_q = C_D + C_O$. Using Lemma 1, we have

$$\begin{aligned} w(C_O) &= \max_{\theta\in\mathbb{R}} \left\| \operatorname{Re}\left(e^{i\theta}C_O\right) \right\| \\ &= \frac{1}{2}\max_{\theta\in\mathbb{R}} \left\| \begin{bmatrix} 0 & u \\ u^* & 0 \end{bmatrix} \right\| \\ &= \frac{1}{2}\max_{\theta\in\mathbb{R}} \|u\| \\ &= \frac{1}{2}\max_{\theta\in\mathbb{R}} \sqrt{|e^{i\theta}b_n - e^{-i\theta}|^2 + \alpha} \\ &= \frac{1}{2}\sqrt{(1+|b_n|)^2 + \alpha}, \end{aligned}$$

where $u = \left[-e^{i\theta}b_n + e^{-i\theta} \ -e^{i\theta}b_{n-1} \ -e^{i\theta}b_{n-2} \cdots -e^{i\theta}b_2 \ -e^{i\theta}b_1 \right]$. Applying inequality (2), we have

$$\begin{aligned} w(C_q) &= w(C_D + C_O) \\ &\leqslant \frac{1}{2}\left(\cos\frac{\pi}{n+1} + \sqrt{\cos^2\frac{\pi}{n+1} + (1+|b_n|)^2 + \alpha} \right). \end{aligned}$$

Now the desired result follows from inequality (1). □

If we apply inequality (3) instead of inequality (2) in the previous theorem, we get

$$|z_j| \leqslant \frac{1}{2}\left(\cos\frac{\pi}{n+1} + \sqrt{\cos^2\frac{\pi}{n+1} + (1+\alpha)^2} \right), \tag{9}$$

For $j = 1,2,\ldots,n$, where $\alpha = \sum_{i=1}^{n} |b_i|^2$. It should be mentioned here that bound (8) improves bound (9).

3. Conclusions

We conclude the paper with the following remarks concerning our results.

REMARK 1. Cauchy's bound, Carmichael and Mason's bound, Abdurakhmanov's bound, Kittaneh's bound, and Fujii-Kubo's bound are not uniformly better than our bounds. For example, for $p(z) = z^5 + z^4 + 2z^3 + 2z^2 + 2z + 2$. Cauchy's bound $= 3$, Carmichael and Mason's bound ≈ 4.24264, Abdurakhmanov's bound ≈ 3.40633, Kittaneh's bound ≈ 3.19779, and Fujii-Kubo's ≈ 3.42758, while the bound given in (4) ≈ 2, the bound given in (5) $= 2.44949$, the bound given in (6) ≈ 2.10799, the bound given in (7) ≈ 2.01900, and the bound given in (8) ≈ 1.91203. Therefore, our bounds lead to improve these bound by taking the minimum of these bounds.

REMARK 2. Our new bounds presented here locate the zeros of p inside discs. The zeros of p can be located inside annuli in those discs by applying these bounds to the polynomial $h(z) = \frac{z^n}{a_1} p\left(\frac{1}{z}\right)$. Indeed, the zeros of h are the reciprocals of those of p. Thus, every upper bound for the zeros of p yields a lower bound counterpart.

REMARK 3. Although the computations are quite involved, it is still possible to derive more bounds for the zeros of p by considering different partitions of C_q, C_q^2, and C_q^3. and estimate the numerical radii of C_q, C_q^2, and C_q^3 in these cases.

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