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ILLUMINATED CAPS OF THE UNIT SPHERES OF BANACH SPACES AND RELATED INEQUALITIES

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Abstract. Geometric properties of illuminated caps of the unit sphere of a finite dimensional Banach space play an important role in Hadwiger's covering problem for centrally symmetric convex bodies. In view of this, fundamental properties of illuminated caps are presented, incenters, inradii, and self-circumradii of illuminated caps are studied, and related inequalities are obtained.

1. Introduction

We denote by $X = (\mathbb{R}^n, \|\cdot\|)$ $(n \geqslant 2)$ a Minkowski space (i.e., a real finite dimensional Banach space) with *origin* o, *unit ball* B_X, and *unit sphere* S_X. Each point in S_X will be called a *unit vector* or a *direction*. The *interior* and *closure* of a subset A of X is denoted by $\operatorname{int}A$ and $\operatorname{cl}A$, respectively. For two unit vectors p and q satisfying $p \neq -q$, the set

$$\operatorname{arc}(p,q) = \{\alpha p + \beta q :\ \alpha, \beta \geqslant 0\} \cap S_X$$

is called the *minor arc connecting p and q*. Let X be a *Minkowski plane* (i.e., a real two-dimensional Banach space), $u \in S_X$, and H^+ and H^- be the two open halfplanes bounded by the line $\langle -u, u \rangle$. Then each of the sets $S_X \cap H^+$ and $S_X \cap H^-$ is called an *open semicircle*.

Let x be a unit vector and u be a direction. If there exists a positive number λ such that $x + \lambda u \in \operatorname{int}B_X$ then we say that the direction u *illuminates* x. For each direction u, we denote by $\mathrm{IC}(u)$ (called the *cap illuminated by* u) the set of points in S_X that are illuminated by u, and by $\mathrm{Sbd}(u)$ (called the *shadow boundary in the direction of* u) the set $S_X \setminus (\mathrm{IC}(u) \cup \mathrm{IC}(-u))$. I.e., $\mathrm{Sbd}(u)$ is the set of points in S_X that cannot be illuminated either by u or by $-u$. It is easy to see that $\mathrm{Sbd}(u)$ is precisely the set of unit vectors that are Birkhoff orthogonal to u (cf. [15, Proposition 2 and Proposition 3]), where a vector x is said to be *Birkhoff orthogonal to* another vector y (denoted by $x \perp_B y$) if

$$\|x + \lambda y\| \geqslant \|x\|, \quad \forall \lambda \in \mathbb{R}.$$

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See [1] for more information on this and other orthogonality types in normed linear spaces. Clearly, for each $u \in S_X$ we have (cf. [15, Proposition 2]) that

$$S_X = \mathrm{IC}(u) \cup \mathrm{IC}(-u) \cup \mathrm{Sbd}(u), \ -u \in \mathrm{IC}(u), \ u \in \mathrm{IC}(-u), \ \mathrm{IC}(u) = -\mathrm{IC}(-u),$$

and that both $\mathrm{IC}(u)$ and $\mathrm{IC}(-u)$ are connected and relatively open with respect to S_X. Moreover, it can be seen that the relative boundary of $\mathrm{IC}(-u)$ with respect to S_X is $\mathrm{cl}\,(\mathrm{IC}(-u)) \cap \mathrm{Sbd}(u)$.

Our work is partially motivated by Hadwiger's covering problem described below. A compact convex set in $\mathbb{R}^n$ having interior points is called a *convex body*. For each convex body K in $\mathbb{R}^n$, we denote by $c(K)$ the least number of translates of $\mathrm{int}\,K$ needed to cover K. Clearly, $c(K)$ is an affine invariant with respect to K. The famous *covering problem of Hadwiger* (cf. [8], [2], [5], [11], [3], and [4]) asks whether $c(K)$ is bounded from above by 2^n for each convex body K in $\mathbb{R}^n$. This problem is already very difficult when K is assumed to be symmetric with respect to o. The answer for this subcase of the problem is affirmative when $n = 3$ (cf. [10]), and this problem is open when $n \geqslant 4$. We note that, when K is symmetric with respect to o, it is the unit ball B_X for a normed space X, and that $c(K)$ equals to the least number of directions needed to illuminate S_X (cf. §34 in [5] or [11]). In other words, $c(K)$ equals to the least number of illuminated caps needed to cover S_X. This observation shows that it is of fundamental importance to study properties of illuminated caps.

In Section 2 we mainly focus on the fundamental geometric structure of illuminated caps. In Section 3 we study the inradius and the circumradius of illuminated caps and present some related inequalities. New characterizations of inner product spaces are obtained in both Section 2 and 3. Note that, although we mainly focus on the geometry of Minkowski spaces, many of our results are also valid for the infinite-dimensional cases.

2. Fundamental properties of illuminated caps

We start with two simple lemmas.

LEMMA 1. *Let X be a Minkowski space and $u \in S_X$. If $v \in \mathrm{IC}(-u)$, then there exists a number $\lambda_0 > 0$ such that*

$$\|v - \lambda u\| < 1, \ \forall \lambda \in (0, \lambda_0); \tag{1}$$

moreover,

$$\|v + \lambda u\| > 1, \ \forall \lambda > 0.$$

Proof. Since $v \in \mathrm{IC}(-u)$, there exists a positive number λ_0 such that $\|v - \lambda_0 u\| < 1$ or, equivalently, $v - \lambda_0 u \in \mathrm{int}\,B_X$. For each $\lambda \in (0, \lambda_0)$, we have

$$v - \lambda u = \left(1 - \frac{\lambda}{\lambda_0}\right) v + \frac{\lambda}{\lambda_0}(v - \lambda_0 u) \in \mathrm{int}\,B_X.$$

Thus (1) holds.

For each number $\lambda > 0$, we have

$$1 = \|v\| = \left\| \frac{\lambda}{\lambda_0 + \lambda}(v - \lambda_0 u) + \frac{\lambda_0}{\lambda_0 + \lambda}(v + \lambda u) \right\| < \frac{\lambda}{\lambda_0 + \lambda} + \frac{\lambda_0}{\lambda_0 + \lambda} \|v + \lambda u\|.$$

It follows that $\|v + \lambda u\| > 1$. $\square$

LEMMA 2. *Let X be a Minkowski space. For each unit vector u and each pair of points $x, y \in \mathrm{IC}(u)$, we have $\|x - y\| < 2$.*

Proof. By the definition of $\mathrm{IC}(u)$, there exists a positive number λ such that

$$x + \lambda u \in \operatorname{int} B_X \quad \text{and} \quad y + \lambda u \in \operatorname{int} B_X.$$

Thus

$$\|x - y\| = \|x + \lambda u - (y + \lambda u)\| < 2. \quad \square$$

Let X be a Minkowski space and $A \subseteq S_X$. If there exists a point $a_0 \in A$ such that

$$a \in A, \quad a \neq -a_0 \Rightarrow \operatorname{arc}(a_0, a) \subseteq A,$$

then A is called a *spherically star-shaped set* and a_0 is called a *spherical star center* of A. If the implication

$$u, v \in A, \quad u \neq -v \Rightarrow \operatorname{arc}(u, v) \subseteq A$$

holds, then we say that A is *spherically convex*.

PROPOSITION 1. *For each direction u, $\mathrm{IC}(-u)$ is a spherically star-shaped set having u as a spherical star center.*

Proof. We only need to show that $\operatorname{arc}(u, v) \subseteq \mathrm{IC}(-u)$ holds for each point $v \in \mathrm{IC}(-u)$. The case $u = v$ is trivial. Otherwise, Lemma 2 shows that $u \neq -v$. For each point $w \in \operatorname{arc}(u, v) \setminus \{u, v\}$, there exists a number $\lambda \in (0, 1)$ such that

$$w = \frac{\lambda u + (1 - \lambda)v}{\|\lambda u + (1 - \lambda)v\|}.$$

Then, by Lemma 1, we have

$$\left\| w - \frac{\lambda}{\|\lambda u + (1 - \lambda)v\|} u \right\| = \frac{(1 - \lambda)}{\|\lambda u + (1 - \lambda)v\|} = \frac{1}{\left\| \frac{\lambda}{1 - \lambda} u + v \right\|} < 1,$$

which shows that $w \in \mathrm{IC}(-u)$. $\square$

PROPOSITION 2. *Let u be a direction, p and q be two points in $\mathrm{IC}(-u)$. Then, for each $\lambda \in [0, 1]$,*

$$[\lambda p + (1 - \lambda)q, \lambda p + (1 - \lambda)q + u\rangle \cap S_X \subseteq \mathrm{IC}(-u).$$

Proof. Since $p,q \in \mathrm{IC}(-u)$, there exists a positive number γ such that $p-\gamma u$, $q-\gamma u \in \mathrm{int} B_X$. Therefore

$$\lambda p+(1-\lambda)q-\gamma u=\lambda(p-\gamma u)+(1-\lambda)(q-\gamma u)\in \mathrm{int} B_X, \tag{2}$$

which shows that $\langle \lambda p+(1-\lambda)q, \lambda p+(1-\lambda)q+u\rangle \cap \mathrm{int} B_X \neq \emptyset$. Therefore the set $[\lambda p+(1-\lambda)q, \lambda p+(1-\lambda)q+u\rangle \cap S_X$ contains precisely one point, namely r. Clearly, there exists a number $\alpha \geqslant 0$ such that $r=\lambda p+(1-\lambda)q+\alpha u$. From (2) it follows that

$$r-(\gamma+\alpha)u=\lambda p+(1-\lambda)q-\gamma u \in \mathrm{int} B_X.$$

Thus $r \in \mathrm{IC}(-u)$. □

We continue with a simple characterization of the shape of illuminated caps of the unit circle of a Minkowski plane.

PROPOSITION 3. *If X is a Minkowski plane then, for each direction u, the set* $\mathrm{IC}(-u)$ *(containing u) is either an open semicircle or a minor arc* $\mathrm{arc}(p,q)$ *without endpoints for two points $p,q \in$* $\mathrm{Sbd}(u)$. *Moreover, if X is strictly convex, then each illuminated cap of S_X is an open semicircle.*

Proof. By Proposition 1, $u \in \mathrm{IC}(-u)$. Let v be an arbitrary point in $\mathrm{Sbd}(u)$,

$$\alpha=\sup\left\{\lambda \in [0,1]: \frac{\lambda v+(1-\lambda)u}{\|\lambda v+(1-\lambda)u\|} \in \mathrm{IC}(-u)\right\},$$

and

$$\beta=\sup\left\{\lambda \in [0,1]: \frac{-\lambda v+(1-\lambda)u}{\|-\lambda v+(1-\lambda)u\|} \in \mathrm{IC}(-u)\right\}.$$

Put

$$p=\frac{\alpha v+(1-\alpha)u}{\|\alpha v+(1-\alpha)u\|} \quad \text{and} \quad q=\frac{-\beta v+(1-\beta)u}{\|-\beta v+(1-\beta)u\|}.$$

The definitions of α and β and Proposition 1 show that

$$(\mathrm{arc}(u,p)\setminus\{p\})\cup(\mathrm{arc}(u,q)\setminus\{q\})\subseteq \mathrm{IC}(-u).$$

Since both $\mathrm{IC}(-u)$ and $\mathrm{IC}(u)$ are relatively open, $p,q \notin \mathrm{IC}(-u)\cup \mathrm{IC}(u)$, which implies that $p,q \in \mathrm{Sbd}(u)$. Therefore $\alpha\cdot\beta \neq 0$.

If S_X does not contain non-trivial segments parallel to the line $\langle -u,u\rangle$, then $\mathrm{IC}(-u)$ is $\mathrm{arc}(v,u)\cup \mathrm{arc}(-v,u)\setminus\{v,-v\}$, which is an open semicircle determined by the line $\langle -v,v\rangle$.

Now suppose that S_X contains non-trivial segments parallel to $\langle -u,u\rangle$. We claim that $\alpha+\beta<2$. Otherwise, $\mathrm{IC}(-u)$ and $\mathrm{IC}(u)$ are the two open semicircles determined by $\langle -v,v\rangle$. Thus $\mathrm{Sbd}(u)=\{-v,v\}$, a contradiction. Since, as we claimed, $\alpha+\beta<2$, $u \in \mathrm{arc}(p,q)$. Thus $\mathrm{IC}(-u)=\mathrm{arc}(p,q)\setminus\{p,q\}$.

Moreover, if X is strictly convex, then, for each $u \in S_X$, S_X does not contain non-trivial segments parallel to $\langle -u,u\rangle$, which implies that $\mathrm{IC}(-u)$ is an open semicircle. □

Proposition 3 shows that $\mathrm{IC}(-u)$ is spherically convex for each $u \in S_X$ if the underlying space is two-dimensional, while it is not spherically convex in general. See the following example.

EXAMPLE 1. Let $X = (\mathbb{R}^3, \|\cdot\|_\infty)$, $u = (1,1,1)$, $p = (-\frac{1}{2}, -\frac{1}{2}, 1)$, $q = (1, -\frac{1}{2}, -\frac{1}{2})$, $v = (0,-1,1)$, and $w = (1,-1,0)$. Then $p, q \in \mathrm{IC}(-u)$ and $v, w \in \mathrm{arc}(p,q) \cap \mathrm{Sbd}(u)$. Thus $\mathrm{IC}(-u)$ is not spherically convex.

In the following we show that, if the underlying space has dimension at least three and is strictly convex, then each illuminated cap is spherically convex if and only if the underlying space is an inner product space.

LEMMA 3. *Let X be an inner product space whose dimension is at least two. Then for each $u \in S_X$, $v \in \mathrm{IC}(-u)$ if and only if the inner product $(u|v)$ of u and v is positive.*

Proof. Suppose that $v \in \mathrm{IC}(-u)$. Then there exists a positive number λ such that $\|v - \lambda u\| < 1$. Thus

$$(v - \lambda u | v - \lambda u) = \|v\|^2 - 2\lambda (u|v) + \lambda^2 \|u\|^2 = 1 + \lambda^2 - 2\lambda (u|v) < 1,$$

which implies that $(u|v) > 0$.

Conversely, assume that $v \in S_X$ is a point satisfying $(u|v) > 0$. Then for each $\lambda \in (0, (u|v))$ we have

$$\|v - \lambda u\|^2 = (v - \lambda u | v - \lambda u) = 1 + \lambda^2 - 2\lambda (u|v) = 1 + \lambda(\lambda - 2(u|v)) < 1.$$

Thus $v \in \mathrm{IC}(-u)$. $\square$

THEOREM 4. *If X is a strictly convex normed linear space whose dimension is at least three, then each illuminated cap of S_X is spherically convex if and only if X is an inner product space.*

Proof. First suppose that X is an inner product space. Let u be an arbitrary point in S_X, $p, q \in \mathrm{IC}(-u)$, and $v \in \mathrm{arc}(p,q)$. Then there exist two numbers $\alpha, \beta \geqslant 0$ such that $\alpha + \beta > 0$ and $v = \alpha p + \beta q$. Thus, by Lemma 3,

$$(v|u) = (\alpha p + \beta q | u) = \alpha (p|u) + \beta (q|u) > 0,$$

which shows that $v \in \mathrm{IC}(-u)$. Hence $\mathrm{arc}(p,q) \subseteq \mathrm{IC}(-u)$.

Now suppose that each illuminated cap of S_X is spherically convex. To show that X is an inner product space we only need to prove that Birkhoff orthogonality is additive on the left (see, e.g., Theorem 4.18 in [1]). Since Birkhoff orthogonality is homogeneous (see Theorem 4.5 in [1]), it suffices to show that

$$x, y \in X, \quad u \in S_X, \quad x \perp_B u, \quad y \perp_B u \Rightarrow x + y \perp_B u.$$

Obviously, we only need to consider the case when $\|x\| \cdot \|y\| > 0$. Put

$$s = \frac{x}{\|x\|} \quad \text{and} \quad t = \frac{y}{\|y\|}.$$

Since X is strictly convex, $s,t \in \mathrm{cl}(\mathrm{IC}(u)) \cap \mathrm{cl}(\mathrm{IC}(-u))$. For each $n \in \mathbb{N}$, put

$$p_n = \frac{\frac{1}{n}u + \left(1-\frac{1}{n}\right)s}{\left\|\frac{1}{n}u + \left(1-\frac{1}{n}\right)s\right\|}, \quad q_n = \frac{\frac{1}{n}u + \left(1-\frac{1}{n}\right)t}{\left\|\frac{1}{n}u + \left(1-\frac{1}{n}\right)t\right\|},$$

$$p_n' = \frac{-\frac{1}{n}u + \left(1-\frac{1}{n}\right)s}{\left\|-\frac{1}{n}u + \left(1-\frac{1}{n}\right)s\right\|}, \quad \text{and} \quad q_n' = \frac{-\frac{1}{n}u + \left(1-\frac{1}{n}\right)t}{\left\|-\frac{1}{n}u + \left(1-\frac{1}{n}\right)t\right\|}.$$

Then

$$p_n, q_n \in \mathrm{IC}(-u) \quad \text{and} \quad p_n', q_n' \in \mathrm{IC}(u),\ \forall n \in \mathbb{N}.$$

Put

$$v_n = \frac{\|x\|\, p_n + \|y\|\, q_n}{\|\|x\|\, p_n + \|y\|\, q_n\|} \quad \text{and} \quad v_n' = \frac{\|x\|\, p_n' + \|y\|\, q_n'}{\|\|x\|\, p_n' + \|y\|\, q_n'\|},\ \forall n \in \mathbb{N}.$$

The spherical convexity of illuminated caps shows that

$$v_n \in \mathrm{IC}(-u) \quad \text{and} \quad v_n' \in \mathrm{IC}(u),\ \forall n \in \mathbb{N}.$$

Clearly, we have

$$\lim_{n\to\infty} v_n = \frac{\|x\|\, s + \|y\|\, t}{\|\|x\|\, s + \|y\|\, t\|} = \frac{x+y}{\|x+y\|}$$

and

$$\lim_{n\to\infty} v_n' = \frac{\|x\|\, s + \|y\|\, t}{\|\|x\|\, s + \|y\|\, t\|} = \frac{x+y}{\|x+y\|}.$$

It follows that $\frac{x+y}{\|x+y\|}$ is in the intersection of the closures of $\mathrm{IC}(-u)$ and $\mathrm{IC}(u)$, which shows that $\frac{x+y}{\|x+y\|} \perp_B u$. Thus $x+y \perp_B u$. Hence Birkhoff orthogonality on X is additive on the left, which shows that X is an inner product space. □

REMARK 1. We are not sure whether the strict convexity in the hypothesis of Theorem 4 can be removed.

COROLLARY 1. *If X is a strictly convex normed linear space whose dimension is at least three, then X is an inner product space if and only if for each $u \in S_X$ there exists a supporting functional $u^* \in S_{X^*}$ of u such that*

$$\mathrm{IC}(-u) = \{v \in S_X :\ u^*(v) > 0\}. \tag{3}$$

Proof. If X is an inner product space, then $u^* = (u|\cdot)$ is a supporting functional of u. Lemma 3 shows that

$$\mathrm{IC}(-u) = \{v \in S_X :\ (u|v) = u^*(v) > 0\}.$$

Conversely, suppose that for each $u \in S_X$ there exists a supporting functional $u^* \in S_{X^*}$ of u satisfying (3). Let p and q be two arbitrary points in $\mathrm{IC}(-u)$, and w be an arbitrary point in $\mathrm{arc}(p,q)$. Then there exist two numbers $\alpha, \beta \geqslant 0$ such that $\alpha + \beta > 0$ and $w = \alpha p + \beta q$. Therefore, $u^*(w) = u^*(\alpha p + \beta q) > 0$. It follows that $\mathrm{arc}(p,q) \subseteq \mathrm{IC}(-u)$. By Theorem 4, X is an inner product space. $\square$

We continue with the following characterization of illuminated caps.

PROPOSITION 5. *Let $X = (\mathbb{R}^n, \|\cdot\|)$. Then for each point $u \in S_X$,*

$$\mathrm{IC}(-u) = \cup\{\mathrm{relint}\, F : \ F \text{ is a face of } B_X, \mathrm{relint}\, F \cap \mathrm{IC}(-u) \neq \emptyset\},$$

where $\mathrm{relint}\, F$ *denotes the relative interior of* F.

Proof. Put

$$B = \cup\{\mathrm{relint}\, F : \ F \text{ is a face of } B_X, \mathrm{relint}\, F \cap \mathrm{IC}(-u) \neq \emptyset\}.$$

Suppose that F is a face of B_X and there exists a point $a \in \mathrm{relint}\, F \cap \mathrm{IC}(-u)$. Let c be an arbitrary point in $\mathrm{relint}\, F \setminus \{a\}$. Then there exists a point $b \in F$ and a number $\gamma \in (0,1)$ such that $c = \gamma a + (1-\gamma) b$. Since $a \in \mathrm{IC}(-u)$, there exists a number $\lambda \in (0,1)$ such that $a - \lambda u \in \mathrm{int}\, B_X$. It follows that

$$c - \gamma \cdot \lambda u = \gamma a + (1-\gamma) b - \gamma \lambda u = \gamma (a - \lambda u) + (1-\gamma) b \in \mathrm{int}\, B_X.$$

Thus $\mathrm{relint}\, F \subseteq \mathrm{IC}(-u)$. Therefore $B \subseteq \mathrm{IC}(-u)$.

Now suppose that a is an arbitrary point in $\mathrm{IC}(-u)$. Let F be the intersection of all faces of B_X containing a. Then $a \in \mathrm{relint}\, F \subseteq B$ (cf. Theorem 2.6.10 in [14]). Thus $\mathrm{IC}(-u) \subseteq B$. $\square$

In the rest of this section we discuss the so-called maximality of illuminated caps.

DEFINITION 1. An illuminated cap of B_X is said to be *maximal* if it is not a proper subset of another illuminated cap of B_X.

The following proposition is clear but useful.

PROPOSITION 6. *Let $X = (\mathbb{R}^n, \|\cdot\|)$ be a Minkowski space. Then $c(B_X)$ equals to the least number of maximal illuminated caps needed to cover S_X.*

EXAMPLE 2. In $X = (\mathbb{R}^2, \|\cdot\|_\infty)$, $\mathrm{IC}(-(0,1))$ is not a maximal illuminated cap, and it is contained in $\mathrm{IC}(-(1,1))$ and in $\mathrm{IC}(-(-1,1))$.

DEFINITION 2. Let X be a normed linear space. For each point $x \in S_X$, $S(B_X, x)$, which is the set such that $S(B_X, x) + x$ is the intersections of all supporting halfspaces of B_X at x, is called the *support cone of B_X at x*.

LEMMA 4. *Let X be a normed linear space and $u \in S_X$. Then $v \in \mathrm{IC}(-u)$ if and only if $-u \in \operatorname{int} S(B_X, v)$.*

Proof. First suppose that $-u \in \operatorname{int} S(B_X, v)$. We show that there exists a positive number λ such that $\|v - \lambda u\| < 1$. Otherwise,

$$\|v - \lambda u\| \geqslant 1, \ \forall \lambda \geqslant 0.$$

For each norm one functional v^* satisfying $v^*(v) = 1$, and for each number $\lambda < 0$ it follows from $-u \in \operatorname{int} S(B_X, v)$ that

$$\|v - \lambda u\| \geqslant v^*(v - \lambda u) = 1 - \lambda v^*(u) > 1.$$

Thus we have

$$\inf_{\lambda \in \mathbb{R}} \|v - \lambda u\| = 1.$$

In other words, $v \perp_B u$. Therefore there exists a norm one functional w^* such that $w^*(v) = 1$ and $w^*(u) = 0$ (cf. Corollary 4.2 in [1]), which is in contradiction to the choice of u.

Now suppose that $v \in \mathrm{IC}(-u)$. Then there exists a positive number λ such that $\|v - \lambda u\| = \gamma < 1$. For each norm one functional v^* satisfying $v^*(v) = 1$, we have

$$\gamma = \|v - \lambda u\| \geqslant v^*(v - \lambda u) = 1 + \lambda v^*(-u).$$

Therefore,

$$v^*(-u) \leqslant \frac{\gamma - 1}{\lambda} < 0.$$

This implies that $-u \in \operatorname{int} S(B_X, v)$. □

PROPOSITION 7. *Let X be a normed linear space and $u \in S_X$. Then $\mathrm{IC}(-u)$ is maximal if and only if*

$$\forall v \in \operatorname{cl} \mathrm{IC}(-u) \setminus \mathrm{IC}(-u), \operatorname{int} S(B_X, v) \cap \left(\bigcap_{x \in \mathrm{IC}(-u)} \operatorname{int} S(B_X, x) \right) = \emptyset. \tag{4}$$

Proof. Put

$$U = \bigcap_{x \in \mathrm{IC}(-u)} \operatorname{int} S(B_X, x). \tag{5}$$

Lemma 4 shows that $U \cap S_X$ is the set of directions that can illuminate each point in $\mathrm{IC}(-u)$.

First suppose that $\mathrm{IC}(-u)$ is maximal. If there exists a point $v \in \operatorname{cl} \mathrm{IC}(-u) \setminus \mathrm{IC}(-u)$ such that there exists a direction $u' \in \operatorname{int} S(B_X, v) \cap U$, then $\mathrm{IC}(-u) \cup \{v\} \subseteq \mathrm{IC}(u')$, which is in contradiction to the fact that $\mathrm{IC}(-u)$ is maximal.

Conversely, suppose that (4) holds. If $\mathrm{IC}(-u)$ is not maximal, then there exist a direction u' and a point $w \in S_X \setminus \mathrm{IC}(-u)$ such that $\mathrm{IC}(-u) \cup \{w\} \subseteq \mathrm{IC}(u')$. In this case we have $u' \in U$. If u and w are linearly dependent then $w = -u$. It follows that there exists a positive number λ such that $-u + \lambda u' \in \operatorname{int} B_X$ and $u + \lambda u' \in \operatorname{int} B_X$. Thus

$$u \in \left[u - \lambda u', u + \lambda u'\right] \subseteq \operatorname{int} B_X,$$

which is impossible. In the following we distinguish two cases.

Case I: $-u' \notin \mathrm{IC}(-u)$. Set

$$\alpha = \sup\left\{\lambda \in [0,1] : \frac{-\lambda u' + (1-\lambda)u}{\|-\lambda u' + (1-\lambda)u\|} \in \mathrm{IC}(-u)\right\} \quad \text{and} \quad v = \frac{-\alpha u' + (1-\alpha)u}{\|-\alpha u' + (1-\alpha)u\|}.$$

It follows that v is a point of intersection of $\operatorname{arc}(-u', u)$ and the relative boundary of $\mathrm{IC}(-u)$. Since $\operatorname{arc}(-u', u) \subseteq \mathrm{IC}(u')$, $v \in (\operatorname{cl} \mathrm{IC}(-u)) \cap \mathrm{IC}(u')$.

Case II: $-u' \in \mathrm{IC}(-u)$. Let

$$\beta = \sup\left\{\lambda \in [0,1] : \frac{\lambda w - (1-\lambda)u'}{\|\lambda w - (1-\lambda)u'\|} \in \mathrm{IC}(-u)\right\} \quad \text{and} \quad v = \frac{\beta w - (1-\beta)u'}{\|\beta w - (1-\beta)u'\|}.$$

In a similar way one can show that $v \in (\operatorname{cl} \mathrm{IC}(-u)) \cap \mathrm{IC}(u')$.

In each of these two cases, by Lemma 4 we have $u' \in \operatorname{int} S(B_X, v) \cap U$, a contradiction to (4). □

COROLLARY 2. *If X is a smooth normed linear space, then each illuminated cap of S_X is maximal.*

Proof. Suppose the contrary, namely that there exists a point $u \in S_X$ such that $\mathrm{IC}(-u)$ is not maximal. Then there exists a point $v \in \operatorname{cl} \mathrm{IC}(-u) \setminus \mathrm{IC}(-u)$ such that $\operatorname{int} S(B_X, v) \cap U$ contains a point w, where U is given by (5). Let v' be the point of intersection of $\operatorname{arc}(u, -v)$ and the relative boundary of $\mathrm{IC}(-u)$. Let v^* be the unique norm one functional such that $v^*(v) = 1$. Then $-v^*$ is the unique norm one functional such that $-v^*(-v) = 1$. Since $v \in \operatorname{cl} \mathrm{IC}(-u) \setminus \mathrm{IC}(-u)$, $v \perp_B u$. Therefore $v^*(u) = 0$. Since there exists a number $\lambda \geqslant 0$ such that $-v + \lambda u = v'$, we have $-v^*(v') = -v^*(-v + \lambda u) = 1$. Thus $-v^*$ is the unique norm one functional such that $-v^*(v') = 1$. For each $n \in \mathbb{N}$, put

$$s_n = \frac{\frac{1}{n}u + \left(1 - \frac{1}{n}\right)v}{\left\|\frac{1}{n}u + \left(1 - \frac{1}{n}\right)v\right\|}, \quad t_n = \frac{\frac{1}{n}u + \left(1 - \frac{1}{n}\right)v'}{\left\|\frac{1}{n}u + \left(1 - \frac{1}{n}\right)v'\right\|},$$

s_n^* and t_n^* be the unique supporting functional of s_n and t_n, respectively. Since

$$\lim_{n\to\infty} s_n = v \quad \text{and} \quad \lim_{n\to\infty} t_n = v',$$

the smoothness of X shows that $s_n^* \xrightarrow{w*} v^*$ and $t_n^* \xrightarrow{w*} -v^*$ (cf. Corollary 5.4.29 on p. 491 in [13]). Hence

$$v^*(w) = \lim_{n\to\infty} s_n^*(w) \leqslant 0,$$

and

$$-v^*(w) = \lim_{n\to\infty} t_n^*(w) \leqslant 0.$$

Thus $v^*(w) = 0$, a contradiction. □

3. Inradius, self-circumradius, and related inequalities

Suppose that $A,B \subseteq X$, $x \in X$. Put

$$\gamma(A,x) = \sup\{\|x-y\|:\ y \in A\},$$

$$\gamma(A,B) = \inf\{\gamma(A,x):\ x \in B\},$$

and

$$\gamma(A) = \gamma(A,X).$$

If $A \subseteq S_X$ and $x \in A$, we set

$$\gamma'(A,x) = \sup\{\gamma \geqslant 0:\ (x+\gamma B_X)\cap S_X \subseteq A\}$$

and

$$\gamma'(A) = \sup\left\{\gamma'(A,x):\ x \in A\right\}.$$

For each unit vector u, the number $\gamma'(\mathrm{IC}(-u))$ is called the *inradius* of $\mathrm{IC}(-u)$; a point $v \in \mathrm{IC}(-u)$ satisfying $\gamma'(\mathrm{IC}(-u),v) = \gamma'(\mathrm{IC}(-u))$ is called an *incenter* of $\mathrm{IC}(-u)$.

In [15], it is proved that

THEOREM 8. ([15]) *For each $u \in S_X$ we have*

1. $\gamma'(\mathrm{IC}(-u),u) = \inf\{\|u-z\|:\ z \in \mathrm{Sbd}(u)\}$;
2. $1 \leqslant \gamma'(\mathrm{IC}(-u),u) \leqslant 2$, *the equality on the left holds if and only if there exists a unit vector z such that $[z-u,z] \subseteq S_X$, and the equality on the right holds if and only if the unit circle of each two-dimensional subspace L of X that contains u is a parallelogram having u as one of its vertices.*

In a similar way we can show the following:

PROPOSITION 9. *For each $u \in S_X$ and each $v \in \mathrm{IC}(-u)$,*

$$\gamma'(\mathrm{IC}(-u),v) = \inf\{\|v-x\|:\ x \in \mathrm{Sbd}(u)\}.$$

Proof. Put

$$\delta = \inf\{\|v-x\|:\ x \in \mathrm{Sbd}(u)\}.$$

Since $\mathrm{Sbd}(u)$ is a compact set, there exists a point $w \in \mathrm{Sbd}(u)$ such that $\|v-w\| = \delta$. Clearly,

$$w \in (v+\|v-w\|B_X)\cap S_X \not\subseteq \mathrm{IC}(-u).$$

Thus $\gamma'(\mathrm{IC}(-u),v) \leqslant \delta$.

Let ε be an arbitrary number in $(0,\delta)$ and z be an arbitrary point in $(v+(\delta-\varepsilon)B_X)\cap S_X$. Clearly, $z \notin \mathrm{Sbd}(u)$. Suppose that $z \in \mathrm{IC}(u)$. Then $\mathrm{arc}(v,z)$ contains a point $x \in \mathrm{Sbd}(u)$, otherwise $\mathrm{IC}(-u)\cup\mathrm{IC}(u)$ would be path-connected, which is impossible. Thus we have $(v+(\delta-\varepsilon)B_X)\cap S_X \subseteq \mathrm{IC}(-u)$, and therefore $\gamma'(\mathrm{IC}(-u),v) \geqslant \delta-\varepsilon$. It follows that $\gamma'(\mathrm{IC}(-u),v) \geqslant \delta$. $\square$

THEOREM 10. *For each unit vector u, $\mathrm{IC}(-u)$ always has an incenter.*

Proof. By the definition of $\gamma'(\mathrm{IC}(-u))$, for each $k \in \mathbb{N}$ there exists a point $u_k \in \mathrm{IC}(-u)$ such that

$$\gamma'(\mathrm{IC}(-u), u_k) > \gamma'(\mathrm{IC}(-u)) - \frac{1}{k}.$$

Since S_X is a compact set, by choosing a subsequence if necessary, we may assume that $\{u_k\}_{k=1}^{\infty}$ converges to a point $u_0 \in S_X$.

For each number $\varepsilon \in (0, \gamma'(\mathrm{IC}(-u)))$, there exists an $N_0 \in \mathbb{N}$ such that

$$\frac{1}{k} < \frac{1}{2}\varepsilon \quad \text{and} \quad u_0 - u_k \in \frac{1}{2}\left(\varepsilon - \frac{1}{k}\right) B_X$$

hold for each $k \geqslant N_0$. Thus, for each $k \geqslant N_0$, we have

$$\begin{aligned}
&(u_0 + (\gamma'(\mathrm{IC}(-u)) - \varepsilon)B_X) \cap S_X \\
=&(u_0 - u_k + u_k + (\gamma'(\mathrm{IC}(-u)) - \varepsilon)B_X) \cap S_X \\
\subseteq&\left(u_0 - u_k + u_k + \left(\gamma'(\mathrm{IC}(-u), u_k) + \frac{1}{k} - \varepsilon\right) B_X\right) \cap S_X \\
\subseteq&\left(u_k + \left(\gamma'(\mathrm{IC}(-u), u_k) + \frac{1}{2}\left(\frac{1}{k} - \varepsilon\right)\right) B_X\right) \cap S_X \\
\subseteq&\mathrm{IC}(-u).
\end{aligned}$$

Since ε is arbitrary, $\gamma'(\mathrm{IC}(-u), u_0) = \gamma'(\mathrm{IC}(-u))$. $\square$

It is possible that a unit vector v is not an incenter of $\mathrm{IC}(-u)$ for each unit vector u. See the following example.

EXAMPLE 3. Let $X = (\mathbb{R}^2, \|\cdot\|_\infty)$, $v = (1, 1/2)$, and $w = (1,1)$. Then for each unit vector u, $\gamma'(\mathrm{IC}(-u)) \in \{1,2\}$ and v is not an incenter of $\mathrm{IC}(-u)$. It is also interesting to observe that w is the incenter of $\mathrm{IC}(-u)$ for each u in the set

$$([(0,1),(1,1)] \cup [(1,1),(1,0)]) \setminus \{(0,1),(1,0)\}.$$

It is interesting to observe that, the inradius of each maximally illuminated cap of B_X are 2, while there exist illuminated caps whose inradius is 1.

PROPOSITION 11. *Let X be a Minkowski plane and $u \in S_X$. Then $\gamma'(\mathrm{IC}(-u)) = 1$ if and only if $\gamma'(\mathrm{IC}(-u), u) = 1$.*

Proof. If $\gamma'(\mathrm{IC}(-u)) = 1$, then

$$1 \leqslant \gamma'(\mathrm{IC}(-u), u) \leqslant \gamma'(\mathrm{IC}(-u)) = 1,$$

which implies that $\gamma'(\mathrm{IC}(-u), u) = 1$.

Conversely, suppose that $\gamma'(\mathrm{IC}(-u),u)=1$. By Theorem 8, there exists a unit vector z such that $[z-u,z]\subseteq S_X$. It follows that

$$[z,z-u]\cup[-z,u-z]\subseteq \mathrm{Sbd}(u).$$

Therefore

$$\mathrm{IC}(-u)\subseteq \mathrm{arc}(z,u-z).$$

Let v be an arbitrary point in $\mathrm{IC}(-u)$. If $v\in \mathrm{arc}(u,z)$, then, by the Monotonocity Lemma [12, Proposition 31], we have

$$\|v-z\|\leqslant\|u-z\|=1.$$

If $v\in \mathrm{arc}(u-z,u)$, then

$$\|v-(u-z)\|\leqslant\|u-(u-z)\|=1.$$

It follows that $\gamma'(\mathrm{IC}(-u))=1$. □

Now we consider the case when $\gamma'(\mathrm{IC}(-u))=2$. Obviously, $\gamma'(\mathrm{IC}(-u),u)=2$ implies that $\gamma'(\mathrm{IC}(-u))=2$. Example 4 below shows that the reverse implication is not true in general.

THEOREM 12. *Let u be a unit vector. If $\gamma'(\mathrm{IC}(-u))=2$, then there exists a point $v\in\mathrm{IC}(-u)$ satisfying $\|u-v\|<1$ such that the unit circle of each two-dimensional subspace L of X which contains v is a parallelogram having v as one of its vertices.*

Proof. We only need to consider the case when $\gamma'(\mathrm{IC}(-u),u)<2$ since the case when $\gamma'(\mathrm{IC}(-u),u)=2$ follows from Theorem 8. By Theorem 10, there exists a point $v\in\mathrm{IC}(-u)\setminus\{u\}$ such that $\gamma'(\mathrm{IC}(-u),v)=2$. Let C be the relative boundary of $\mathrm{IC}(-u)$ with respect to S_X. Then $\|v-x\|=2$ holds for each point $x\in C$. Let L be an arbitrary two-dimensional subspace of X containing v, and p and q be the two points in $L\cap C$. Then we have $\|p-v\|=\|q-v\|=2$. Clearly, $-p\in\mathrm{Sbd}(u)\cap L$. If $q=-p$, then $\|p-v\|=\|p+v\|=2$ which implies that S_L is a parallelogram having $\pm v$ and $\pm p$ as vertices. If $q\neq -p$, then $q\in\mathrm{arc}(-p,v)$. By the Monotonocity Lemma [12, Proposition 31] we have

$$2\geqslant\|-p-v\|\geqslant\|q-v\|=2.$$

Thus $\|p-v\|=\|p+v\|=2$. Again this implies that S_L is a parallelogram having $\pm v$ and $\pm p$ as vertices. It follows that $q=-p$, a contradiction.

Consider now the two-dimensional subspace L_0 spanned by u and v. Previous arguments show that S_{L_0} is a parallelogram having v as one vertex. Let $\pm w$ be the two vertices of S_{L_0} adjacent to v. Then there exist two numbers α and β satisfying $|\alpha|+|\beta|=1$ such that $u=\alpha v+\beta w$. Suppose that $\|v-u\|\geqslant 1$. Let $\lambda>0$ be an arbitrary number satisfying $1-\lambda\alpha>0$. Then

$$\|v-\lambda u\|=\|(1-\lambda\alpha)v-\lambda\beta w\|=1-\lambda\alpha+\lambda|\beta|=1+\lambda(\|v-u\|-1)\geqslant 1.$$

This shows that $v\notin\mathrm{IC}(-u)$, a contradiction. Thus $\|v-u\|<1$. □

THEOREM 13. *For each Minkowski space X and each $u \in S_X$, we always have*

$$0 \leqslant \gamma'(\mathrm{IC}(-u)) - \gamma'(\mathrm{IC}(-u), u) < 1.$$

Proof. We only need to show the inequality on the right. Otherwise we have $\gamma'(\mathrm{IC}(-u) = 2$ and $\gamma'(\mathrm{IC}(-u), u) = 1$. By Theorem 12, there exists a point $v \in \mathrm{IC}(-u)$ such that the unit circle of each two-dimensional subspace of X is a parallelogram containing v as one vertex and that $\|u-v\| < 1$. Moreover, $\gamma'(\mathrm{IC}(-u), v) = 2$. Therefore, for each point $p \in (\mathrm{int}\,(u + (2 - \|u-v\|)B_X)) \cap S_X$ we have

$$\|v-p\| = \|v-u+u-p\| \leqslant \|u-v\| + \|u-p\| < \|u-v\| + 2 - \|u-v\| = 2,$$

which shows that $p \in \mathrm{IC}(-u)$. Thus

$$\gamma'(\mathrm{IC}(-u), u) \geqslant 2 - \|u-v\| > 1,$$

which is in contradiction to the fact that $\gamma'(\mathrm{IC}(-u), u) = 1$. $\square$

The difference $\gamma'(\mathrm{IC}(-u)) - \gamma'(\mathrm{IC}(-u), u)$ can be arbitrarily close to 1, see the following example.

EXAMPLE 4. Let $X = (\mathbb{R}^2, \|\cdot\|_\infty)$. For each $n \in \mathbb{N}$, put $u_n = (\frac{1}{n}, 1)$. Then

$$\mathrm{IC}(-u_n) = ([(-1,1),(1,1)] \cup [(1,1),(1,-1)]) \setminus \{(-1,1),(1,-1)\}.$$

Clearly,

$$\gamma'(\mathrm{IC}(-u_n)) = 2 \quad \text{and} \quad \gamma'(\mathrm{IC}(-u_n), u_n) = 1 + \frac{1}{n}.$$

It follows that

$$\gamma'(\mathrm{IC}(-u_n)) - \gamma'(\mathrm{IC}(-u_n), u_n) = 1 - \frac{1}{n}.$$

In the rest of this section we study $\gamma(\mathrm{IC}(-u), u)$, which is called the *self-circumradius of* $\mathrm{IC}(-u)$.

PROPOSITION 14. *For each $u \in S_X$,*

$$\gamma(\mathrm{IC}(-u), u) \geqslant 1,$$

and equality holds if and only if in the unit circle of each two-dimensional subspace L of X containing u a segment is contained whose length is not less than 1 and which is parallel to the line $\langle -u, u \rangle$.

Proof. Clearly, for each $v \in \mathrm{cl}(\mathrm{IC}(-u)) \setminus \mathrm{IC}(-u)$ we have $v \perp_B u$. Thus

$$\|u-v\| \geqslant 1,$$

which shows that

$$\gamma(\mathrm{IC}(-u), u) \geqslant 1.$$

Suppose that $\gamma(\mathrm{IC}(-u),u)=1$. Let L be a two-dimensional subspace containing u, and $v\in(\mathrm{cl}(\mathrm{IC}(-u))\setminus\mathrm{IC}(-u))\cap S_L$. Thus $v\perp_B u$. Therefore,

$$1\leqslant\|v-u\|\leqslant\gamma(\mathrm{IC}(-u),u)=1.$$

It follows that

$$v-u\in S_L.$$

For each $\lambda\in[0,1]$, we have

$$1\geqslant\|\lambda(v-u)+(1-\lambda)v\|=\|v-\lambda u\|\geqslant 1,$$

which implies that

$$[v-u,v]\subseteq S_L.$$

Moreover, the length of the segment $[v-u,v]$ is 1.

Conversely, suppose that the unit circle of each two-dimensional subspace L containing u contains a segment parallel to the line $\langle -u,u\rangle$ whose length is not less than 1. Let v be an arbitrary point in $\mathrm{IC}(-u)\setminus\{u\}$, then u and v are linearly independent. Let L be the two-dimensional subspace of X spanned by u and v. Then S_L contains a segment $[s,t]$ whose length is not less than 1 and parallel to the line $\langle -u,u\rangle$. Without loss of generality, we can assume that

$$[s,t]=\langle s,t\rangle\cap B_X,\quad t-u\in[s,t],\quad\text{and}\quad v\in\mathrm{arc}(u,t).$$

It is not difficult to see that $t\in\mathrm{cl}(\mathrm{IC}(-u))$. Again we have by the Monotonocity Lemma (cf. Proposition 31 in [12]) that

$$\|u-v\|\leqslant\|u-t\|=1.$$

It follows that $\gamma(\mathrm{IC}(-u),u)=1$. □

PROPOSITION 15. *If X is a Minkowski space, then*

$$\gamma(\mathrm{IC}(-u),u)\leqslant 2,\forall u\in S_X.$$

Equality holds if and only if there exists a point $v\in S_X$ satisfying the following two conditions:

(1) $[-v,u]\subseteq S_X$,

(2) $\|v+\lambda u\|>1,\forall\lambda>0$.

Proof. We only need to characterize the case when equality holds.

First suppose that $\gamma(\mathrm{IC}(-u),u)=2$. Then there exists $v\in\mathrm{cl}(\mathrm{IC}(-u))\setminus\mathrm{IC}(-u)$ such that $\|v-u\|=2$.

Since $v\perp_B u$, $u\neq -v$. Therefore there exist three different unit vectors $-v$, u, and $\frac{1}{2}(u-v)$ in $[-v,u]$, which shows that $[-v,u]\subseteq S_X$.

If the condition (2) does not hold, then there exists $\lambda_0 > 0$ such that $\|v+\lambda_0 u\| = 1$. Since $v \perp_B u$, we have

$$[v, v+\lambda_0 u] \subseteq S_X,$$

a contradiction to $v \in \mathrm{cl}(\mathrm{IC}(-u))$.

Conversely, suppose that $v \in S_X$ satisfies (1) and (2). Then

$$\|u-v\| = 2.$$

For each $\lambda > 0$, we have

$$\|v-\lambda u\| = (1+\lambda)\left\|\frac{1}{1+\lambda}v + \frac{\lambda}{1+\lambda}(-u)\right\| = 1+\lambda > 1.$$

Thus

$$\|v+\lambda u\| > 1, \ \forall \lambda \neq 0.$$

Let $L = \mathrm{span}\{v,u\}$. Then $\pm v$ are the only two points in S_L that are Birkhoff orthogonal to u, which shows that $v \in \mathrm{cl}(\mathrm{IC}(-u))$. It follows that $\gamma(\mathrm{IC}(-u),u) = 2$. $\square$

REMARK 2. Conditions (1) and (2) in Proposition 15 cannot be replaced by "$v \perp_B u$ and $\|v-u\| = 2$". For example, let $X = (\mathbb{R}^2, \|\cdot\|_{\infty-1})$, where

$$\|(\alpha,\beta)\|_{\infty-1} = \begin{cases} \max\{|\alpha|,|\beta|\}, & \alpha\cdot\beta \geqslant 0, \\ |\alpha|+|\beta|, & \alpha\cdot\beta < 0. \end{cases}$$

Pick $u = (1,0)$ and $v = (0,1)$. Then $v \perp_B u$ and $\|u-v\|_{\infty-1} = 2$. However, $\gamma(\mathrm{IC}(-u),u) = 1$.

DEFINITION 3. For a Minkowski space X, set

$$RC'(X) := \sup\{\gamma(\mathrm{IC}(-u),u) : \ u \in S_X\}.$$

To prove the following theorem we shall make use of *James constant* $J(X)$ and *Schäffer constant* $S(X)$ of X, where

$$J(X) := \sup\{\|x+y\| : \ x,y \in S_X, \|x+y\| = \|x-y\|\},$$
$$S(X) := \inf\{\|x+y\| : \ x,y \in S_X, \|x+y\| = \|x-y\|\}.$$

It is well known that

$$1 \leqslant S(X) \leqslant \sqrt{2} \leqslant J(X) \leqslant 2$$

holds for each normed linear space X, see, e.g., [7] and [9]. It is not difficult to verify that there always exist two unit vectors in X such that $\|x+y\| = \|x-y\| = \sqrt{2}$. This fact will be used in the proof of Theorem 16.

THEOREM 16. *If X is a normed linear space whose dimension is at least two, then*

$$\sqrt{2} \leqslant RC'(X) \leqslant 2,$$

and equality on the left holds if and only if X is an inner product space.

Proof. First we prove that

$$RC'(X) \geqslant \sqrt{2}.$$

Let L be an arbitrary two-dimensional subspace of X, and $u, v \in S_X$ be two points satisfying $\|u+v\| = \|u-v\| = \sqrt{2}$. If $v \in \mathrm{cl}(\mathrm{IC}(-u))$ or $v \in \mathrm{cl}(\mathrm{IC}(u))$, then

$$RC'(X) \geqslant \|u-v\| = \|u+v\| = \sqrt{2}.$$

Otherwise, there exist two different points s and t such that

1. $[s,t] \subseteq S_L$ is parallel to $\langle -u, u\rangle$;
2. v is a relative interior point of $[s,t]$;
3. $(t-s)/\|t-s\| = u$.

By the Monotonocity Lemma (cf. Proposition 31 in [12]), we have

$$\|u-s\| \geqslant \|u-v\| = \sqrt{2}.$$

If $\|u-s\| = \sqrt{2}$, $[s,v]$ is contained in a line which intersects the line $\langle -u,u\rangle$ (cf. Theorem 6 in [16]), a contradiction to the facts that $[s,v] \subseteq \langle s,t\rangle$ and $\langle s,t\rangle$ is parallel to $\langle -u,u\rangle$. Therefore,

$$\|u-s\| > \sqrt{2}.$$

There exists a sufficiently large $n \in \mathbb{N}$ such that

$$\left\| s - \frac{(1-\frac{1}{n})u + \frac{1}{n}s}{\|(1-\frac{1}{n})u + \frac{1}{n}s\|} \right\| \geqslant \|s-u\| - \left\| u - \frac{(1-\frac{1}{n})u + \frac{1}{n}s}{\|(1-\frac{1}{n})u + \frac{1}{n}s\|} \right\| > \sqrt{2}. \tag{6}$$

Set

$$u' = \frac{(1-\frac{1}{n})u + \frac{1}{n}s}{\|(1-\frac{1}{n})u + \frac{1}{n}s\|}.$$

For each p in $[s,t] \setminus \{s\}$, there exists $\alpha > 0$ such that $p = s + \alpha u$. Then

$$\begin{aligned} p - \frac{2n-1}{2(n-1)}\alpha \left\| \left(1-\frac{1}{n}\right)u + \frac{1}{n}s \right\| u' &= s + \alpha u - \frac{2n-1}{2(n-1)}\alpha\left(1-\frac{1}{n}\right)u - \frac{2n-1}{2(n-1)}\alpha\frac{1}{n}s \\ &= \left(1 - \frac{2n-1}{2n(n-1)}\alpha\right)s + \frac{1}{2n}\alpha u. \end{aligned}$$

When n is sufficiently large, we have

$$1 - \frac{2n-1}{2n(n-1)}\alpha > 0, \quad \frac{1}{2n}\alpha > 0$$

and

$$1 - \frac{2n-1}{2n(n-1)}\alpha + \frac{1}{2n}\alpha = 1 - \frac{1}{2(n-1)}\alpha \in (0,1).$$

Therefore,

$$p-\frac{2n-1}{2(n-1)}\alpha\left\|\left(1-\frac{1}{n}\right)u+\frac{1}{n}s\right\|u'\in \operatorname{int}B_X.$$

It follows that

$$p\in \mathrm{IC}(-u').$$

Since $s\in \operatorname{cl}([s,t]\setminus\{s\})$,

$$s\in \operatorname{cl}(\mathrm{IC}(-u')).$$

Then the inequality (6) shows that

$$RC'(X)\geqslant \|s-u'\|>\sqrt{2}.$$

In the following we characterize the case when $RC'(X)=\sqrt{2}$.

If X is an inner product space, then for each $u\in S_X$, $\operatorname{cl}(\mathrm{IC}(-u))$ is the intersection of the closed ball centered at u having radius $\sqrt{2}$ and S_X. Therefore,

$$RC'(X)=\sqrt{2}.$$

Conversely, suppose that $RC'(X)=\sqrt{2}$. To show X is an inner product space, we only need to prove $\|u-v\|\leqslant\sqrt{2}$ holds for each pair of points $u,v\in S_X$ satisfying $v\perp_B u$ (cf. [6] or Theorem 4.22 in [1]). Let L be the two-dimensional subspace spanned by u and v. If $\|u-v\|>\sqrt{2}$, then, since $RC'(X)=\sqrt{2}$,

$$v\notin \operatorname{cl}(\mathrm{IC}(u))\cup \operatorname{cl}(\mathrm{IC}(-u)).$$

Therefore, there exist two different points s and t such that

1. $[s,t]\subseteq S_L$, and $\langle s,t\rangle$ is parallel to $\langle -u,u\rangle$,
2. v is a relative interior point of $[s,t]$,
3. $(t-s)/\|t-s\|=u$.

In a similar way as in the first part of the proof, we can construct a point u', such that $\gamma(\mathrm{IC}(-u'),u')>\sqrt{2}$, which is in contradiction to the fact that $RC'(X)=\sqrt{2}$. Thus $\|u-v\|\leqslant\sqrt{2}$, as claimed. It follows that X is an inner product space. □

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