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A NEW SCHAUDER BASIS FOR $L^r((0,1)^n)$, $n=2,3$

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Abstract. We show that under suitable conditions on p, q, and summability, the system of generalized trigonometric functions $\{\Pi_{i=1}^{d}\sin_{p,q}(n_i\pi_{p,q}x_i)\}_{n_1,\dots,n_d}$ is a basis for $L^r((0,1)^d)$ for any $r\in(1,\infty)$ where $d=2$ or $d=3$.

1. Introduction

It is a truth universally acknowledged, that Fourier series play a fundamental role in different areas of mathematics, and then one must be in want to generalize it. The focus of this paper is in the specific direction of generalizing the multi-dimensional Fourier series of trigonometric functions to multi-dimensional Fourier series of generalized trigonometric functions. The generalized trigonometric functions considered in this study are obtained as eigenfunctions of the one-dimensional p-Laplacian and they have a long history ([8]). They appear in classical studies of exact constants for integral operators (see: [7], [9]), and also in Approximation Theory ([6]).

We start by recalling same basic terms. For a real number $p\in(1,\infty)$ and a bounded domain $\Omega\subset\mathbb{R}^n$, the eigenvalue problem for the p-Laplacian

$$\Delta_p(u):=\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$$

is given by:

$$\Delta_p(u)=-\lambda|u|^{p-2}u. \tag{1}$$

A sequence (s_j) in a Banach space X is said to be a Schauder basis (or simply a basis, if there is no room for confusion, as it will be the case in the sequel) for X if for any $x\in X$ there exists a unique sequence of scalars (x_j) with $x=\sum_1^\infty x_js_j$.

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2. Generalized trigonometric functions

For $1 < p,q < \infty$ the generalized trigonometric functions are defined as follows: Set

$$\sin_{p,q}^{-1} x := \int_0^x (1-t^q)^{-1/p}\, dt, \quad x \in (0,1) \tag{2}$$

and

$$\pi_{p,q} := 2\int_0^1 (1-x^q)^{-\frac{1}{p}}\, dx. \tag{3}$$

Then we extend $\sin_{p,q} : [0,\pi_{p,q}/2] \to [0,1]$ symmetrically about the line $x = \frac{\pi_{p,q}}{2}$ into $[\frac{\pi_{p,q}}{2}, \pi_{p,q}]$, next as an odd function to the interval $[-\pi_{p,q}, 0]$ and finally, periodically from the interval $[-\pi_{p,q}, \pi_{p,q}]$ to $(-\infty,\infty)$. In the sequel we set $\pi_p := \pi_{p,p}$ and $\sin_p(x) := \sin_{p,p}(x)$ for $1 < p < \infty$. Let us note that $\sin_p(n\pi_p x)$, $n \in \mathbb{N}$ are eigenfunctions of (1) on the interval $(0,1)$.

We observe in passing that natural extensions of the preceding definitions exist for the full range $(p,q) \in [1,\infty] \times [1,\infty]$. Since the end-point cases fall beyond the scope of this work, we omit every mention to the case when either of the subindices p or q is 1 or ∞. The reader is referred to [6] for more details in connection with this remark.

The aim of this work is to exhibit sufficient conditions on the subindexes p and q which guarantee that for each $r \in (1,\infty)$, the systems

$$\left\{\sin_{p,q}(n\pi_{p,q}x)\sin_{p,q}(m\pi_{p,q}y)\right\}_{(n,m)\in\mathbb{N}^2} \tag{4}$$

and

$$\left\{\sin_{p,q}(n\pi_{p,q}x)\sin_{p,q}(m\pi_{p,q}y)\sin_{p,q}(k\pi_{p,q}z)\right\}_{(n,m,k)\in\mathbb{N}^3} \tag{5}$$

constitute a basis for $L^r((0,1)^2)$ and $L^r((0,1)^3)$, respectively. The one dimensional case has been extensively treated for example in [2] and [5], among others.

In what follows we denote a multi-index by $\mathbf{k} := (k_1,k_2,\ldots,k_n) \in \mathbb{N}^n$ and by $\mathbf{k} \leqslant \mathbf{l}$ we mean that $k_i \leqslant l_i$ for each i: $1 \leqslant i \leqslant n$. It is well known that for $f \in L^r((-1,1)^n)$

$$\left\| f - \sum_{\mathbf{k}\leqslant\mathbf{l}} \hat{f}(k_1,\ldots,k_n) e^{2\pi i k_j x_j} \right\|_{L^r((-1,1)^n)} \longrightarrow 0 \tag{6}$$

as $\min\{l_1,l_2,\ldots,l_n\} \to \infty$, where, as is customary the ordinary Fourier coefficients of f will be written as

$$\hat{f}(k_1,k_2,\ldots k_n) := \int_{(-1,1)^n} f(x_1,\ldots,x_n)\,\Pi_{j=1}^n e^{2\pi i k_j x_j}\, dx_1 \ldots dx_n.$$

Since any $f \in L^r((0,1)^n)$ can be uniquely extended to $(-1,1)^n$ as an odd function, it is readily concluded that

THEOREM 1. *For $r \in (1,\infty)$ and $f \in L^r((0,1)^n)$ the sine Fourier partial sums $S_{\mathbf{l}}$ converge in L^r-norm to f in the Pringsheim sense, i.e.:*

$$\|f - S_{\mathbf{l}}\|_{L^r((0,1)^n)} \longrightarrow 0 \quad as \ \min\{l_1, l_2, \dots, l_n\} \to \infty,$$

where $\mathbf{l} := (l_1, l_2, \dots, l_n) \in \mathbb{N}^n$,

$$S_{\mathbf{l}} := \sum_{\mathbf{k} \leqslant \mathbf{l}} \hat{f}(k_1, \dots, k_n) \Pi_{j=1}^n \sin \pi k_j x_j.$$

and

$$\hat{f}(k_1, k_2, \dots k_n) := 2^n \int_{(0,1)^n} f(x_1, \dots, x_n) \Pi_{j=1}^n \sin \pi k_j x_j \, dx_1 \dots dx_n.$$

In the sequel we set $\mathbf{x} = (x_1, x_2, \dots, x_n)$, $x_j \in \mathbb{R}$ for $j = 1, 2, \dots, n$; for $\mathbf{k} \in \mathbb{N}^n$ we define the function $g_{\mathbf{k},p,q} \in L^r((0,1)^n)$ by

$$g_{\mathbf{k},p,q}(\mathbf{x}) = \Pi_{j=1}^n \sin_{p,q} \pi_{p,q} k_j x_j; \tag{7}$$

the corresponding Fourier coefficients are given by

$$\begin{aligned} \hat{g}_{\mathbf{k},p,q}(l_1, \dots, l_n) &= 2^n \int_{[0,1]^n} \Pi_{j=1}^n \sin_{p,q} \pi_{p,q} k_j x_j \, \Pi_{i=1}^n \sin \pi l_i x_i d\mathbf{x} \\ &= 2^n \Pi_{j=1}^n \int_0^1 \sin_{p,q} \pi_{p,q} k_j x_j \sin \pi l_j x_j \, dx_j. \end{aligned} \tag{8}$$

It is easy to see that because of the symmetry of $\sin_{p,q} x$ about the vertical line $x = \pi_{p,q}/2$, one has

$$\hat{g}_{\mathbf{1},p,q}(\mathbf{k}) = 0$$

when $\mathbf{k}$ has at least one even component. The next lemma is a direct consequence of this observation.

LEMMA 1.

$$\begin{aligned} \hat{g}_{\mathbf{k},p,q}(l_1, \dots l_n) &= \Pi_{j=1}^n \widehat{\sin_{p,q}(k_j \pi_{p,q}} x_j)(l_j) \\ &= \Pi_{j=1}^n \widehat{\sin_{p,q}(\pi_{p,q}} x_j)(l_j/k_j) \end{aligned}$$

if l_j/k_j is odd ($j = 1, \dots n$) and 0 otherwise.

For the sake of completeness we state the following lemma which follows immediately from Proposition 4.1 in [5]:

LEMMA 2. *Let $1 < p, q < \infty$ and m odd:*

$$\left|\hat{f}_{1,p,q}(m)\right| \leqslant 4\pi_{p,q}/(\pi m)^2.$$

The next lemma is a direct consequence of Proposition 4.2 in [5].

LEMMA 3. *For $1 < p,q < \infty$, one has*

$$\widehat{\sin}_{p,q}\pi_{p,q}(\cdot)(1) =: \tau_{p,q}(1) \geqslant 8/\pi^2.$$

DEFINITION 1. For a function $f : [0,1]^n \to \mathbb{R}$ we define its extension as the function $\tilde{f} : [0,\infty)^n \to \mathbb{R}$ as follows:

$$\tilde{f}(\mathbf{x}) = -\tilde{f}(2\mathbf{k} - \mathbf{x}) \ \text{ for } \ \mathbf{x} \in \Pi_{j=1}^n [k_j, k_j + 1) \ k_j \in \mathbb{N}^n,$$

$$\text{and } \ \tilde{f} \equiv f \text{ on } [0,1)^n.$$

It is a matter of routine to verify that given $r \in (1,\infty)$, for each $\mathbf{k} \in \mathbb{N}^n$, the map

$$M_{\mathbf{k}} : L^r((0,1)^n) \to L^r((0,1)^n)$$

which is defined as $M_{\mathbf{k}}(g)(\mathbf{x}) := \tilde{g}(\mathbf{x}\mathbf{k})$ is well defined, linear and in fact, an isometry (i.e. we have $\|M_{\mathbf{k}}\| = 1$). Note that here $\mathbf{x}\mathbf{k} = (x_1, x_2, \dots x_n)(k_1, k_2, \dots k_n) = (x_1k_1, x_2k_2, \dots x_nk_n)$.

Let us set

$$\begin{aligned} \tau_{p,q}(\mathbf{k}) &:= 2^n \int_{(0,1)^n} \Pi_{j=1}^n \sin_{p,q} \pi_{p,q} k_j x_j \sin \pi k_j x_j \, d\mathbf{x} \\ &= 2^n \int_0^1 \Pi_{j=1}^n \sin_{p,q} \pi_{p,q} k_j x_j \sin k_j \pi x_j \, dx_j \\ &= \Pi_{j=1}^n \tau_{p,q}^{\circ}(k_j), \end{aligned} \tag{9}$$

where $\tau_{p,q}^{\circ}(k_j) =: 2\int_0^1 \sin_{p,q} \pi_{p,q} x_j \sin k_j \pi x_j \, dx_j$. Then the (linear) operator

$$T : L^r((0,1)^n) \longrightarrow L^r((0,1)^n)$$

defined by:

$$T(g) := \sum_{\mathbf{k} \in \mathbb{N}^n} \tau_{p,q}(\mathbf{k}) M_{\mathbf{k}}(g)$$

is well defined and bounded (just observe that $\sum_{\mathbf{k} \in \mathbb{N}^n} |\tau_{p,q}(\mathbf{k})| < \infty$).

Next we point out that

$$
\begin{aligned}
\|T - id\cdot\tau_{p,q}(\mathbf{1})\| &\leqslant n[\tau^{\circ}_{p,q}(1)]^{1}\sum_{k_2>1,..,k_n>1}\Pi_2^n\tau^{\circ}_{p,q}(k_j)\\
&\quad+\ldots\binom{n}{s}[\tau^{\circ}_{p,q}(1)]^{s}\sum_{k_s>1,..,k_n>1}\Pi_{j+1}^n\tau^{\circ}_{p,q}(k_j)\\
&\quad+\ldots+\sum_{k_1>1,k_2>1,..,k_n>1}\Pi_{j=1}^n\tau^{\circ}_{p,q}(k_j)\\
&=\sum_{k=1}^{n-1}\binom{n}{k}[\tau^{\circ}_{p,q}(1)]^{k}\left(\sum_{j=3}^{\infty}\tau^{\circ}_{p,q}(j)\right)^{n-k}+\left(\sum_{j=3}^{\infty}\tau^{\circ}_{p,q}(j)\right)^{n}\\
&\leqslant\sum_{k=1}^{n-1}\binom{n}{k}[\tau^{\circ}_{p,q}(1)]^{k}\left(\frac{4\pi_{p,q}}{\pi^2}\left(\frac{\pi^2}{8}-1\right)\right)^{n-k}+\left(\frac{4\pi_{p,q}}{\pi^2}\left(\frac{\pi^2}{8}-1\right)\right)^{n}\\
&\leqslant\left(\frac{4\pi_{p,q}}{\pi^2}\right)^{n}\left[\sum_{k=1}^{n-1}\binom{n}{k}\left(\frac{\pi^2}{8}-1\right)^{n-k}+\left(\frac{\pi^2}{8}-1\right)^{n}\right] \qquad (10)\\
&=\left(\frac{4\pi_{p,q}}{\pi^2}\right)^{n}\left(\left(\frac{\pi^2}{8}\right)^{n}-1\right).
\end{aligned}
$$

LEMMA 4. *For $1<q<q'<\infty,\ 1<p<p'<\infty$, the function*

$$w(x)=\frac{\sin^{-1}_{p,q}(x)}{\sin^{-1}_{p'q'}(x)}$$

is strictly increasing on $(0,1)$.

Proof. A simple calculation reveals that the sign of $w'(x)$ at any point $x\in(0,1)$ is the same as that of

$$v(x)=\sin^{-1}_{p'q'}x-(1-x^{q})^{1/p}(1-x^{q'})^{-1/p'}\sin^{-1}_{p,q}x; \qquad (11)$$

in turn,

$$v'(x)=r(x)\left[\frac{q'}{p'}\frac{x^{q'}}{1-x^{q'}}-\frac{q}{p}\frac{x^{q}}{1-x^{q}}\right], \qquad (12)$$

where $r(x)<0$ on $(0,1)$. Setting

$$s(x)=\frac{x^{q'}(1-x^{q})}{x^{q}(1-x^{q'})} \qquad (13)$$

it is easy to verify that

$$s'(x)=x^{q+q'}\left((q'-q)+qx^{q'}-q'x^{q}\right)x^{-2q}(1-x^{q'})^{-2}. \qquad (14)$$

Since $1 < q < q'$, the function

$$x \to (q'-q) + qx^{q'} - q'x^q$$

is strictly decreasing on $(0,1)$, from which one obtains immediately the inequality, valid for $x \in (0,1)$:

$$s'(x) > 0. \tag{15}$$

Hence, s is increasing on $(0,1)$ and one has there:

$$-\frac{q}{p} < \frac{q'}{p'}s(x) - \frac{q}{p} < q\left(\frac{1}{p'} - \frac{1}{p}\right) < 0 \tag{16}$$

given that $p < p'$. We conclude that $v' > 0$ and thus v is increasing on $(0,1)$, which implies that $v(x) > 0$ on $(0,1)$, which yields the lemma. $\square$

COROLLARY 1. *If* $1 < p < p' < \infty$, $1 < q < q' < \infty$ *one has*

(i)

$$\frac{\sin_{p,q}^{-1} x}{\pi_{p,q}} < \frac{\sin_{p'q'}^{-1} x}{\pi_{p',q'}}$$

for $x \in (0,1)$.

(ii) *If* $x \in (0,1/2)$, *then*

$$\sin_{p',q'} \pi_{p',q'} x < \sin_{p,q} \pi_{p,q} x.$$

(iii) *Uniformly on* $(0,1)$*:*

$$1 < \frac{\frac{\sin_{p',q'}^{-1} x}{\pi_{p',q'}}}{\frac{\sin_{p,q}^{-1} x}{\pi_{p,q}}} < \frac{\pi_{p,q}}{\pi_{p',q'}}$$

Proof. Claim (i) follows immediately since

$$\sin_{p',q'}^{-1}(1) = \frac{\pi_{p',q'}}{2}$$

and

$$\sin_{p,q}^{-1}(1) = \frac{\pi_{p,q}}{2}.$$

With regard to (ii) it is sufficient to compare the inverse functions

$$\left(\frac{\sin_{p',q'}^{-1}(\cdot)}{\pi_{p',q'}}\right)^{-1}$$

and

$$\left(\frac{\sin_{p,q}^{-1}(\cdot)}{\pi_{p,q}}\right)^{-1}$$

on the interval $(0,\frac{1}{2})$ using the information provided by (i). Claim (iii) follows from (i). $\square$

COROLLARY 2. *For* $1 < p < 2,\ 1 < q < 2,$

$$\tau_{p,q}(\mathbf{1}) > 1.$$

Proof. By virtue of (9) and Corollary 1 (ii), one has

$$\begin{aligned}\tau_{p,q}(\mathbf{1}) &= \Pi_{j=1}^{n}\tau_{p,q}(1)\tau_{p,q}^{\circ}(1) = \left(\tau_{p,q}^{\circ}(1)\right)^{n} \\ &= \left(2\int_0^1 \sin_{p,q}\pi_{p,q}t\sin\pi t\,dt\right)^n \\ &> \left(2\int_0^1 \sin^2\pi t\,dt\right)^n = 1. \quad \square\end{aligned}$$

COROLLARY 3. *The system*

$$\left\{\sin_{pq}k_1\pi_{pq}x_1\sin_{pq}k_2\pi_{pq}x_2\ldots\sin_{pq}k_n\pi_{pq}x_n,\ (k_1,k_2,\ldots k_n)\in\mathbb{N}^n\right\}$$

is a basis in $L^r((0,1)^n)$ *if* $1 < p < 2,\ 1 < q < 2$ *and*

$$\pi_{p,q} < \frac{2\pi^2}{(\pi^{2n}-8^n)^{1/n}}$$

or if either $p \geqslant 2$ *or* $q \geqslant 2$, *and*

$$\pi_{p,q} < \frac{16}{(\pi^{2n}-8^n)^{1/n}}.$$

Proof. Both claims follow, respectively, from Lemma 3, Corollary 2, formula (10) and the standard functional-analytic argument that if K is an operator with norm strictly less than one on a Banach space X, then $I+K$ is invertible on X. $\square$

COROLLARY 4. *In particular, if* p_0 *and* p_1 *are defined by the equalities*

$$\pi_{p_0} = \frac{2\pi^2}{(\pi^4-8^2)^{1/2}} \quad (p_0 \approx 1.85) \tag{17}$$

$$\pi_{p_1} = \frac{16}{(\pi^4-8^2)^{1/2}} \quad (p_1 \approx 2.33) \tag{18}$$

then

(i) *for* $p = q \in (p_0,2)\cup(p_1,\infty)$, *the system*

$$\{\sin_p(m\pi_p x)\sin_p(n\pi_p y)\}_{(m,n)\in\mathbb{N}^2}$$

constitutes a basis for $L^r((0,1)^2),\ r\in(1,\infty)$.

(*ii*) *For* $p = q$, $r \in (1,\infty)$ *the system*

$$\{\sin_p(m\pi px)\sin_p(n\pi_p y)\sin_p(k\pi_p x)\}_{(m,n,k)\in\mathbb{N}^3}$$

is a basis in $L^r((0,1)^3)$ *for* $p > p_2$, *where* $p_2 \approx 6.5$ *is given by*

$$\pi_{p_2} = \frac{16}{(\pi^6 - 8^3)^{1/3}}. \tag{19}$$

REMARK 1. Notice that

$$\lim_{n\to\infty} \frac{2\pi^2}{(\pi^{2n} - 8^n)^{1/n}} = 2.$$

Therefore the highest dimension n for which a conclusion can be reached using Corollary 3 is $n = 3$.

We now set about to improving Corollary 4 (i). In fact, we will establish that the basis property holds for any $p \in (p_0,\infty)$. We start with the following simple observation that follows from the right-hand-side inequality in Corollary 1 (iii): For $2 < p$ one has:

$$\sin_p^{-1} x < \sin^{-1} x \tag{20}$$

on $(0,1)$. Since $\pi_p < \pi$, (20) forces the following relation between the inverse functions, on the interval $(0, \frac{\pi_p}{2})$:

$$\sin x < \sin_p x, \tag{21}$$

which in turn implies the following estimate on $(0,1/2)$:

$$\sin_p \pi_p x > \sin \pi_p x. \tag{22}$$

In conclusion,

$$\begin{aligned} \tau(p) &= 4\int_0^{1/2} \sin_p \pi_p x \sin \pi x\, dx > 4\int_0^{1/2} \sin \pi_p x \sin \pi x\, dx \\ &= 2\left(\frac{1}{\pi - \pi_p}\sin\frac{\pi - \pi_p}{2} - \frac{1}{\pi + \pi_p}\sin\frac{\pi + \pi_p}{2}\right) \\ &= \frac{4\pi_p \cos\frac{\pi_p}{2}}{(\pi + \pi_p)(\pi - \pi_p)} = \gamma(\pi_p). \end{aligned} \tag{23}$$

Since

$$\gamma(x) = \frac{4x\cos\frac{x}{2}}{(\pi + x)(\pi - x)}$$

is increasing in $(\pi/2,\pi)$, π_p increases to π as p decreases to 2 and

$$\lim_{p\to 2^+} \gamma(\pi_p) = 1,$$

it is immediate that if for any $\delta > 0$ and $\pi_{p^*} > \pi/2$ it held that

$$\gamma(\pi_{p^*}) > 1 - \delta$$

it would follow that

$$\gamma(\pi_p) > 1 - \delta$$

for any p with $2 \leqslant p < p^*$. On the other hand,

$$\gamma(\pi_{2.33}) = \gamma\left(\frac{2\pi}{2.33 \sin \frac{\pi}{2.33}}\right) > \frac{93}{100} > \frac{8}{\pi^2}$$

and which implies

$$\gamma(\pi_p) = \tau(p) > \frac{93}{100}$$

for $p \in [2, 2.33]$. Since the inequality

$$\left(\frac{4\pi_p}{\pi^2}\right)^2 \left(\left(\frac{\pi^2}{8}\right)^2 - 1\right) < 0.93^2 \tag{24}$$

is satisfied whenever $\pi_p < 3.17$, i.e, whenever $p < 2.33$, one has the following result:

LEMMA 5. *The system* (4) *is a basis for* $L^r((0,1)^2)$ *for* $p = q \in (p_0, \infty)$.

THEOREM 2. *For* $r \in (1, \infty)$, *the system* (4) *is a basis for* $L^r((0,1)^2)$ *if* $(p,q) \in (p_0, 2) \times (p_0, 2) \cup (p_1, \infty) \times (p_1, \infty)$.

Proof. If $p_1 \leqslant p, q < \infty$ and $s = \min\{p, q\}$, then it is clear from Corollary 3 and since π_{pq} decreases when one of the subindexes is fixed and the remaing one increases, that

$$\pi_{p,q} < \pi_{s,s} < \frac{16}{\sqrt{\pi^4 - 64}}.$$

Hence the the basis property holds for (p,q). Likewise, if $p \in (p_0, 2)$, $q \in (p_0, 2)$, then

$$\pi_{p,q} < \pi_{\min\{p,q\},\min\{p,q\}} < \frac{2\pi^2}{\sqrt{\pi^4 - 64}},$$

which in view of Corollary 3 completes the proof. □

Invoking Corollary 4 (ii), a similar reasoning yield the following:

THEOREM 3. *If* $n = 3$, $r \in (1, \infty)$ *and* p_2 *is defined as in* (19), *the system* (5) *is a basis for* $L^r((0,1)^3)$ *when* $(p,q) \in (p_2, \infty) \times (p_2, \infty)$.

3. Concluding remarks

The two-dimensional generalized Fourier system opens the way for the use of non-orthogonal systems in the treatment of signal processing, which conceivably could be a valuable tool in studying image processing in the case of discontinuous gradient (see [1], [3], [4]), due to the fact that generalized trigonometric functions have a lesser degree of smoothness than the usual trigonometric functions ($p = q = 2$). In fact, the smoothness of generalized trigonometric function can in principle, be controlled by a suitable variation of the parameters p and q.

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