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COMPLETELY MONOTONIC FUNCTIONS RELATED TO GURLAND'S RATIO FOR THE GAMMA FUNCTION

CHAO-PING CHEN AND JUNESANG CHOI

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Abstract. The subject of complete monotonicity for functions has gained considerable popularity and importance from a rather long ago up to now, due mainly to its demonstrated applications in getting diverse inequalities. Here, we investigate some completely monotonic functions related to Gurland's ratio for the gamma function. Certain relevant connections of the results presented here with those earlier ones are also pointed out. Further, an interesting open conjecture regarding our present concern is posed.

1. Introduction

A function f is said to be completely monotonic on an interval I if it has derivatives of all orders on I and satisfies the following inequality:

$$(-1)^n f^{(n)}(x) \geqslant 0 \qquad (x \in I;\ n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\},\ \mathbb{N} := \{1,2,3,\ldots\}). \tag{1}$$

Dubourdieu [3, p. 98] pointed out that, if a non-constant function f is completely monotonic on $I = (a,\infty)$, then the strict inequality in (1) holds true (for a simpler proof of this result, see [5]). Bernstein's theorem asserts that f is completely monotonic on $[0,\infty)$ if and only if

$$f(x) = \int_0^\infty e^{-xt}\,\mathrm{d}\mu(t),$$

where $\mu(t)$ is bounded and non-decreasing and the integral converges for all $0 \leqslant x < \infty$ (see [11, p. 161]). The main properties of completely monotonic functions are given in [11, Chapter IV]. An extensive list of references on completely monotonic functions can be found in [2].

The gamma function Γ is one of the most important functions in mathematical analysis and has many applications in diverse areas. The psi (or digamma) function ψ defined by the logarithmic derivative of the gamma function and the polygamma functions $\psi^{(m)}$ $(m \in \mathbb{N})$ have the following integral representations (see, e.g., [1, p. 259, Entry 6.3.21] and [1, p. 260, Entry 6.4.1], respectively):

$$\psi(x) = \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-xt}}{1-e^{-t}} \right) \mathrm{d}t \quad (x>0) \tag{2}$$

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and

$$\psi^{(m)}(x) = (-1)^{m+1} \int_0^\infty \frac{t^m}{1-e^{-t}} e^{-xt}\, \mathrm{d}t \quad (x > 0;\ m \in \mathbb{N}). \tag{3}$$

The ratio of gamma functions

$$T(x,y) := \frac{\Gamma(x)\Gamma(y)}{\Gamma^2((x+y)/2)} \quad (x,\ y > 0) \tag{4}$$

was investigated by Gurland [4] who presented the following inequality:

$$\frac{\Gamma(x)\Gamma(x+2\beta)}{\Gamma^2(x+\beta)} \geqslant 1 + \frac{\beta^2}{x} \qquad (x > 0;\ x+2\beta > 0).$$

In probability theory and its applications, the Gurland's ratio $T(1/\lambda, 3/\lambda)$ appears in a form of ratio of the variance and squared absolute expectation of a generalized gamma random variable with the shape parameter λ (cf. [10]) which is also known as the generalized Gaussian ratio [8] and has interesting applications in the domain of image recognition [6, 8].

In the present sequel of the earlier works about the function $T(x,y)$ in (4), we investigate some completely monotonic functions related to the Gurland's ratio for the gamma function. Certain relevant connections of the results presented here with those earlier ones are also indicated. Further, an interesting open conjecture which arises naturally in the present investigation is posed.

2. Main results

We begin by defining a function $F(x)$ by

$$F(x) = T\left(\frac{1}{x}, \frac{3}{x}\right) = \frac{\Gamma(1/x)\Gamma(3/x)}{\Gamma^2(2/x)} \qquad (x > 0) \tag{5}$$

and its related function $L(x)$ by

$$L(x) = \begin{cases} \ln\Gamma(x) + \ln\Gamma(3x) - 2\ln\Gamma(2x) - \ln\frac{4}{3}, & x > 0, \\ 0, & x = 0. \end{cases} \tag{6}$$

It is obvious to see that

$$L(x) = \ln F\left(\frac{1}{x}\right) - \ln\frac{4}{3} \qquad (x > 0). \tag{7}$$

Merkle [7, Theorem 6] proved that the function $F(x)$ in (5) is convex and monotonically decreasing in x with

$$\lim_{x\to 0^+} F(x) = \infty \quad \text{and} \quad \lim_{x\to\infty} F(x) = \frac{4}{3}. \tag{8}$$

The function $L(x)$ in (6) is also known to be continuously differentiable of any order, convex and monotonically increasing on $[0,\infty)$ with $L'(0)=0$.

In the course of proof of convexity of $F(x)$, Merkle [7, p. 401] presented the following second derivative of $\ln F(x)$:

$$\big(\ln F(x)\big)''=t^3\left(6\ln 3-8\ln 2+\psi'\left(t+\frac{1}{3}\right)+\psi'\left(t+\frac{2}{3}\right)-2\psi'\left(t+\frac{1}{2}\right)\right)\qquad (t=1/x),$$

which may be corrected as in (12) below. We provide a corrected proof of the log-convexity of $F(x)$ in Theorem 1.

THEOREM 1. *The function $F(x)$ in (5) is log-convex on $(0,\infty)$.*

Proof. Using the duplication and triplication formulas for the gamma function (see [1, p. 256]; see also [9, p. 6]):

$$\Gamma(2z)=(2\pi)^{-\frac{1}{2}}2^{2z-\frac{1}{2}}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right)$$

and

$$\Gamma(3z)=(2\pi)^{-1}3^{3z-\frac{1}{2}}\Gamma(z)\Gamma\left(z+\frac{1}{3}\right)\Gamma\left(z+\frac{2}{3}\right),$$

we get

$$\begin{aligned} L(x)=&\left(3x-\frac{1}{2}\right)\ln 3-(4x-1)\ln 2+\ln\Gamma\left(x+\frac{1}{3}\right)\\ &+\ln\Gamma\left(x+\frac{2}{3}\right)-2\ln\Gamma\left(x+\frac{1}{2}\right)-\ln\frac{4}{3}. \end{aligned}\tag{9}$$

A simple computation yields

$$L'(x)=\begin{cases}3\ln 3-4\ln 2+\psi\left(x+\frac{1}{3}\right)+\psi\left(x+\frac{2}{3}\right)-2\psi\left(x+\frac{1}{2}\right), & x>0,\\ 0, & x=0\end{cases}$$

and

$$L''(x)=\psi'\left(x+\frac{1}{3}\right)+\psi'\left(x+\frac{2}{3}\right)-2\psi'\left(x+\frac{1}{2}\right).\tag{10}$$

Using the relation

$$\ln F(x)=L(1/x)+\ln(4/3),\tag{11}$$

we have

$$\big(\ln F(x)\big)''=\frac{1}{x^3}\left[\frac{1}{x}L''\left(\frac{1}{x}\right)+2L'\left(\frac{1}{x}\right)\right]=y^3G(y)\qquad (y=1/x),\tag{12}$$

where

$$\begin{aligned} G(y) &:= yL''(y) + 2L'(y) = \Big(yL(y)\Big)'' \\ &= y\left[\psi'\left(y+\frac{1}{3}\right) + \psi'\left(y+\frac{2}{3}\right) - 2\psi'\left(y+\frac{1}{2}\right)\right] \\ &\quad + 2\left[\psi\left(y+\frac{1}{3}\right) + \psi\left(y+\frac{2}{3}\right) - 2\psi\left(y+\frac{1}{2}\right)\right] \\ &\quad + 6\ln 3 - 8\ln 2 \end{aligned} \tag{13}$$

and

$$G(0) = 2L'(0) = 0. \tag{14}$$

We see from (12) that, for $x > 0$,

$$\Big(\ln F(x)\Big)'' > 0 \quad \text{if and only if} \quad G(x) = \Big(xL(x)\Big)'' > 0. \tag{15}$$

Differentiating $G(y)$ yields

$$\begin{aligned} \frac{G'(y)}{y} &= \psi''\left(y+\frac{1}{3}\right) + \psi''\left(y+\frac{2}{3}\right) - 2\psi''\left(y+\frac{1}{2}\right) \\ &\quad + \frac{3}{y}\left[\psi'\left(y+\frac{1}{3}\right) + \psi'\left(y+\frac{2}{3}\right) - 2\psi'\left(y+\frac{1}{2}\right)\right]. \end{aligned}$$

Using (2) and the following identity:

$$\frac{1}{y} = \int_0^\infty e^{-yt}\mathrm{d}t,$$

we have

$$\frac{G'(y)}{y} = -\int_0^\infty \frac{t^2 p(t)}{e^t - 1} e^{-yt}\,\mathrm{d}t + 3\int_0^\infty e^{-yt}\mathrm{d}t \int_0^\infty \frac{t\,p(t)}{e^t-1} e^{-yt}\,\mathrm{d}t,$$

where

$$p(t) := e^{2t/3} + e^{t/3} - 2e^{t/2}. \tag{16}$$

Using the convolution theorem for Laplace transforms, we have

$$\begin{aligned} \frac{G'(y)}{y} &= -\int_0^\infty \frac{t^2 p(t)}{e^t-1} e^{-yt}\,\mathrm{d}t + 3\int_0^\infty \left[\int_0^t \frac{up(u)}{e^u-1} du\right] e^{-yt}\mathrm{d}t \\ &= \int_0^\infty q(t)e^{-yt}\mathrm{d}t, \end{aligned}$$

where

$$q(t) := 3\int_0^t \frac{up(u)}{e^u-1}\mathrm{d}u - \frac{t^2 p(t)}{e^t-1}.$$

Differentiation yields

$$q'(t) = \frac{t}{3(e^t-1)^2} r(t),$$

where

$$\begin{aligned} r(t) &:= 3e^{5t/3} - 3e^{2t/3} + 3e^{4t/3} - 3e^{t/3} - 6e^{3t/2} + 6e^{t/2} + te^{5t/3} \\ &\quad + 2te^{2t/3} + 2te^{4t/3} + te^{t/3} - 3te^{3t/2} - 3te^{t/2} \\ &= \frac{17}{2592}t^5 + \frac{17}{2592}t^6 + \frac{151}{43740}t^7 + \frac{443}{349920}t^8 + \frac{679969}{1881169920}t^9 \\ &\quad + \frac{160289}{1881169920}t^{10} + \sum_{n=11}^{\infty} \frac{a_n}{n!}t^n \end{aligned} \tag{17}$$

with

$$\begin{aligned} a_n = &\left[(n+5)\left(\frac{5}{3}\right)^{n-1} - 3(n+3)\left(\frac{3}{2}\right)^{n-1}\right] + \left[2(n+2)\left(\frac{4}{3}\right)^{n-1} - 3(n-1)\left(\frac{1}{2}\right)^{n-1}\right] \\ &+ 2(n-1)\left(\frac{2}{3}\right)^{n-1} + (n-1)\left(\frac{1}{3}\right)^{n-1}. \end{aligned}$$

By induction on n, it is easy to show that, for $n \geqslant 11$,

$$\left(\frac{10}{9}\right)^{n-1} > \frac{3(n+3)}{n+5} \quad \text{and} \quad \left(\frac{8}{3}\right)^{n-1} > \frac{3(n-1)}{2(n+2)}.$$

Hence $a_n > 0$ for $n \geqslant 11$. Then, in view of (17), we have $r(t) > 0$ for $t > 0$ so that $q'(t) > 0$ on $(0, \infty)$. Since $q(t)$ is strictly increasing on $(0, \infty)$, $q(t) > q(0) = 0$ for all $t \in (0, \infty)$. Likewise we find that $G'(y) > 0$ and $G(y) > G(0) = 0$ for all $y > 0$. This implies $\big(\ln F(x)\big)'' > 0$ for all $x > 0$. Hence the proof is complete. □

REMARK 1. Merkle [7] proved that the function $L(x)$ in (6) is convex on $(0, \infty)$. Furthermore we see that the function $x \mapsto L''(x)$ is completely monotonic on $(0, \infty)$. Indeed, using (3), we obtain from (10) that

$$L''(x) = \int_0^\infty \frac{t\,p(t)}{e^t - 1} e^{-xt}\, dt, \tag{18}$$

where $p(t)$ is given in (16). We note that, for all t,

$$e^{2t/3} + e^{t/3} \geqslant 2\sqrt{e^{2t/3} \cdot e^{t/3}} = 2\,e^{t/2},$$

where the equality holds when $e^{2t/3} = e^{t/3}$ if and only if $t = 0$. Thus we find from (16) that $p(t) > 0$ for all $t > 0$ and $p(0) = 0$. We thus find from (18) that

$$(-1)^n \big(L''(x)\big)^{(n)} = \int_0^\infty \frac{t^{n+1} p(t)}{e^t - 1} e^{-xt}\, dt > 0 \qquad (x > 0;\ n \in \mathbb{N}_0).$$

Hence the complete monotonicity of the function $L''(x)$ on $(0, \infty)$ has been proved.

THEOREM 2. *The function* $x \mapsto \big(xL(x)\big)^{(3)}$ *is completely monotonic on* $(0,\infty)$.

Proof. By using Malmstén's formula (see, e.g., [1, p. 258, Entry (2.1.50)]; see also [9, p. 27, Eq. (25)]):

$$\ln\Gamma(z) = \int_0^\infty \left[e^{-t}(z-1) + \frac{e^{-zt}-e^{-t}}{1-e^{-t}}\right]\frac{\mathrm{d}t}{t} \qquad (\Re(z)>0), \tag{19}$$

we get

$$L(x) = (3\ln 3 - 4\ln 2)x + \ln(\sqrt{3}/2) + \int_0^\infty \omega(t)e^{-xt}\,\mathrm{d}t,$$

where

$$\omega(t) := \frac{e^{2t/3}+e^{t/3}-2e^{t/2}}{t(e^t-1)}.$$

Differentiation yields

$$\omega'(t) = -\frac{r(t)}{3t^2(e^t-1)^2} < 0 \qquad (t>0),$$

where $r(t)$ is given in (17).

An integration by parts yields

$$\begin{aligned} xL(x) &= (3\ln 3 - 4\ln 2)x^2 + x\ln(\sqrt{3}/2) - \int_0^\infty \omega(t)\mathrm{d}(e^{-xt}) \\ &= (3\ln 3 - 4\ln 2)x^2 + x\ln(\sqrt{3}/2) - \left[\frac{\omega(t)}{e^{xt}}\right]_0^\infty + \int_0^\infty \omega'(t)e^{-xt}\mathrm{d}t \\ &= (3\ln 3 - 4\ln 2)x^2 + x\ln(\sqrt{3}/2) + \frac{1}{36} + \int_0^\infty \omega'(t)e^{-xt}\mathrm{d}t. \end{aligned}$$

We thus have

$$(-1)^n\big(xL(x)\big)^{(n+3)} = \int_0^\infty t^{n+3}\big(-\omega'(t)\big)e^{-xt}\mathrm{d}t > 0 \qquad (x>0;\ n\in\mathbb{N}_0).$$

The proof is complete. □

REMARK 2. In particular, we have

$$\big(xL(x)\big)^{(3)} = \int_0^\infty t^3\big(-\omega'(t)\big)e^{-xt}\mathrm{d}t > 0 \qquad (x>0).$$

Hence $x \mapsto \big(xL(x)\big)''$ is strictly increasing on $(0,\infty)$, and we have

$$\big(xL(x)\big)'' > \Big[\big(xL(x)\big)''\Big]_{x=0} = G(0) = 0 \qquad (x>0). \tag{20}$$

It follows from (15) that the function $x \mapsto \ln F(x)$ is convex on $(0,\infty)$. Thus the log-convexity of $F(x)$ is proved again.

3. Concluding remarks

Computer experiments indicate that $x \mapsto \ln F(x)$ is not only decreasing and convex but also completely monotonic on $(0,\infty)$. So we are posing an open problem stated in the following conjecture.

CONJECTURE 1. The following inequality holds true:

$$(-1)^n \left(\ln F(x)\right)^{(n)} > 0 \qquad (x>0;\ n\in\mathbb{N}_0). \tag{21}$$

Considering the course of proof of Theorem 1, it seems convenient and potentially useful to give a general rule for (12) asserted by Theorem 3.

THEOREM 3. *Let a function $f(x)$ $(x\in\mathbb{R})$ have derivatives of all orders. Then, for all $x\neq 0$ and $n\in\mathbb{N}$,*

$$(-1)^n \frac{\mathrm{d}^n}{\mathrm{d}x^n} f\left(\frac{1}{x}\right) = y^{n+1}\frac{\mathrm{d}^n}{\mathrm{d}y^n}\left\{y^{n-1}f(y)\right\} \qquad (y=1/x). \tag{22}$$

Proof. We proceed to prove (22) by using the principle of mathematical induction on $n\in\mathbb{N}$. It is easy to see that (22) is true for $n=1$. Assume that (22) is true for some $n\in\mathbb{N}$. We begin with

$$\mathscr{L}_{n+1}(x) := (-1)^{n+1}\frac{\mathrm{d}^{n+1}}{\mathrm{d}x^{n+1}} f\left(\frac{1}{x}\right) = -\frac{\mathrm{d}}{\mathrm{d}x}\left\{(-1)^n \frac{\mathrm{d}^n}{\mathrm{d}x^n} f\left(\frac{1}{x}\right)\right\}.$$

Then we find from induction hypothesis that

$$\begin{aligned}
\mathscr{L}_{n+1}(x) &= -\frac{\mathrm{d}}{\mathrm{d}x}\left[y^{n+1}\frac{\mathrm{d}^n}{\mathrm{d}y^n}\left\{y^{n-1}f(y)\right\}\right] \qquad (y=1/x)\\
&= -\frac{\mathrm{d}y}{\mathrm{d}x}\left\{\frac{\mathrm{d}}{\mathrm{d}y}\left[y^{n+1}\frac{\mathrm{d}^n}{\mathrm{d}y^n}\left\{y^{n-1}f(y)\right\}\right]\right\}\\
&= -\left(-y^2\right)\left[(n+1)y^n\frac{\mathrm{d}^n}{\mathrm{d}y^n}\left\{y^{n-1}f(y)\right\} + y^{n+1}\frac{\mathrm{d}^{n+1}}{\mathrm{d}y^{n+1}}\left\{y^{n-1}f(y)\right\}\right]\\
&= y^{n+2}\left[(n+1)\frac{\mathrm{d}^n}{\mathrm{d}y^n}\left\{y^{n-1}f(y)\right\} + y\frac{\mathrm{d}^{n+1}}{\mathrm{d}y^{n+1}}\left\{y^{n-1}f(y)\right\}\right].
\end{aligned}$$

On the other hand, it follows from the Leibniz's general product rule for differentiation that

$$\begin{aligned}
\frac{\mathrm{d}^{n+1}}{\mathrm{d}y^{n+1}}\left\{y\cdot y^{n-1}f(y)\right\} &= \sum_{k=0}^{n+1}\binom{n+1}{k}\frac{\mathrm{d}^k}{\mathrm{d}y^k}y\,\frac{\mathrm{d}^{n+1-k}}{\mathrm{d}y^{n+1-k}}\left\{y^{n-1}f(y)\right\}\\
&= y\frac{\mathrm{d}^{n+1}}{\mathrm{d}y^{n+1}}\left\{y^{n-1}f(y)\right\} + (n+1)\frac{\mathrm{d}^n}{\mathrm{d}y^n}\left\{y^{n-1}f(y)\right\}.
\end{aligned}$$

We thus have shown that

$$\mathscr{L}_{n+1}(x) = y^{n+2}\frac{d^{n+1}}{dy^{n+1}}\{y^n f(y)\} \qquad (y = 1/x).$$

Hence, by the principle of mathematical induction, (22) holds true for all $n \in \mathbb{N}$. □

Now it is easy to see from Theorem 3 that the following equivalent statements hold.

COROLLARY 1. *Let a function $f(x)$ have derivatives of all orders on $(0,\infty)$. Then, for all $x > 0$ and $n \in \mathbb{N}_0$,*

$$(-1)^n\left(f\left(\frac{1}{x}\right)\right)^{(n)} > 0 \quad \textit{if and only if} \quad \left(x^{n-1}f(x)\right)^{(n)} > 0. \tag{23}$$

In view of (23), we find that $x \mapsto f(1/x)$ is completely monotonic on $(0,\infty)$ if and only if $\left(x^{n-1}f(x)\right)^{(n)} > 0$ for all $x > 0$ and $n \in \mathbb{N}_0$.

It is easy to see that

$$(-1)^n\left(\ln F(x)\right)^{(n)} > 0 \Longleftrightarrow (-1)^n\left(L\left(\frac{1}{x}\right)\right)^{(n)} > 0 \Longleftrightarrow \left(x^{n-1}L(x)\right)^{(n)} > 0. \tag{24}$$

Hence Conjecture 1 is seen to be equivalent to the following Conjecture 2.

CONJECTURE 2. The following inequality holds true:

$$\left(x^{n-1}L(x)\right)^{(n)} > 0 \qquad (x > 0;\ n \in \mathbb{N}_0). \tag{25}$$

Merkle [7] proved (25) for $n = 0$ and $n = 1$. The inequality (20) shows that (25) is true for $n = 2$.

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Chao-Ping Chen
School of Mathematics and Informatics
Henan Polytechnic University
Jiaozuo City 454000, Henan Province, China
e-mail: chenchaoping@sohu.com

Junesang Choi
Department of Mathematics
Dongguk University
Gyeongju 38066, Republic of Korea
e-mail: junesang@mail.dongguk.ac.kr