Mathematical
Inequalities
& Applications
Volume 20, Number 3 (2017), 755–772
doi:10.7153/mia-20-48

UL'YANOV–TYPE INEQUALITIES AND EMBEDDINGS BETWEEN BESOV SPACES: THE CASE OF PARAMETERS WITH LIMIT VALUES

OSCAR DOMÍNGUEZ

(*Communicated by Z. Ditzian*)

Abstract. In this paper we obtain some limit cases of inequalities of Ul'yanov-type for modulus of smoothness between Lorentz-Zygmund spaces on $\mathbb{T}^n$. Corresponding embedding theorems for the Besov spaces are investigated.

1. Introduction

For a periodic function $f \in L^p(\mathbb{T})$, Ul'yanov proved in [40] the weak-type inequality

$$\omega(f,\delta)_{p^*} \leqslant C\left(\int_0^\delta (t^{-\sigma}\omega(f,t)_p)^{p^*}\frac{dt}{t}\right)^{1/p^*} \tag{1}$$

where $\sigma = 1/p - 1/p^*$ and $1 \leqslant p < p^* < \infty$. Here $\omega(f,\delta)_p = \omega_1(f,\delta)_p$ and the modulus of smoothness of order $k \in \mathbb{N}$ is given by

$$\omega_k(f,\delta)_p = \sup_{|h|\leqslant\delta} \|\Delta_h^k f\|_p \tag{2}$$

with $\Delta_h^k f(x) = \Delta_h^{k-1}(\Delta_h f(x))$, $k > 1$, and $\Delta_h^1 f(x) = \Delta_h f(x) = f(x+h) - f(x)$. It is known that the estimate (1) also holds for modulus of smoothness of an integer order of functions on the n-dimensional torus $\mathbb{T}^n$ (see [18, (2.1)]). The inequality (1) has important applications in the theory of function spaces, approximation theory and interpolation theory. See, for example, the papers by DeVore, Riemenschneider and Sharpley [14], Gol'dman [29], Kolyada [32], Simonov and Tikhonov [35], Trebels [38] and Haroske and Triebel [31].

The estimate (1) gives optimal embedding results for functions with a certain degree of smoothness. For instance, it is sharp over the class of functions satisfying that $\omega(f,\delta)_p \leqslant C\delta^\alpha$, $0 < \alpha < 1$. However, inequality (1) is not sharp in general even for functions $f \in C^\infty(\mathbb{T})$ since $\omega(f,\delta)_r \leqslant C\delta$, $1 \leqslant r < \infty$. To overcome this obstruction, one possibility is to make use of modulus of smoothness $\omega_\kappa(f,\delta)_p$ of fractional order

Mathematics subject classification (2010): 41A17, 46E30, 46E35, 46M35.

Keywords and phrases: Modulus of smoothness, K-functionals, Ul'yanov-type inequalities, generalized Lorentz-Zygmund spaces, Besov spaces, embedding theorems.

The author has been supported in part by the Spanish Ministerio de Economía y Competitividad (MTM2013-42220-P) and by the FPU grant AP2012-0779 of the Ministerio de Economía y Competitividad.

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$\kappa > 0$ (see Section 2 for precise definitions). Thus, a sharp Ul'yanov-type inequality states that if $f \in L^p(\mathbb{T})$ then

$$\omega_\kappa(f,\delta)_{p^*} \leqslant C\left(\int_0^\delta (t^{-\sigma}\omega_{\kappa+\sigma}(f,t)_p)^{p^*}\frac{dt}{t}\right)^{1/p^*}, \tag{3}$$

where $\sigma = 1/p - 1/p^*$ and $1 < p < p^* < \infty$ (see [35] and [38]).

As mentioned above, there exists a strong connection between Ul'yanov-type inequalities and embedding theorems for smooth function spaces. Since Lorentz-Zygmund spaces $L^{p,q}(\log L)^\gamma$ arise in a natural way in some limit cases of embeddings (see, for example, the papers [4] and [8]), it is natural to consider sharp Ul'yanov inequalities in this setting. Recently, Gogatishvili, Opic, Tikhonov and Trebels obtained in [26, Theorem 1.2(a)] the following generalization of (3): if $f \in L^{p,r}(\log L)^{\alpha-\gamma}(\mathbb{T}^n)$ then

$$\omega_\kappa(f,\delta)_{p^*,s;\alpha} \leqslant C\left(\int_0^\delta (t^{-\sigma}(1-\log t)^\gamma \omega_{\kappa+\sigma}(f,t)_{p,r;\alpha-\gamma})^s\frac{dt}{t}\right)^{1/s}, \quad \delta \to 0+,$$

with $\sigma = n(1/p - 1/p^*)$, $1 < p < p^* < \infty$, $1 \leqslant r \leqslant s \leqslant \infty$, $\alpha \in \mathbb{R}$ and $\gamma \geqslant 0$. In the case that $\gamma < 0$, the previous estimate only holds in the trivial case that f is constant. They also studied the limit case $p = p^*$ [26, Theorem 1.2(b)]. Namely, assume that $1 < r \leqslant s < \infty$ and $\gamma > 0$, then

$$\begin{aligned}\omega_\kappa(f,\delta)_{p,s;\alpha} \leqslant C\left(\int_0^\delta ((1-\log t)^{\gamma-1/s}\omega_\kappa(f,t)_{p,r;\alpha-\gamma})^s\frac{dt}{t}\right)^{1/s} \\ + C(1-\log\delta)^\gamma\omega_\kappa(f,\delta)_{p,r;\alpha-\gamma},\end{aligned} \tag{4}$$

when $\delta \to 0+$ and if $1 \leqslant s < r < \infty$ and $\gamma > 1/r - 1/s$, then

$$\begin{aligned}\omega_\kappa(f,\delta)_{p,s;\alpha} \leqslant C\left(\int_0^\delta ((1-\log t)^{\gamma-1/r}\omega_\kappa(f,t)_{p,r;\alpha-\gamma})^s\frac{dt}{t}\right)^{1/s} \\ + C(1-\log\delta)^{\gamma+1/s-1/r}\omega_\kappa(f,\delta)_{p,r;\alpha-\gamma}\end{aligned} \tag{5}$$

when $\delta \to 0+$. As application, they derived in [26, Corollary 3.6] limiting embeddings between Besov-type spaces $B^{(p,r;\beta),\xi}_{\sigma,\gamma}(\mathbb{T}^n)$ based on $L^{p,r}(\log L)^\beta(\mathbb{T}^n)$ with classical smoothness σ and logarithmic smoothness with exponent γ (detailed in Section 2). Namely, if $1 < p < \infty$, $\alpha \in \mathbb{R}$, $\xi > 0$, and either $1 < r \leqslant s < \infty$ or $1 \leqslant s < r < \infty$. Put $\gamma + \max\{1/s - 1/r, 0\} > 0$. Then

$$B^{(p,r;\alpha-\gamma),\xi}_{\lambda,\mu+\gamma+\max\{1/s-1/r,0\}}(\mathbb{T}^n) \hookrightarrow B^{(p,s;\alpha),\xi}_{\lambda,\mu}(\mathbb{T}^n), \quad \lambda > 0, \quad \mu \in \mathbb{R}, \tag{6}$$

and

$$B^{(p,r;\alpha-\gamma),\xi}_{0,\mu+\gamma+\max\{1/\xi-1/s,0\}+\max\{1/s-1/r,0\}}(\mathbb{T}^n) \hookrightarrow B^{(p,s;\alpha),\xi}_{0,\mu}(\mathbb{T}^n), \quad \mu > -1/\xi. \tag{7}$$

It is natural to investigate inequalities (4) and (5) in the limit cases when $\gamma=0$ and $\gamma=1/r-1/s$, respectively. Accordingly, we study in this paper such a question.

To get this aim we shall use generalized Lorentz-Zygmund spaces $L^{p,q}(\log L)^{\gamma}(\log\log L)^{\beta}$ introduced by Edmunds, Gurka and Opic [20, 21]. These spaces allow one more tier than the Lorentz-Zygmund spaces and they have been useful to solve some limiting problems in connection with double exponential integrability of the Bessel potential [20, 21], fine interpolation theorems for quasilinear operators [24] or Hausdorff-Young type estimates for functions in spaces close to L_1 [10].

We show that if $1<r\leqslant s<\infty$, then

$$\omega_{\kappa}(f,\delta)_{p,s;\alpha,1/s}\leqslant C\left(\int_0^{\delta}((1-\log t)^{-1/s}\omega_{\kappa}(f,t)_{p,r;\alpha})^s\frac{dt}{t}\right)^{1/s} + C(1+\log(1-\log\delta))^{1/s}\omega_{\kappa}(f,\delta)_{p,r;\alpha} \tag{8}$$

when $\delta\to 0+$ and if $1\leqslant s<r<\infty$, then

$$\omega_{\kappa}(f,\delta)_{p,s;\alpha,1/r}\leqslant C\left(\int_0^{\delta}((1-\log t)^{-1/s}\omega_{\kappa}(f,t)_{p,r;\alpha+1/s-1/r})^s\frac{dt}{t}\right)^{1/s} + C(1+\log(1-\log\delta))^{1/s}\omega_{\kappa}(f,\delta)_{p,r;\alpha+1/s-1/r} \tag{9}$$

when $\delta\to 0+$. Note that there are differences between (4) (respectively, (5)) and the corresponding limit inequality (8) (respectively, (9)). To be more precise, we have an additional double logarithmic integrability on the left-hand side of (8) and (9) which arises when introducing the term $(1+\log(1-\log\delta))^{1/s}$ on their right-hand sides. To establish these estimates we follow an approach based on limiting interpolation (see [11], [12], [8]), on the characterizations of the K-functional associated to the couple formed by $L^{p,q}(\log L)^{\gamma}(\log\log L)^{\beta}$ and the Riesz-potential space $H_{\lambda}^{p,q;\gamma,\beta}$ and Nikolskii-type inequalities for trigonometric polynomials.

As application, we derive limiting embeddings corresponding to (6) and (7) when $\gamma+\max\{1/s-1/r,0\}=0$. Namely, if $1<p<\infty$, $\alpha\in\mathbb{R}$, $\xi>0$, and either $1<r\leqslant s<\infty$ or $1\leqslant s<r<\infty$, then

$$B_{\lambda,\mu,1/s}^{(p,r;\alpha+\max\{1/s-1/r,0\}),\xi}(\mathbb{T}^n)\hookrightarrow B_{\lambda,\mu}^{(p,s;\alpha,1/\max\{r,s\}),\xi}(\mathbb{T}^n),\quad \lambda>0,\quad \mu\in\mathbb{R}, \tag{10}$$

and

$$B_{0,\mu+\max\{1/\xi-1/s,0\},1/s}^{(p,r;\alpha+\max\{1/s-1/r,0\}),\xi}(\mathbb{T}^n)\hookrightarrow B_{0,\mu}^{(p,s;\alpha,1/\max\{r,s\}),\xi}(\mathbb{T}^n),\quad \mu>-1/\xi. \tag{11}$$

Note that the double logarithmic components in smoothness on the left-hand side of (10) and (11) lead to double logarithmic integrability on their right-hand sides. Furthermore, we also obtain the limit case of (11) when $\mu=-1/\xi$. We show that

$$B_{0,-1/\xi+\max\{1/\xi-1/s,0\},1/s+\max\{1/\xi-1/s,0\}}^{(p,r;\alpha+\max\{1/s-1/r,0\}),\xi}(\mathbb{T}^n)\hookrightarrow B_{0,-1/\xi}^{(p,s;\alpha,1/\max\{r,s\}),\xi}(\mathbb{T}^n).$$

The plan of the paper is as follows. In Section 2 we review the interpolation methods that we use in the paper, equivalence results for the modulus of smoothness,

K-functionals and realization functionals, and we introduce the function spaces that we consider here. Section 3 is devoted to Ul'yanov-type inequalities in the case of parameters with limit values. Needed versions of Nikolskii-type inequalities for trigonometric polynomials are also given. Finally, in Section 4 we establish limiting embeddings between Besov-type spaces.

2. Preliminaries

In what follows, if X,Y are non-negative quantities depending on certain parameters, we write $X \lesssim Y$ if there is a constant $c>0$ independent of the parameters in X and Y such that $X \leqslant cY$. If $X \lesssim Y$ and $Y \lesssim X$, we write $X \sim Y$.

Let (A_0,A_1) be a couple of Banach spaces with $A_1 \hookrightarrow A_0$, that is, the embedding from A_1 into A_0 is continuous. For $t>0$, Peetre's K-functional is defined by

$$K(t,a)=K(t,a;A_0,A_1)=\inf\{\|a_0\|_{A_0}+t\|a_1\|_{A_1} : a=a_0+a_1, a_j \in A_j\}, \quad a \in A_0.$$

For $0<\theta<1$ and $1 \leqslant q<\infty$, the real interpolation space $(A_0,A_1)_{\theta,q}$ is the set of all elements $a \in A_0$ having a finite norm

$$\|a\|_{(A_0,A_1)_{\theta,q}} = \left(\int_0^\infty (t^{-\theta}K(t,a))^q \frac{dt}{t}\right)^{1/q}.$$

See [3, 39, 2].

The following extension of the real interpolation method is useful. Let b be a slowly varying function on $(0,\infty)$ (see, for example, [19]). Define the interpolation space $(A_0,A_1)_{\theta,q,b}$ by

$$(A_0,A_1)_{\theta,q,b} = \left\{ a \in A_0 : \|a\|_{(A_0,A_1)_{\theta,q,b}} = \left(\int_0^\infty (t^{-\theta}b(t)K(t,a))^q \frac{dt}{t}\right)^{1/q} < \infty \right\}.$$

See [30, 27]. Under suitable assumptions on b and q, spaces $(A_0,A_1)_{\theta,q,b}$ are well-defined even if $\theta=0$ or $\theta=1$. Put $\ell(t)=1+|\log t|$ and $\ell\ell(t)=\ell(\ell(t))=1+\log(1+|\log t|)$. In the particular case that $b(t)=\ell^\gamma(t), \gamma \in \mathbb{R}$ (respectively, $b(t)=\ell^\gamma(t)\ell\ell^\beta(t)$, $\gamma,\beta \in \mathbb{R}$) we simply write $(A_0,A_1)_{\theta,q,\gamma}$ (respectively, $(A_0,A_1)_{\theta,q,\gamma,\beta}$) instead of $(A_0,A_1)_{\theta,q,b}$. For $\mathbb{A}=(\alpha_0,\alpha_\infty) \in \mathbb{R}^2$, let

$$\ell^{\mathbb{A}}(t) = \begin{cases} \ell^{\alpha_0}(t) & \text{for } t \in (0,1], \\ \ell^{\alpha_\infty}(t) & \text{for } t \in (1,\infty). \end{cases}$$

In the special case $b(t)=\ell^{\mathbb{A}}(t)$, the space $(A_0,A_1)_{\theta,q,b}$ coincides with the logarithmic space $(A_0,A_1)_{\theta,q,\mathbb{A}}$ considered in [23, 25].

For $\eta \in \mathbb{R}$, the limit interpolation space $(A_0,A_1)_{(0,\eta),q}$ is formed by all $a \in A_0$ for which

$$\|a\|_{(A_0,A_1)_{(0,\eta),q}} = \left(\int_0^1 (\ell^\eta(t)K(t,a))^q \frac{dt}{t}\right)^{1/q} < \infty$$

(see [11, 8]). Note that $(A_0,A_1)_{(0,\eta),q} = A_0$ if $\eta < -1/q$ and so, the only case of interest is when $\eta \geqslant -1/q$.

It is clear that any of these three interpolation methods has the interpolation property for bounded linear operators.

The following reiteration formula (with a slightly different notation) was proved in [8, Lemma 2.5] by using the connection between limiting interpolation spaces $(A_0,A_1)_{(0,\eta),q}$ and logarithmic spaces $(A_0,A_1)_{\theta,q,\mathbb{A}}$ with $\theta = 0$ [22, Proposition 1], and reiteration results for logarithmic spaces [23, Theorems 5.9*, 4.7*, 5.7 and 4.7].

Let $\eta > -1/q$, then

$$(A_0,A_1)_{\theta,q,\eta+1/\min\{p,q\}} \hookrightarrow ((A_0,A_1)_{\theta,p},A_1)_{(0,\eta),q} \hookrightarrow (A_0,A_1)_{\theta,q,\eta+1/\max\{p,q\}}.$$

The limit case when $\eta = -1/q$ in the previous formula was obtained very recently in [10, Lemma 3.1]. In this case, we have that

$$\begin{aligned}(A_0,A_1)_{\theta,q,-1/q+1/\min\{p,q\},1/\min\{p,q\}} &\hookrightarrow ((A_0,A_1)_{\theta,p},A_1)_{(0,-1/q),q} \\ &\hookrightarrow (A_0,A_1)_{\theta,q,-1/q+1/\max\{p,q\},1/\max\{p,q\}}.\end{aligned} \tag{12}$$

Let $\mathbb{T}^n$ be the n-dimensional torus. Let $1 \leqslant p,q < \infty$ and $\gamma,\beta \in \mathbb{R}$. The generalized Lorentz-Zygmund space $L^{p,q}(\log L)^{\gamma}(\log\log L)^{\beta}(\mathbb{T}^n)$ is the set of all measurable functions f on $\mathbb{T}^n$ such that

$$\|f\|_{p,q;\gamma,\beta} = \left(\int_0^1 (t^{1/p}\ell^{\gamma}(t)\ell\ell^{\beta}(t)f^*(t))^q \frac{dt}{t}\right)^{1/q}$$

is finite. See [20, 21] (see also [19, 3.4.1]). Here f^* denotes the non-increasing rearrangement of f. The generalized Lorentz-Zygmund spaces are rearrangement-invariant spaces if $p > 1$. If $\beta = 0$ we get the Lorentz-Zygmund space $L^{p,q}(\log L)^{\gamma}(\mathbb{T}^n)$. In the special case when $\gamma = 0$, we obtain the Lorentz space $L^{p,q}(\mathbb{T}^n)$, which coincides with the Lebesgue space $L^p(\mathbb{T}^n)$ if $p = q$.

Next we recall some notions of the theory of multiple Fourier series. See [36, 34]. To every $f \in L^1(\mathbb{T}^n)$ we assign the Fourier series

$$f(x) \sim \sum_{m\in\mathbb{Z}^n} \widehat{f}_m e^{2\pi imx}$$

where

$$\widehat{f}_m = \int_{\mathbb{T}^n} f(x)e^{-2\pi imx}dx, \quad m \in \mathbb{Z}^n,$$

are the Fourier coefficients of f. For $N \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}$, we denote by $\mathscr{T}_N$ the set formed by all trigonometric polynomials of degree less than or equal to N. To be more precise,

$$\mathscr{T}_N = \left\{\sum_{|m|\leqslant N} c_m e^{2\pi imx} : c_m \in \mathbb{C}\right\}.$$

The N-th error of approximation of $f \in L^{p,q}(\log L)^{\gamma}(\log\log L)^{\beta}(\mathbb{T}^n)$ by elements from $\mathscr{T}_N$ is given by

$$E_N(f)_{p,q;\gamma,\beta} = \inf\{\|f-g\|_{p,q;\gamma,\beta} : g \in \mathscr{T}_N\}.$$

Let $\lambda > 0$. The periodic Riesz-potential space $H^{p,q;\gamma,\beta}_{\lambda}(\mathbb{T}^n)$ is formed by all $f \in L^{p,q}(\log L)^{\gamma}(\log\log L)^{\beta}(\mathbb{T}^n)$ such that

$$|f|_{H^{p,q;\gamma,\beta}_{\lambda}} = \|(-\Delta)^{\lambda/2} f\|_{p,q;\gamma,\beta} < \infty.$$

Here $(-\Delta)^{\lambda/2}$ denotes the Riesz potential operator which is given by

$$(-\Delta)^{\lambda/2} f(x) \sim \sum_{m \in \mathbb{Z}^n} |m|^{\lambda} \widehat{f}_m e^{2\pi i m x}.$$

The modulus of smoothness of fractional order $\kappa > 0$ of $f \in L^p(\mathbb{T}^n)$ is given by [5, 41]

$$\omega_{\kappa}(f,\delta)_p = \sup_{|h| \leqslant \delta} \|\Delta^{\kappa}_h f\|_p, \quad \delta > 0,$$

where

$$\Delta^{\kappa}_h f(x) = \sum_{\nu=0}^{\infty} (-1)^{\nu} \binom{\kappa}{\nu} f(x + \nu h)$$

is the κ-th difference of f with step h at the point x. Here,

$$\binom{\kappa}{\nu} = \frac{\kappa(\kappa-1)\cdots(\kappa-\nu+1)}{\nu!} \quad \text{for} \quad \nu > 1,$$

$\binom{\kappa}{1} = \kappa$ and $\binom{\kappa}{0} = 1$. It is clear that if $\kappa \in \mathbb{N}$ then we have the classical modulus of smoothness given in (2). Analogously, we can define the modulus of smoothness of $f \in L^{p,q}(\log L)^{\gamma}(\log\log L)^{\beta}(\mathbb{T}^n)$ by

$$\omega_{\kappa}(f,\delta)_{p,q;\gamma,\beta} = \sup_{|h| \leqslant \delta} \|\Delta^{\kappa}_h f\|_{p,q;\gamma,\beta}, \quad \delta > 0.$$

Consider the generalized de la Vallée-Poussin means V_t defined by

$$V_t f(x) = \sum_{|m| \leqslant 2/t} \chi(t|m|) \widehat{f}_m e^{2\pi i m x}, \quad t > 0,$$

where $\chi \in C^{\infty}[0,\infty)$ with $\chi(u) = 1$ for $0 \leqslant u \leqslant 1$ and $\chi(u) = 0$ for $u \geqslant 2$.

The K-functional associated to the couple

$$(L^{p,q}(\log L)^{\gamma}(\log\log L)^{\beta}(\mathbb{T}^n), H^{p,q;\gamma,\beta}_{\lambda}(\mathbb{T}^n))$$

plays an important role in our arguments. The following characterizations hold

$$\begin{aligned} &K(t^{\lambda}, f; L^{p,q}(\log L)^{\gamma}(\log\log L)^{\beta}(\mathbb{T}^n), H^{p,q;\gamma,\beta}_{\lambda}(\mathbb{T}^n)) \\ &\sim \|f - V_t f\|_{p,q;\gamma,\beta} + t^{\lambda} |V_t f|_{H^{p,q;\gamma,\beta}_{\lambda}} \sim \omega_{\lambda}(f,t)_{p,q;\gamma,\beta} \end{aligned} \tag{13}$$

for $1 < p < \infty$. These equivalences were shown in [26, Lemma 3.1] for Lorentz-Zygmund spaces, but the method of proof carries over to generalized Lorentz-Zygmund spaces.

Let $\sigma \geqslant 0$, $1 \leqslant p,r < \infty$, $\xi > 0$ and $\beta,\gamma,\delta,\eta \in \mathbb{R}$. The Besov-type space $B^{(p,r;\beta,\eta),\xi}_{\sigma,\gamma,\delta}(\mathbb{T}^n)$ consists of all $f \in L^{p,r}(\log L)^{\beta}(\log\log L)^{\eta}(\mathbb{T}^n)$ having a finite semi-norm

$$|f|_{B^{(p,r;\beta,\eta),\xi}_{\sigma,\gamma,\delta}} = \left(\int_0^1 (t^{-\sigma}\ell^{\gamma}(t)\ell\ell^{\delta}(t)\omega_{\lambda}(f,t)_{p,r;\beta,\eta})^{\xi}\frac{dt}{t}\right)^{1/\xi}$$

where $\lambda > \sigma$. It becomes a Banach space when equipped with the norm

$$\|f\|_{B^{(p,r;\beta,\eta),\xi}_{\sigma,\gamma,\delta}} = \|f\|_{p,r;\beta,\eta} + |f|_{B^{(p,r;\beta,\eta),\xi}_{\sigma,\gamma,\delta}}.$$

The notation is justified by the fact that the definition is independent of λ (see [26, pages 1030 and 1041] and [9, Theorem 3.1]). If $\eta = 0$ (respectively, $\delta = 0$) then we simply write $B^{(p,r;\beta),\xi}_{\sigma,\gamma,\delta}(\mathbb{T}^n)$ (respectively, $B^{(p,r;\beta,\eta),\xi}_{\sigma,\gamma}(\mathbb{T}^n)$) to denote the corresponding Besov space. In the case that $\sigma = 0$, we get Besov-type spaces involving only logarithmic smoothness $\ell^{\gamma}(t)\ell\ell^{\delta}(t)$ (see [14], [6], [7], [8], [9], [10] and the references within). Note that in this limit case, we are only interested when $\gamma \geqslant -1/\xi$, otherwise $B^{(p,r;\beta,\eta),\xi}_{0,\gamma,\delta}(\mathbb{T}^n) = L^{p,r}(\log L)^{\beta}(\log\log L)^{\eta}(\mathbb{T}^n)$.

3. Ul'yanov-type inequalities

We start with a limit case left open in [26, Lemma 3.5].

LEMMA 1. *Let $1 < p,r < \infty$, $1 \leqslant s < \infty$ and $\alpha \in \mathbb{R}$. Then,*

$$B^{(p,r;\alpha),s}_{0,-1/s}(\mathbb{T}^n) \hookrightarrow L^{p,s}(\log L)^{-1/s+\alpha+1/\max\{r,s\}}(\log\log L)^{1/\max\{r,s\}}(\mathbb{T}^n).$$

In particular, we have

$$B^{(p,r;\alpha),s}_{0,-1/s}(\mathbb{T}^n) \hookrightarrow L^{p,s}(\log L)^{\alpha}(\log\log L)^{1/s}(\mathbb{T}^n) \quad \text{if} \quad 1 < r \leqslant s < \infty,$$

and

$$B^{(p,r;\alpha+1/s-1/r),s}_{0,-1/s}(\mathbb{T}^n) \hookrightarrow L^{p,s}(\log L)^{\alpha}(\log\log L)^{1/r}(\mathbb{T}^n) \quad \text{if} \quad 1 \leqslant s < r < \infty.$$

Proof. Let $W^1L^{p,r}(\log L)^{\alpha}(\mathbb{T}^n)$ be the Sobolev space built upon $L^{p,r}(\log L)^{\alpha}(\mathbb{T}^n)$ (see [19, 3.6.1]). It follows from

$$\begin{aligned} &K(t,f;L^{p,r}(\log L)^{\alpha}(\mathbb{T}^n),W^1L^{p,r}(\log L)^{\alpha}(\mathbb{T}^n)) \\ &\sim t\|f\|_{p,r;\alpha} + \omega_1(f,t)_{p,r;\alpha}, \quad 0 < t < 1, \end{aligned}$$

(see [28, (1.6)]) that

$$B^{(p,r;\alpha),s}_{0,-1/s}(\mathbb{T}^n) = (L^{p,r}(\log L)^{\alpha}(\mathbb{T}^n),W^1L^{p,r}(\log L)^{\alpha}(\mathbb{T}^n))_{(0,-1/s),s}. \tag{14}$$

If $n \geqslant 2$, choose p_1 such that

$$\max\left\{1, \frac{np}{n+p}\right\} < p_1 < \min\{p,n\}.$$

Since $L^{p,r}(\log L)^{\alpha}(\mathbb{T}^n) \hookrightarrow L^{p_1}(\mathbb{T}^n)$ (see [1, Theorem 9.1]) we derive that $W^1L^{p,r}(\log L)^{\alpha}(\mathbb{T}^n) \hookrightarrow W^1L^{p_1}(\mathbb{T}^n)$. By the Sobolev embedding theorem

$$W^1L^{p_1}(\mathbb{T}^n) \hookrightarrow L^{p_1^*}(\mathbb{T}^n), \;\; 1/p_1^* = 1/p_1 - 1/n,$$

and then,

$$W^1L^{p,r}(\log L)^{\alpha}(\mathbb{T}^n) \hookrightarrow L^{p_1^*}(\mathbb{T}^n).$$

If $n=1$, it is clear that $W^1L^1(\mathbb{T}) \hookrightarrow C(\mathbb{T})$ and as a consequence,

$$W^1L^{p,r}(\log L)^{\alpha}(\mathbb{T}) \hookrightarrow W^1L^1(\mathbb{T}) \hookrightarrow L^{p_1^*}(\mathbb{T})$$

for any $p_1^* \in [1,\infty)$. Hence, there exists p_1^* such that $p < p_1^*$ and $W^1L^{p,r}(\log L)^{\alpha}(\mathbb{T}^n) \hookrightarrow L^{p_1^*}(\mathbb{T}^n)$.

Let θ be given by the equation $1/p = 1-\theta+\theta/p_1^*$ and put $\alpha_0 = \alpha/(1-\theta)$. Using the characterization (14) and the reiteration formula (12), we obtain that

$$\begin{aligned}
B^{(p,r;\alpha),s}_{0,-1/s}(\mathbb{T}^n) &\hookrightarrow (L^{p,r}(\log L)^{\alpha}(\mathbb{T}^n), L^{p_1^*}(\mathbb{T}^n))_{(0,-1/s),s} \\
&= ((L^{1,1}(\log L)^{\alpha_0}(\mathbb{T}^n), L^{p_1^*}(\mathbb{T}^n))_{\theta,r}, L^{p_1^*}(\mathbb{T}^n))_{(0,-1/s),s} \\
&\hookrightarrow (L^{1,1}(\log L)^{\alpha_0}(\mathbb{T}^n), L^{p_1^*}(\mathbb{T}^n))_{\theta,s,-1/s+1/\max\{r,s\},1/\max\{r,s\}} \\
&= L^{p,s}(\log L)^{-1/s+\alpha+1/\max\{r,s\}}(\log\log L)^{1/\max\{r,s\}}(\mathbb{T}^n)
\end{aligned}$$

where in the last equivalence we have used [27, Lemma 5.5]. □

REMARK 1. In the case of Besov spaces based on Lebesgue spaces on the one-dimensional torus $\mathbb{T}$, that is, when $p=r$ and $\alpha=0$, the embedding given by the previous result was shown in [10, Theorem 4.2].

Let $1<p<\infty$, $1 \leqslant r,s \leqslant \infty$ and $\alpha \in \mathbb{R}$. The following Nikolskii-type inequalities were proved in [26, Lemma 3.4 and (3.16)],

$$\|g\|_{p,s;\alpha} \lesssim \ell^{\gamma}(N)\|g\|_{p,r;\alpha-\gamma} \quad \text{if} \quad r \leqslant s \quad \text{and} \quad \gamma > 0, \tag{15}$$

and

$$\|g\|_{p,s;\alpha} \lesssim \ell^{\gamma+1/s-1/r}(N)\|g\|_{p,r;\alpha-\gamma} \quad \text{if} \quad s<r \quad \text{and} \quad \gamma > 1/r-1/s, \tag{16}$$

for all $g \in \mathscr{T}_N$. Next we establish the corresponding estimates in the limit cases of (15) and (16). More precisely, when $\gamma = 0$ in (15) and $\gamma = 1/r-1/s$ in (16).

LEMMA 2. *Let* $1<p<\infty$, $1 \leqslant r \leqslant s < \infty$ *and* $\alpha \in \mathbb{R}$. *Then,*

$$\|g\|_{p,s;\alpha,1/s} \lesssim \ell\ell^{1/s}(N)\|g\|_{p,r;\alpha}$$

for all $g \in \mathscr{T}_N$.

Proof. We have that (see [17] and [33])

$$g^*(0) \leqslant c g^*(N^{-n}) \tag{17}$$

where the constant c is independent of g and N (but depends on the dimension n). Then

$$\begin{aligned}\|g\|_{p,s;\alpha,1/s}^s &= \int_0^{N^{-n}} (t^{1/p}\ell^\alpha(t)\ell\ell^{1/s}(t)g^*(t))^s \frac{dt}{t} \\ &\quad + \int_{N^{-n}}^1 (t^{1/p}\ell^\alpha(t)\ell\ell^{1/s}(t)g^*(t))^s \frac{dt}{t} \\ &= I_1 + I_2.\end{aligned}$$

Using (17) and [19, Proposition 3.4.33/(v)], we derive

$$\begin{aligned}I_1 &\leqslant g^*(0)^s \int_0^{N^{-n}} (t^{1/p}\ell^\alpha(t)\ell\ell^{1/s}(t))^s \frac{dt}{t} \\ &\lesssim g^*(N^{-n})^s N^{-ns/p}\ell^{\alpha s}(N)\ell\ell(N) \\ &\sim g^*(N^{-n})^s \ell\ell(N) \left(\int_0^{N^{-n}} (t^{1/p}\ell^\alpha(t))^r \frac{dt}{t}\right)^{s/r} \\ &\leqslant \ell\ell(N) \left(\int_0^{N^{-n}} (t^{1/p}\ell^\alpha(t)g^*(t))^r \frac{dt}{t}\right)^{s/r} \\ &\leqslant \ell\ell(N)\|g\|_{p,r;\alpha}^s.\end{aligned}$$

As for I_2, we have that

$$\begin{aligned}I_2 &\leqslant \sup_{N^{-n}\leqslant u\leqslant 1} \ell\ell(u) \int_{N^{-n}}^1 (t^{1/p}\ell^\alpha(t)g^*(t))^s \frac{dt}{t} \\ &\lesssim \ell\ell(N)\|g\|_{p,s;\alpha}^s \lesssim \ell\ell(N)\|g\|_{p,r;\alpha}^s\end{aligned}$$

where in the last estimate we have used the fact that $L^{p,r}(\log L)^\alpha(\mathbb{T}^n) \hookrightarrow L^{p,s}(\log L)^\alpha(\mathbb{T}^n)$ because $r \leqslant s$. Consequently,

$$\|g\|_{p,s;\alpha,1/s}^s \lesssim \ell\ell(N)\|g\|_{p,r;\alpha}^s. \quad \square$$

LEMMA 3. *Let* $1 < p < \infty$, $1 \leqslant s < r < \infty$ *and* $\alpha \in \mathbb{R}$. *Then,*

$$\|g\|_{p,s;\alpha,1/r} \lesssim \ell\ell^{1/s}(N)\|g\|_{p,r;\alpha+1/s-1/r}$$

for all $g \in \mathscr{T}_N$.

Proof. We have that

$$\begin{aligned}\|g\|_{p,s;\alpha,1/r}^s &= \int_0^{N^{-n}} (t^{1/p}\ell^\alpha(t)\ell\ell^{1/r}(t)g^*(t))^s \frac{dt}{t} \\ &\quad + \int_{N^{-n}}^1 (t^{1/p}\ell^\alpha(t)\ell\ell^{1/r}(t)g^*(t))^s \frac{dt}{t} \\ &= I_1 + I_2.\end{aligned}$$

We estimate I_1 by using (17),

$$\begin{aligned} I_1 &\leqslant g^*(0)^s \int_0^{N^{-n}} (t^{1/p}\ell^{\alpha}(t)\ell\ell^{1/r}(t))^s \frac{dt}{t} \\ &\lesssim g^*(N^{-n})^s N^{-ns/p}\ell^{\alpha s}(N)\ell\ell^{s/r}(N) \\ &\sim g^*(N^{-n})^s \ell^{(1/r-1/s)s}(N)\ell\ell^{s/r}(N)\left(\int_0^{N^{-n}} (t^{1/p}\ell^{\alpha+1/s-1/r}(t))^r \frac{dt}{t}\right)^{s/r} \\ &\leqslant \ell^{(1/r-1/s)s}(N)\ell\ell^{s/r}(N)\left(\int_0^{N^{-n}} (t^{1/p}\ell^{\alpha+1/s-1/r}(t)g^*(t))^r \frac{dt}{t}\right)^{s/r} \\ &\leqslant \ell^{(1/r-1/s)s}(N)\ell\ell^{s/r}(N)\|g\|^s_{p,r;\alpha+1/s-1/r} \\ &\leqslant \ell\ell(N)\|g\|^s_{p,r;\alpha+1/s-1/r} \end{aligned}$$

since $s<r$. Finally, by Hölder's inequality,

$$\begin{aligned} I_2 &= \int_{N^{-n}}^1 (t^{1/p-1/r}\ell^{\alpha+1/s-1/r}(t)g^*(t))^s (t^{1/r-1/s}\ell^{1/r-1/s}(t)\ell\ell^{1/r}(t))^s dt \\ &\leqslant \left(\int_{N^{-n}}^1 (t^{1/p-1/r}\ell^{\alpha+1/s-1/r}(t)g^*(t))^r dt\right)^{s/r} \\ &\quad \times \left(\int_{N^{-n}}^1 (t^{1/r-1/s}\ell^{1/r-1/s}(t)\ell\ell^{1/r}(t))^{\frac{sr}{r-s}} dt\right)^{1-s/r} \\ &\leqslant \left(\int_{N^{-n}}^1 \ell\ell^{\frac{s}{r-s}}(t)\frac{dt}{t\ell(t)}\right)^{1-s/r} \|g\|^s_{p,r;\alpha+1/s-1/r} \\ &\lesssim \ell\ell(N)\|g\|^s_{p,r;\alpha+1/s-1/r}. \quad \square \end{aligned}$$

Now we can state the main result of this section.

THEOREM 1. *Let* $\kappa>0$, $1<p<\infty$ *and* $\alpha\in\mathbb{R}$.

(i) *If* $1<r\leqslant s<\infty$ *then*

$$\omega_\kappa(f,\delta)_{p,s;\alpha,1/s} \lesssim \left(\int_0^\delta (\ell^{-1/s}(t)\omega_\kappa(f,t)_{p,r;\alpha})^s \frac{dt}{t}\right)^{1/s} + \ell\ell^{1/s}(\delta)\omega_\kappa(f,\delta)_{p,r;\alpha} \tag{18}$$

for all $f\in L^{p,r}(\log L)^{\alpha}(\mathbb{T}^n)$ *when* $\delta\to 0+$.

(ii) *If* $1\leqslant s<r<\infty$ *then*

$$\omega_\kappa(f,\delta)_{p,s;\alpha,1/r} \lesssim \left(\int_0^\delta (\ell^{-1/s}(t)\omega_\kappa(f,t)_{p,r;\alpha+1/s-1/r})^s \frac{dt}{t}\right)^{1/s} + \ell\ell^{1/s}(\delta)\omega_\kappa(f,\delta)_{p,r;\alpha+1/s-1/r} \tag{19}$$

for all $f\in L^{p,r}(\log L)^{\alpha+1/s-1/r}(\mathbb{T}^n)$ *when* $\delta\to 0+$.

Proof. (i) Assume first that $r \leqslant s$. For $N \in \mathbb{N}$, by (13),

$$\omega_\kappa(f,1/N)_{p,s;\alpha,1/s} \sim \|f - V_{1/N}f\|_{p,s;\alpha,1/s} + N^{-\kappa}|V_{1/N}f|_{H_\kappa^{p,s;\alpha,1/s}} \\ = I + II. \tag{20}$$

Applying the estimate given by Lemma 2 for $(-\Delta)^{\kappa/2}V_{1/N}f \in \mathscr{T}_{2N}$ we obtain that

$$\begin{aligned}|V_{1/N}f|_{H_\kappa^{p,s;\alpha,1/s}} &= \|(-\Delta)^{\kappa/2}V_{1/N}f\|_{p,s;\alpha,1/s} \\ &\lesssim \ell\ell^{1/s}(N)\|(-\Delta)^{\kappa/2}V_{1/N}f\|_{p,r;\alpha} \\ &= \ell\ell^{1/s}(N)|V_{1/N}f|_{H_\kappa^{p,r;\alpha}}.\end{aligned}$$

Then, using again (13) we derive

$$\begin{aligned}II &\lesssim N^{-\kappa}\ell\ell^{1/s}(N)|V_{1/N}f|_{H_\kappa^{p,r;\alpha}} \\ &\lesssim \ell\ell^{1/s}(N)\omega_\kappa(f,1/N)_{p,r;\alpha}.\end{aligned} \tag{21}$$

We proceed to estimate I. Under the assumptions on the parameters, it follows from Lemma 1 that

$$\begin{aligned}\|f\|_{p,s;\alpha,1/s} &\lesssim \|f\|_{p,r;\alpha} + \left(\int_0^1 (\ell^{-1/s}(t)\omega_1(f,t)_{p,r;\alpha})^s \frac{dt}{t}\right)^{1/s} \\ &\sim \|f\|_{p,r;\alpha} + \left(\sum_{j=1}^{\infty} (\ell^{-1/s}(j)E_j(f)_{p,r;\alpha})^s \frac{1}{j}\right)^{1/s}\end{aligned} \tag{22}$$

where the last equivalence is a consequence of Jackson and Bernstein inequalities in Lorentz-Zygmund spaces (see [26, (3.5)], [16, Theorem 2.1] and [15, Theorem 2.3]) in the same way as it is done in [26, page 1043]. For $f \in L^{p,r}(\log L)^\alpha(\mathbb{T}^n)$, we denote by $T_N^{p,r;\alpha}(f) \in \mathscr{T}_N$ the best approximant of f by elements from $\mathscr{T}_N$ (see [13, Theorem 3.1.1]). Set $g = f - T_N^{p,r;\alpha}(f)$. By construction,

$$E_j(g)_{p,r;\alpha} \leqslant \|g\|_{p,r;\alpha} = \|f - T_N^{p,r;\alpha}(f)\|_{p,r;\alpha} = E_N(f)_{p,r;\alpha} \tag{23}$$

for $0 \leqslant j \leqslant N$. Next we show that

$$E_j(g)_{p,r;\alpha} = E_j(f)_{p,r;\alpha} \quad \text{if} \quad j \geqslant N. \tag{24}$$

Indeed, for arbitrary $t \in \mathscr{T}_j$, we have that

$$\|g - t\|_{p,r;\alpha} = \|f - (T_N^{p,r;\alpha}(f) + t)\|_{p,r;\alpha} \geqslant E_j(f)_{p,r;\alpha}.$$

Taking the infimum we derive that $E_j(g)_{p,r;\alpha} \geqslant E_j(f)_{p,r;\alpha}$. Conversely,

$$\|f - t\|_{p,r;\alpha} = \|g + T_N^{p,r;\alpha}(f) - t\|_{p,r;\alpha} \geqslant E_j(g)_{p,r;\alpha}$$

which implies that $E_j(f)_{p,r;\alpha} \geqslant E_j(g)_{p,r;\alpha}$. By (22)-(24) and Jackson inequality given in [16, Theorem 2.1], we get

$$\begin{aligned}
E_N(f)_{p,s;\alpha,1/s} &\leqslant \|f - T_N^{p,r;\alpha}(f)\|_{p,s;\alpha,1/s} \\
&\lesssim \|f - T_N^{p,r;\alpha}(f)\|_{p,r;\alpha} + \left(\sum_{j=1}^{\infty} (\ell^{-1/s}(j) E_j(f - T_N^{p,r;\alpha}(f))_{p,r;\alpha})^s \frac{1}{j}\right)^{1/s} \\
&\lesssim E_N(f)_{p,r;\alpha} + \left(\sum_{j=1}^{N} \ell^{-1}(j) \frac{1}{j}\right)^{1/s} E_N(f)_{p,r;\alpha} + \left(\sum_{j=N+1}^{\infty} (\ell^{-1/s}(j) E_j(f)_{p,r;\alpha})^s \frac{1}{j}\right)^{1/s} \\
&\sim \ell\ell^{1/s}(N) E_N(f)_{p,r;\alpha} + \left(\sum_{j=N+1}^{\infty} (\ell^{-1/s}(j) E_j(f)_{p,r;\alpha})^s \frac{1}{j}\right)^{1/s} \\
&\lesssim \ell\ell^{1/s}(N) \omega_\kappa(f, 1/N)_{p,r;\alpha} + \left(\sum_{j=N+1}^{\infty} (\ell^{-1/s}(j) \omega_\kappa(f, 1/j)_{p,r;\alpha})^s \frac{1}{j}\right)^{1/s} \\
&\sim \ell\ell^{1/s}(N) \omega_\kappa(f, 1/N)_{p,r;\alpha} + \left(\int_0^{1/N} (\ell^{-1/s}(t) \omega_\kappa(f,t)_{p,r;\alpha})^s \frac{dt}{t}\right)^{1/s}
\end{aligned}$$

where the latter estimate is due to the monotonicity properties of the modulus of smoothness. We use the fact that the de la Vallée-Poussin sum satisfies

$$\|V_t f\|_{p,s;\alpha,\beta} \leqslant C \|f\|_{p,s;\alpha,\beta} \text{ for all } f \in L^{p,s}(\log L)^{\alpha}(\log\log L)^{\beta}(\mathbb{T}^n)$$

with constant $C > 0$ independent of f and t, and $V_{1/N} t = t$ for all $t \in \mathscr{T}_N$. Therefore, we obtain

$$\begin{aligned}
\|f - V_{1/N} f\|_{p,s;\alpha,1/s} &\leqslant \|f - T_N^{p,s;\alpha,1/s}(f)\|_{p,s;\alpha,1/s} + \|T_N^{p,s;\alpha,1/s}(f) - V_{1/N} f\|_{p,s;\alpha,1/s} \\
&= \|f - T_N^{p,s;\alpha,1/s}(f)\|_{p,s;\alpha,1/s} + \|V_{1/N}(T_N^{p,s;\alpha,1/s}(f) - f)\|_{p,s;\alpha,1/s} \\
&\lesssim \|f - T_N^{p,s;\alpha,1/s}(f)\|_{p,s;\alpha,1/s} \\
&= E_N(f)_{p,s;\alpha,1/s} \\
&\lesssim \ell\ell^{1/s}(N) \omega_\kappa(f, 1/N)_{p,r;\alpha} + \left(\int_0^{1/N} (\ell^{-1/s}(t) \omega_\kappa(f,t)_{p,r;\alpha})^s \frac{dt}{t}\right)^{1/s}.
\end{aligned}$$

Inserting this estimate and (21) in (20), one has

$$\begin{aligned}
\omega_\kappa(f, 1/N)_{p,s;\alpha,1/s} \lesssim &\left(\int_0^{1/N} (\ell^{-1/s}(t) \omega_\kappa(f,t)_{p,r;\alpha})^s \frac{dt}{t}\right)^{1/s} \\
&+ \ell\ell^{1/s}(N) \omega_\kappa(f, 1/N)_{p,r;\alpha}.
\end{aligned}$$

This establishes (i).

(ii) Suppose now that $s < r$. By (13) we have

$$\begin{aligned}
\omega_\kappa(f, 1/N)_{p,s;\alpha,1/r} &\sim \|f - V_{1/N} f\|_{p,s;\alpha,1/r} + N^{-\kappa} |V_{1/N} f|_{H_\kappa^{p,s;\alpha,1/r}} \qquad (25) \\
&= I + II.
\end{aligned}$$

The same argument as in (i) but now applying Lemma 3 instead of Lemma 2 yields that

$$II \lesssim \ell\ell^{1/s}(N)\omega_\kappa(f,1/N)_{p,r;\alpha+1/s-1/r}. \tag{26}$$

Let us estimate I. Using Lemma 1, we derive

$$\|f\|_{p,s;\alpha,1/r} \lesssim \|f\|_{p,r;\alpha+1/s-1/r} + \left(\sum_{j=1}^{\infty}(\ell^{-1/s}(j)E_j(f)_{p,r;\alpha+1/s-1/r})^s\frac{1}{j}\right)^{1/s}. \tag{27}$$

The estimate (27) together with the corresponding formulae to (23) and (24) in $L^{p,r}(\log L)^{\alpha+1/s-1/r}(\mathbb{T}^n)$ imply that

$$\begin{aligned}
E_N(f)_{p,s;\alpha,1/r} &\leqslant \|f - T_N^{p,r;\alpha+1/s-1/r}(f)\|_{p,s;\alpha,1/r}\\
&\lesssim \|f - T_N^{p,r;\alpha+1/s-1/r}(f)\|_{p,r;\alpha+1/s-1/r}\\
&\quad + \left(\sum_{j=1}^{\infty}(\ell^{-1/s}(j)E_j(f - T_N^{p,r;\alpha+1/s-1/r}(f))_{p,r;\alpha+1/s-1/r})^s\frac{1}{j}\right)^{1/s}\\
&\lesssim E_N(f)_{p,r;\alpha+1/s-1/r} + \left(\sum_{j=1}^{N}\ell^{-1}(j)\frac{1}{j}\right)^{1/s} E_N(f)_{p,r;\alpha+1/s-1/r}\\
&\quad + \left(\sum_{j=N+1}^{\infty}(\ell^{-1/s}(j)E_j(f)_{p,r;\alpha+1/s-1/r})^s\frac{1}{j}\right)^{1/s}\\
&\sim \ell\ell^{1/s}(N)E_N(f)_{p,r;\alpha+1/s-1/r}\\
&\quad + \left(\sum_{j=N+1}^{\infty}(\ell^{-1/s}(j)E_j(f)_{p,r;\alpha+1/s-1/r})^s\frac{1}{j}\right)^{1/s}\\
&\lesssim \ell\ell^{1/s}(N)\omega_\kappa(f,1/N)_{p,r;\alpha+1/s-1/r}\\
&\quad + \left(\int_0^{1/N}(\ell^{-1/s}(t)\omega_\kappa(f,t)_{p,r;\alpha+1/s-1/r})^s\frac{dt}{t}\right)^{1/s}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\|f - V_{1/N}f\|_{p,s;\alpha,1/r} &\lesssim E_N(f)_{p,s;\alpha,1/r}\\
&\lesssim \ell\ell^{1/s}(N)\omega_\kappa(f,1/N)_{p,r;\alpha+1/s-1/r}\\
&\quad + \left(\int_0^{1/N}(\ell^{-1/s}(t)\omega_\kappa(f,t)_{p,r;\alpha+1/s-1/r})^s\frac{dt}{t}\right)^{1/s}.
\end{aligned}$$

Applying this estimate in (25) together with (26), we get

$$\begin{aligned}
\omega_\kappa(f,1/N)_{p,s;\alpha,1/r} &\lesssim \left(\int_0^{1/N}(\ell^{-1/s}(t)\omega_\kappa(f,t)_{p,r;\alpha+1/s-1/r})^s\frac{dt}{t}\right)^{1/s}\\
&\quad + \ell\ell^{1/s}(N)\omega_\kappa(f,1/N)_{p,r;\alpha+1/s-1/r}. \qquad \square
\end{aligned}$$

REMARK 2. The two terms on the right-hand side of the inequality (18) are independent of each other. Put $p=r$ and $\alpha=0$. Let $\beta<-1/s$. There exist $f_0\in L^p(\mathbb{T})$ and $\delta_0>0$, such that

$$\begin{aligned}\omega_\kappa(f_0,t)_p &\sim \ell\ell^{-1/s}(t)\ell\ell\ell^{\beta}(t)\\ &=(1+\log(1+|\log t|))^{-1/s}(1+\log(1+\log(1+|\log t|)))^{\beta},\ \ 0<t<\delta_0,\end{aligned}$$

(see [37, Theorem 2.5]). Then, it is clear that the second term in (18) is equivalent to $\ell\ell\ell^{\beta}(\delta)$, while the integral term satisfies

$$\left(\int_0^\delta \ell\ell\ell^{\beta s}(t)\frac{dt}{t\ell(t)\ell\ell(t)}\right)^{1/s}\sim \ell\ell\ell^{\beta+1/s}(\delta).$$

On the other hand, there are $f_1\in L^p(\mathbb{T})$ and $\delta_1>0$ such that $\omega_\kappa(f_1,t)_p\sim t^\kappa$, $0<t<\delta_1$. Therefore, the integral term behaves like $\delta^\kappa\ell^{-1/s}(\delta)$, while the second one like $\delta^\kappa\ell\ell^{1/s}(\delta)$.

Analogously, putting $p=r$ and $\alpha=1/r-1/s$, we can show the independence of the two terms on the right-hand side of (19).

4. Embedding theorems for Besov spaces

In this section we give an application of Theorem 1 to embeddings between Besov spaces. The following result is a limit case of [26, Corollary 3.6] which was left open.

THEOREM 2. *Suppose that* $1<p<\infty$, $\xi>0$, $\alpha\in\mathbb{R}$, *and either* $1<r\leqslant s<\infty$ *or* $1\leqslant s<r<\infty$. *Then*

$$B^{(p,r;\alpha+\max\{1/s-1/r,0\}),\xi}_{\lambda,\mu,1/s}(\mathbb{T}^n)\hookrightarrow B^{(p,s;\alpha,1/\max\{r,s\}),\xi}_{\lambda,\mu}(\mathbb{T}^n),\lambda>0,\mu\in\mathbb{R},\tag{28}$$

$$B^{(p,r;\alpha+\max\{1/s-1/r,0\}),\xi}_{0,\mu+\max\{1/\xi-1/s,0\},1/s}(\mathbb{T}^n)\hookrightarrow B^{(p,s;\alpha,1/\max\{r,s\}),\xi}_{0,\mu}(\mathbb{T}^n),\mu>-1/\xi,\tag{29}$$

and

$$B^{(p,r;\alpha+\max\{1/s-1/r,0\}),\xi}_{0,-1/\xi+\max\{1/\xi-1/s,0\},1/s+\max\{1/\xi-1/s,0\}}(\mathbb{T}^n)\hookrightarrow B^{(p,s;\alpha,1/\max\{r,s\}),\xi}_{0,-1/\xi}(\mathbb{T}^n).\tag{30}$$

Proof. By Theorem 1 we have that

$$\begin{aligned}|f|_{B^{(p,s;\alpha,1/\max\{r,s\}),\xi}_{\lambda,\mu}} &=\left(\int_0^1(t^{-\lambda}\ell^{\mu}(t)\omega_\kappa(f,t)_{p,s;\alpha,1/\max\{r,s\}})^{\xi}\frac{dt}{t}\right)^{1/\xi}\\ &\lesssim\left(\int_0^1\left[t^{-\lambda}\ell^{\mu}(t)\left(\int_0^t[\ell^{-1/s}(u)\omega_\kappa(f,u)_{p,r;\alpha+\max\{1/s-1/r,0\}}]^s\frac{du}{u}\right)^{1/s}\right]^{\xi}\frac{dt}{t}\right)^{1/\xi}\\ &\quad+\left(\int_0^1[t^{-\lambda}\ell^{\mu}(t)\ell\ell^{1/s}(t)\omega_\kappa(f,t)_{p,r;\alpha+\max\{1/s-1/r,0\}}]^{\xi}\frac{dt}{t}\right)^{1/\xi}=I+II.\end{aligned}$$

Assume that $\lambda > 0$ and $\mu \in \mathbb{R}$. Since $\omega_\kappa(f,u)_{p,r;\alpha+\max\{1/s-1/r,0\}}/u^\kappa$ is equivalent to a decreasing function we can still apply the extension of the Hardy inequality given in [1, Theorem 6.4] to derive that

$$\begin{aligned} I &= \left(\int_0^1 \left(t^{-\lambda s}\ell^{\mu s}(t)\int_0^t [\ell^{-1/s}(u)\omega_\kappa(f,u)_{p,r;\alpha+\max\{1/s-1/r,0\}}]^s \frac{du}{u}\right)^{\xi/s}\frac{dt}{t}\right)^{1/\xi} \\ &\leqslant \left(\int_0^1 \left(t^{-\lambda s}\ell^{\mu s-1}(t)\int_0^t [\omega_\kappa(f,u)_{p,r;\alpha+\max\{1/s-1/r,0\}}]^s \frac{du}{u}\right)^{\xi/s}\frac{dt}{t}\right)^{1/\xi} \\ &\lesssim \left(\int_0^1 \left(t^{1-\lambda s}\ell^{\mu s-1}(t)[\omega_\kappa(f,t)_{p,r;\alpha+\max\{1/s-1/r,0\}}]^s \frac{1}{t}\right)^{\xi/s}\frac{dt}{t}\right)^{1/\xi} \\ &= \left(\int_0^1 (t^{-\lambda}\ell^{\mu-1/s}(t)\omega_\kappa(f,t)_{p,r;\alpha+\max\{1/s-1/r,0\}})^\xi \frac{dt}{t}\right)^{1/\xi} \\ &\leqslant II = |f|_{B^{(p,r;\alpha+\max\{1/s-1/r,0\}),\xi}_{\lambda,\mu,1/s}}. \end{aligned}$$

This establishes the embedding (28).

Next we prove (29). Let $\lambda = 0$ and $\mu > -1/\xi$. Assume first that $\xi \geqslant s$. Applying Hardy's inequality given in [1, Theorem 6.5] we obtain

$$\begin{aligned} I &= \left(\int_0^1 \left(\ell^{\mu s}(t)\int_0^t [\ell^{-1/s}(u)\omega_\kappa(f,u)_{p,r;\alpha+\max\{1/s-1/r,0\}}]^s \frac{du}{u}\right)^{\xi/s}\frac{dt}{t}\right)^{1/\xi} \\ &\lesssim \left(\int_0^1 \left(t\ell^{1+\mu s}(t)[\ell^{-1/s}(t)\omega_\kappa(f,t)_{p,r;\alpha+\max\{1/s-1/r,0\}}]^s \frac{1}{t}\right)^{\xi/s}\frac{dt}{t}\right)^{1/\xi} \\ &= \left(\int_0^1 (\ell^{\mu}(t)\omega_\kappa(f,t)_{p,r;\alpha+\max\{1/s-1/r,0\}})^\xi \frac{dt}{t}\right)^{1/\xi} \\ &\leqslant II = |f|_{B^{(p,r;\alpha+\max\{1/s-1/r,0\}),\xi}_{0,\mu,1/s}}. \end{aligned}$$

Suppose now $\xi < s$. Let $K(t,f) = K(t,f;L^{p,r}(\log L)^{\alpha+\max\{1/s-1/r,0\}}(\mathbb{T}^n), H_\kappa^{p,r;\alpha+\max\{1/s-1/r,0\}}(\mathbb{T}^n))$. For $t>0$, we denote by $\chi_{(0,t)}$ the characteristic function of the interval $(0,t)$. Then, using the equivalence (13), the embedding $L^{1/\kappa,\xi}(\log L)^{-1/s}(\mathbb{T}^n) \hookrightarrow L^{1/\kappa,s}(\log L)^{-1/s}(\mathbb{T}^n)$ (see [1, Theorem 9.3]) and that $K(u^\kappa,f)/u^\kappa$ is a decreasing function, we get

$$\begin{aligned} I &= \left(\int_0^1 \ell^{\mu\xi}(t)\left(\int_0^t [\ell^{-1/s}(u)\omega_\kappa(f,u)_{p,r;\alpha+\max\{1/s-1/r,0\}}]^s \frac{du}{u}\right)^{\xi/s}\frac{dt}{t}\right)^{1/\xi} \\ &\sim \left(\int_0^1 \ell^{\mu\xi}(t)\left(\int_0^t [\ell^{-1/s}(u)K(u^\kappa,f)]^s \frac{du}{u}\right)^{\xi/s}\frac{dt}{t}\right)^{1/\xi} \end{aligned}$$

$$
\begin{aligned}
&= \left(\int_0^1 \ell^{\mu\xi}(t) \left\| \frac{K(u^\kappa, f)}{u^\kappa} \chi_{(0,t)}(u) \right\|_{1/\kappa,s;-1/s}^{\xi} \frac{dt}{t} \right)^{1/\xi} \\
&\lesssim \left(\int_0^1 \ell^{\mu\xi}(t) \left\| \frac{K(u^\kappa, f)}{u^\kappa} \chi_{(0,t)}(u) \right\|_{1/\kappa,\xi;-1/s}^{\xi} \frac{dt}{t} \right)^{1/\xi} \\
&= \left(\int_0^1 \ell^{\mu\xi}(t) \int_0^t [\ell^{-1/s}(u) K(u^\kappa, f)]^\xi \frac{du}{u} \frac{dt}{t} \right)^{1/\xi} \\
&\sim \left(\int_0^1 \ell^{\mu\xi}(t) \int_0^t [\ell^{-1/s}(u) \omega_\kappa(f,u)_{p,r;\alpha+\max\{1/s-1/r,0\}}]^\xi \frac{du}{u} \frac{dt}{t} \right)^{1/\xi} \\
&= \left(\int_0^1 [\ell^{-1/s}(u) \omega_\kappa(f,u)_{p,r;\alpha+\max\{1/s-1/r,0\}}]^\xi \int_u^1 \ell^{\mu\xi}(t) \frac{dt}{t} \frac{du}{u} \right)^{1/\xi} \\
&\lesssim \left(\int_0^1 [\ell^{-1/s}(u) \omega_\kappa(f,u)_{p,r;\alpha+\max\{1/s-1/r,0\}}]^\xi \ell^{\mu\xi+1}(u) \frac{du}{u} \right)^{1/\xi} \\
&\leqslant |f|_{B^{(p,r;\alpha+\max\{1/s-1/r,0\}),\xi}_{0,\mu+1/\xi-1/s,1/s}}
\end{aligned}
$$

because $\mu > -1/\xi$. In addition, since $\xi < s$, it is clear that $II \leqslant |f|_{B^{(p,r;\alpha+\max\{1/s-1/r,0\}),\xi}_{0,\mu+1/\xi-1/s,1/s}}$.

Finally, let us show (30). Let $\lambda = 0$ and $\mu = -1/\xi$. If $\xi \geqslant s$, we use Hardy's inequality given in [24, Lemma 3.3] to derive

$$
\begin{aligned}
I &= \left(\int_0^1 \left(\int_0^t [\ell^{-1/s}(u) \omega_\kappa(f,u)_{p,r;\alpha+\max\{1/s-1/r,0\}}]^s \frac{du}{u} \right)^{\xi/s} \frac{dt}{t\ell(t)} \right)^{1/\xi} \\
&\lesssim \left(\int_0^1 \left(t\ell(t)\ell\ell(t) [\ell^{-1/s}(t) \omega_\kappa(f,t)_{p,r;\alpha+\max\{1/s-1/r,0\}}]^s \frac{1}{t} \right)^{\xi/s} \frac{dt}{t\ell(t)} \right)^{1/\xi} \\
&= \left(\int_0^1 (\ell\ell^{1/s}(t) \omega_\kappa(f,t)_{p,r;\alpha+\max\{1/s-1/r,0\}})^\xi \frac{dt}{t\ell(t)} \right)^{1/\xi} \\
&= |f|_{B^{(p,r;\alpha+\max\{1/s-1/r,0\}),\xi}_{0,-1/\xi,1/s}} = II.
\end{aligned}
$$

Assume now that $\xi < s$. Using the equivalence (13) and an embedding result between Lorentz-Zygmund spaces, we get

$$
\begin{aligned}
I &\lesssim \left(\int_0^1 \int_0^t [\ell^{-1/s}(u) \omega_\kappa(f,u)_{p,r;\alpha+\max\{1/s-1/r,0\}}]^\xi \frac{du}{u} \frac{dt}{t\ell(t)} \right)^{1/\xi} \\
&= \left(\int_0^1 [\ell^{-1/s}(u) \omega_\kappa(f,u)_{p,r;\alpha+\max\{1/s-1/r,0\}}]^\xi \int_u^1 \frac{dt}{t\ell(t)} \frac{du}{u} \right)^{1/\xi} \\
&\lesssim \left(\int_0^1 [\ell^{-1/s}(u) \omega_\kappa(f,u)_{p,r;\alpha+\max\{1/s-1/r,0\}}]^\xi \ell\ell(u) \frac{du}{u} \right)^{1/\xi} \\
&= |f|_{B^{(p,r;\alpha+\max\{1/s-1/r,0\}),\xi}_{0,-1/s,1/\xi}}.
\end{aligned}
$$

On the other hand, it is not hard to check that $II \lesssim |f|_{B^{(p,r;\alpha+\max\{1/s-1/r,0\}),\xi}_{0,-1/s,1/\xi}}$ since $\xi < s$. The proof is complete.

Acknowledgements. It is a pleasure to thank Professor Fernando Cobos for some helpful comments during the preparation of this paper. The author would like to thank the referees for their comments.

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(Received February 29, 2016)

Oscar Domínguez
Departamento de Análisis Matemático
Facultad de Matemáticas
Universidad Complutense de Madrid
Plaza de Ciencias 3, 28040 Madrid, Spain
e-mail: oscar.dominguez@ucm.es