Mathematical
Inequalities
& Applications
Volume 20, Number 3 (2017), 815–824 doi:10.7153/mia-20-51

SOME EXACT BERNSTEIN–SZEGŐ INEQUALITIES ON THE STANDARD TRIANGLE

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(*Communicated by M. A. Hernandez Cifre*)

Abstract. An actual problem in the theory of approximations is to extend the univariate inequality of Bernstein to the multivariate setting. This question is satisfactorily settled in the case of a centrally symmetric convex body. In spite of the presence of good estimates, exact inequalities of Bernstein's type for nonsymmetric convex bodies are not known.

We prove that the approach based on the Krein-Milman theorem can be applied to maximize the nonlinear functional, which corresponds to the estimate of Bernstein-Szegő type for the gradients of arbitrary polynomials on convex bodies.

As applications we prove exact Bernstein-Szegő inequalities for some classes of bivariate polynomials on the standard triangle Δ. Note that in a certain sense Δ is the least symmetric convex body in $\mathbb{R}^2$.

1. Introduction and statement of the results

Denote by π_n^d the set of all real algebraic polynomials of d variables and of total degree not exceeding n. Let K be a compact set in $\mathbb{R}^d$ and let $\|f\|_{C(K)} := \max_{X\in K}|f(X)|$ be the uniform norm on K of a continuous function $f : K\subset\mathbb{R}^d \longrightarrow \mathbb{R}$. We use the notation $B_n(K)$ for the unit ball of π_n^d with respect to $\|\cdot\|_{C(K)}$, i.e., $B_n(K) = \{p\in\pi_n^d : \|p\|_{C(K)} \leqslant 1\}$.

An interesting generalization of the Bernstein inequality is obtained by Szegő (see [3] or [2]). It states that

$$|p'(x)| \leqslant n\frac{\sqrt{\|p\|^2_{C[a,b]} - p^2(x)}}{\sqrt{(x-a)(b-x)}}, \text{ for every } x\in(a,b), \text{and } p\in\pi_n^1. \tag{1}$$

The above inequality is sharp since it is satisfied as equality for every $x\in(a,b)$ if $p=\overline{T}_n$, where $\overline{T}_n$ is the Chebyshev polynomial of degree n for the interval $[a,b]$. Recall that $\overline{T}_n(t) := T_n(l(t))$, where l denotes the linear transformation from $[a,b]$ to $[-1,1]$ and $T_n(t) := \cos(n\arccos t)$, $t\in[-1,1]$. Moreover, if for a fixed $x\in(a,b)$, a polynomial $p\in\pi_n^1$ provides equality in (1) then either $p=c\overline{T}_n$ or p is an arbitrary polynomial such that $|p(x)| = \|p\|_{C[a,b]}$.

Mathematics subject classification (2010): 41A17, 26C05, 26B25.
Keywords and phrases: Bernstein-Szegő inequalities, polynomials, extreme points.
The research was supported by the Sofia University Science Foundation under Grant 75/2015.

Paper MIA-20-51

The multivariate extensions of the Bernstein-Szegő inequalities to convex bodies in $\mathbb{R}^d$ (compact convex sets with nonempty interior) are challenging problems in the theory of approximations.

There are only a few exact Bernstein-type inequalities in the multivariate case. The sets K in these inequalities are centrally symmetric convex bodies. We refer to the papers [8], [1], [6].

To the best of our knowledge, there are not exact inequalities of Bernstein type for nonsymmetric convex bodies.

Let K be a convex body in $\mathbb{R}^d$. For every $X \in \operatorname{int} K$ we denote by

$$BS_n(K;X) := \frac{1}{n} \sup \left\{ \frac{|Dp(X)|_2}{\sqrt{\|p\|_{C(K)}^2 - p^2(X)}} : p \in \pi_n^d,\, |p(X)| < \|p\|_{C(K)} \right\},$$

the Bernstein-Szegő factor for K. Note that $Dp(X)$ stands for the gradient of p at the point X. The above definition implies the following Bernstein-Szegő estimate:

$$|Dp(X)|_2 \leqslant nBS_n(K;X)\sqrt{\|p\|_{C(K)}^2 - p^2(X)},$$

for all $X \in \operatorname{int} K$ and $p \in \pi_n^d$.

Using the method of proof of [4, Theorem 1] and applying the univariate Bernstein-Szegő inequality instead of Bernstein inequality, one can prove the estimate

$$BS_n(K;X) \leqslant \frac{2\sqrt{2}}{w(K)\sqrt{1-\alpha^2(K;X)}}, \tag{2}$$

where $w(K)$ is the minimal distance between two parallel supporting hyperplanes of K and $\alpha(K;X)$ denotes the generalized Minkowski functional (see e.g. [4, p. 137]). In fact, the estimate (2) first appeared in [6, Theorem 2.7]. It can be improved when K is the standard triangle in $\mathbb{R}^2$, see [6, Theorem 4.6].

The first goal of the present paper is to prove a general representation of $BS_n(K;X)$ by using the extreme points of the unit ball in the polynomial space under consideration. Our approach is based on the Krein-Milman theorem. The main difficulty is that the domain of definition of the corresponding functional (see below) is not convex.

Let us denote by $E_n(K)$ the set of all extreme points of $B_n(K)$. Recall that a point p of a convex set B is said to be *extreme* if the equality $p = \lambda p_1 + (1-\lambda)p_2$ for some $p_1, p_2 \in B$ and $\lambda \in (0,1)$ implies $p = p_1 = p_2$. According to the Krein-Milman theorem, $B_n(K)$ is the convex hull of $E_n(K)$. As a consequence we have the equality

$$\max_{p \in B_n(K)} f(p) = \max_{p \in E_n(K)} f(p), \tag{3}$$

provided f is a convex function defined on $B_n(K)$.

We introduce the nonlinear functional

$$F(p) := F_n(p) := F_n(p;X) := \frac{1}{n} \frac{|Dp(X)|_2}{\sqrt{\|p\|_{C(K)}^2 - p^2(X)}}.$$

The domain of definition of F is

$$S := \{p \in \pi_n^d : |p(X)| < \|p\|_{C(K)}\}.$$

In Section 2 we prove the following result.

THEOREM 1. *Let K be a convex body in $\mathbb{R}^d$. For every $X \in \operatorname{int} K$ we have*

$$BS_n(K;X) = \sup_{p \in E_n(K) \cap S} F(p).$$

As it is discussed in [4], there is a precise mathematical background which allows us to say that in $\mathbb{R}^d$ the least symmetric convex body is the standard simplex

$$\Delta_d := \left\{ (x_1, \ldots, \mathbf{x}_d) \in \mathbb{R}^d : 0 \leqslant x_i \leqslant 1, i = 1, \ldots, d, \sum_{i=1}^{d} x_i \leqslant 1 \right\}.$$

Therefore, it is of interest to deal with this special case. For example, the exact yield of the inscribed ellipse method for the simplex was found in [6].

Since the dimension of π_n^d grows rapidly in both the parameters n and d, we shall obtain some exact inequalities for $d = 2$.

We shall use the abbreviated notations $\Delta := \Delta_2$ and $\|\cdot\| := \|\cdot\|_{C(\Delta)}$. We also set

$$M(x,y) := \max\left\{ \frac{1}{\sqrt{x(1-x)}}, \frac{1}{\sqrt{y(1-y)}}, \frac{\sqrt{2}}{\sqrt{(x+y)(1-x-y)}} \right\},$$

for every $(x,y) \in \operatorname{int}\Delta$.

THEOREM 2. *For every $(x,y) \in \operatorname{int}\Delta$ we have $BS_1(\Delta;x,y) = M(x,y)$. The extremal polynomials belong to the set $\{cp_i\}_{i=1}^3$, where $c \in \mathbb{R} \setminus \{0\}$, and $p_1(x,y) := 1-2x$, $p_2(x,y) := 1-2y$, $p_3(x,y) := 1-2(x+y)$.*

REMARK 1. The explicit expression for $BS_1(\Delta;x,y)$ given in Theorem 2, was announced without proof in [5].

Let us define the functions φ and ψ and the domains D_i, $i = 1,2,3$, as

$$\varphi(x) := \frac{1}{2} + x - \sqrt{2x^2 + \frac{1}{4}}, \quad x \in [0,1],$$

$$\psi(y) := \frac{1}{2} + y - \sqrt{2y^2 + \frac{1}{4}}, \quad y \in [0,1],$$

$$D_1 := \{(x,y) : y \in (0,1), 0 < x < \psi(y)\},$$

$$D_2 := \{(x,y) : x \in (0,1), 0 < y < \varphi(x)\},$$

$$D_3 := \operatorname{int}\Delta \setminus \{D_1 \cup D_2\}.$$

The next corollary gives the explicit form of the Bernstein-Szegő factor $BS_1(\Delta;x,y)$ depending on the point $(x,y) \in \operatorname{int}\Delta$.

COROLLARY 1. *We have*

$$BS_1(\Delta;x,y) = \begin{cases} \frac{1}{\sqrt{x(1-x)}}, & \text{if } (x,y)\in D_1; \\ \frac{1}{\sqrt{y(1-y)}}, & \text{if } (x,y)\in D_2; \\ \frac{\sqrt{2}}{\sqrt{(x+y)(1-x-y)}}, & \text{if } (x,y)\in D_3. \end{cases}$$

REMARK 2. It follows from the proof of Corollary 1 that if $p\in S$, $\|p\|=1$, and $F(p)=BS_1(\Delta;x,y)$ then $p=\pm p_i$, provided $(x,y)\in \operatorname{int} D_i$, $i=1,2,3$, and $p\in\{\pm p_i,\pm p_3\}$, provided $(x,y)\in \overline{D}_i\cap D_3$, $i=1,2$ (here $\overline{A}$ denotes the closure of the set A; see Figure 1).

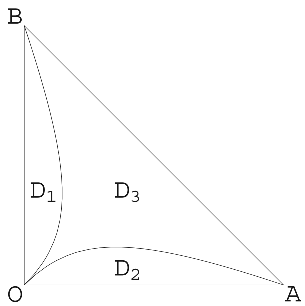

Figure 1: *The areas where $F(p_i)$, $i=1,2,3$, are maximal.*

We set $\mathscr{R}_n := \{p\in\pi_n^2 : p(x,y)=P(ax+by+c), P\in\pi_n^1\}$. The following theorem gives a sharp inequality of Bernstein-Szegő type for polynomials lying in $\mathscr{R}_n$.

THEOREM 3. *For every $(x,y)\in \operatorname{int}\Delta$ and $p\in\mathscr{R}_n$ we have the inequality*

$$|Dp(x,y)|_2 \leqslant nM(x,y)\sqrt{\|p\|^2-p^2(x,y)}. \tag{4}$$

The equality is attained if and only if $|p(x,y)|=\|p\|$ or $p=cq_i$, provided $(x,y)\in \operatorname{int} D_i$, $i=1,2,3$, or $p\in\{cq_i,cq_3\}$, provided $(x,y)\in\overline{D}_i\cap D_3$, $i=1,2$, where $q_1(x,y):=T_n(2x-1)$, $q_2(x,y):=T_n(2y-1)$ and $q_3(x,y):=T_n(2(x+y)-1)$.

2. Proof of Theorem 1

We shall need the following lemmas. From now on, $\|\cdot\|_{C(K)}$ will be denoted by $\|\cdot\|$.

LEMMA 1. *Let us fix a point $X\in \operatorname{int} K$. Then*

$$\sup_{p\in S} F(p) = \sup_{p\in S^1} F^1(p), \tag{5}$$

where

$$F^1(p) := \frac{1}{n}\frac{|Dp(X)|_2}{\sqrt{1-p^2(X)}},$$

and

$$S^1 := \{p \in \pi_n^d : |p(X)| < 1, \|p\|_{C(K)} \leqslant 1\}$$

is the domain of definition of F^1*.*

Proof. Let us denote by $B_n^1(K;X)$ the supremum in the right-hand side of (5). We have to prove that $BS_n(K;X) = B_n^1(K;X)$.

For every $p \in S$ the polynomial $q := \frac{p}{\|p\|}$ belongs to S^1. The homogeneity of F and $\|q\| = 1$ imply $F(p) = F(q) = F^1(q) \leqslant B_n^1(K;X)$. Taking the supremum over all $p \in S$ we get $BS_n(K;X) \leqslant B_n^1(K;X)$.

It remains to prove the converse inequality. Let q be an arbitrary polynomial from S^1. There are two cases.

Case 1. $q \in S$. It is easily seen that the function $f(\lambda) := \dfrac{1}{\sqrt{\lambda^2 - q^2(X)}}$ decreases for $\lambda > |q(X)|$, which gives $F^1(q) \leqslant F(q) \leqslant BS_n(K;X)$.

Case 2. $q \notin S$, i.e. $\|q\| = |q(X)| < 1$. Since q attains its norm at the interior point X, we have $Dq(X) = \mathbf{0}$, hence $0 = F^1(q) \leqslant BS_n(K;X)$.

Thus, $F^1(q) \leqslant BS_n(K;X)$ for every $q \in S^1$, which implies $B_n^1(K;X) \leqslant BS_n(K;X)$. The proof of Lemma 1 is completed. □

Let us define the functional

$$\Phi(p) := (F^1(p))^2 = \frac{1}{n^2}\frac{|Dp(X)|_2^2}{1-p^2(X)}, \quad p \in S^1.$$

LEMMA 2. *Φ is a convex functional on* $\operatorname{int} B_n(K)$.

Proof. Let $p_1, p_2 \in \operatorname{int} B_n(K)$. Without loss of generality we can assume that $p_1 \neq p_2$. It is sufficient to prove that $g(\lambda) := \Phi(\lambda p_1 + (1-\lambda)p_2)$, $\lambda \in [0,1]$ is a convex function. The coordinate representation of the gradient yields $g(\lambda) = \frac{1}{n^2}\sum_{i=1}^d h_i(\lambda)$, where

$$h_i(\lambda) = \frac{[\lambda A_i + (1-\lambda)B_i]^2}{1-[\lambda A + (1-\lambda)B]^2},$$

and

$$A := p_1(X), \quad B := p_2(X), \quad A_i := \frac{\partial p_1}{\partial x_i}(X), \quad B_i := \frac{\partial p_2}{\partial x_i}(X), \quad i = 1,\dots,d.$$

The claim will be established if we show that $h_i''(\lambda) \geqslant 0$, for every $i = 1,\dots,d$ and $\lambda \in [0,1]$. A computation gives

$$h_i''(\lambda) = \frac{2M(\lambda)}{(1-U^2)^3},$$

where

$$M(\lambda) := 4V^2U_i^2U^2 + 4VV_iUU_i(1-U^2) + V^2U_i^2(1-U^2) + V_i^2(1-U^2)^2,$$

and

$$U := \lambda A + (1-\lambda)B, \quad U_i := \lambda A_i + (1-\lambda)B_i, \quad i = 1,\dots,d,$$
$$V := A - B, \quad V_i := A_i - B_i, \quad i = 1,\dots,d.$$

Note that $1 - U^2 > 0$ because of $\|p_i\| < 1$, $i = 1,2$. Setting $Z := (1-U^2)V_i$ and $Y := 2UU_iV$, we get

$$M(\lambda) = (Z+Y)^2 + V^2U_i^2(1-U^2) \geqslant 0,$$

which finishes the proof. □

Furthermore, we shall need the functional $\overline{\Phi} = \operatorname{cl}\Phi$, defined by

$$\overline{\Phi}(p) := \liminf_{\substack{q\to p,\\ q\in \operatorname{int} B_n(K)}} \Phi(q), \; p \in B_n(K).$$

Because of the convexity of Φ (Lemma 2), $\overline{\Phi}$ is a convex functional on $B_n(K)$ (see [7], p. 52). The definition of Φ implies that it is a continuous function on S^1, hence $\overline{\Phi}(p) = \Phi(p)$ for $p \in S^1$. The next lemma provides information about the values of $\overline{\Phi}$ at the remaining points of $B_n(K)$.

LEMMA 3. $\overline{\Phi}(p_0) = 0$, *for every* $p_0 \in B_n(K) \setminus S^1$.

Proof. Since $\Phi(p) \geqslant 0$ for every $p \in S^1$, we have $\overline{\Phi}(p_0) \geqslant 0$. Let us consider the polynomials $p_\varepsilon(Y) := (1-\varepsilon)p_0(Y)$ for $\varepsilon \in (0,1)$. It follows from $|p_0(X)| = 1$ that $1 - p_\varepsilon^2(X) = \varepsilon(2-\varepsilon) > 0$. Note also that p_0 attains its norm at the point $X \in \operatorname{int} K$ which implies $Dp_0(X) = \mathbf{0}$. As a consequence, $Dp_\varepsilon(X) = (1-\varepsilon)Dp_0(X) = \mathbf{0}$.

Thus, the definition of Φ gives $\Phi(p_\varepsilon) = 0$ for every $\varepsilon \in (0,1)$, and hence $\lim_{\varepsilon\to 0}\Phi(p_\varepsilon) = 0$. Lemma 3 is proved. □

Proof of Theorem 1. Using Lemma 1 and the definition of Φ, we get

$$BS_n(K;X) = \sup_{p\in S^1} F^1(p) = \Big\{\sup_{p\in S^1}\Phi(p)\Big\}^{\frac{1}{2}}. \tag{6}$$

We claim that

$$\sup_{p\in S^1}\Phi(p) = \sup_{p\in B_n(K)}\overline{\Phi}(p) = \sup_{p\in E_n(K)}\overline{\Phi}(p) = \sup_{p\in E_n(K)\cap S^1}\Phi(p). \tag{7}$$

Indeed, let us set

$$A := \sup_{p\in S^1}\Phi(p), \quad \overline{A} := \sup_{p\in B_n(K)}\overline{\Phi}(p), \quad \overline{C} := \sup_{p\in E_n(K)}\overline{\Phi}(p),$$

and

$$C := \sup_{p \in E_n(K) \cap S^1} \Phi(p).$$

The inequality $A \leqslant \overline{A}$ follows from the fact that $\overline{\Phi}$ is a continuation of Φ from S^1 to $B_n(K)$. Let us suppose that $A < \overline{A}$. Then there exists a polynomial $p_0 \in B_n(K)$ such that $A < \overline{\Phi}(p_0)$. The definition of A excludes the possibility $p_0 \in S^1$, hence $p_0 \in B_n(K) \setminus S^1$ and, by Lemma 3, we have $\overline{\Phi}(p_0) = 0$, a contradiction.

The second equality in (7) is a consequence of (3). Finally, again by Lemma 3, we clearly get $C = \overline{C}$. This completes the proof of (7).

It follows from (6) and (7) that

$$BS_n(K;X) = \Big\{ \sup_{p \in E_n(K) \cap S^1} \Phi(p) \Big\}^{\frac{1}{2}} = \sup_{p \in E_n(K) \cap S^1} F^1(p).$$

Since $\|p\| = 1$ for every $p \in E_n(K)$, we have $E_n(K) \cap S^1 = E_n(K) \cap S =: D$ and $F^1(p) = F(p)$, for every $p \in D$. Theorem 1 is proved. $\square$

3. Proofs of Theorems 2 and 3

Proof of Theorem 2. We shall apply Theorem 1. Our first goal is to find $E_1(\Delta)$, i.e., the set of all extreme points of $B_1(\Delta)$. We claim that a polynomial $p \in \pi_1^2$ belongs to $E_1(\Delta)$ if and only if p satisfies the conditions:

$$|p(O)| = |p(A)| = |p(B)| = 1, \tag{8}$$

where $O(0,0)$, $A(1,0)$, and $B(0,1)$ are the vertices of Δ. Indeed, let $p \in E_1(\Delta)$. On the contrary, we assume that (8) are not satisfied and, without loss of generality, let $|p(O)| < 1$. There is an unique $\lambda \in (0,1)$ such that $p(O) = \lambda \cdot 1 + (1-\lambda)\cdot(-1)$. Then we define the first-degree polynomials q_1 and q_2 by the conditions:

$$q_i(O) = (-1)^{i-1}, \quad q_i(A) = p(A), \quad q_i(B) = p(B), \ \text{for} \ \ i = 1,2.$$

Clearly $q_i \in B_1(\Delta)$, $i = 1,2$, and since $p = \lambda q_1 + (1-\lambda) q_2$ we conclude that p is not an extreme point of $B_1(\Delta)$, a contradiction.

Let us suppose now that $p \in \pi_1^2$ and satisfies (8). It is sufficient to prove that the equality $p = \lambda q_1 + (1-\lambda) q_2$, with some $q_1, q_2 \in B_1(\Delta)$ and $\lambda \in (0,1)$ implies $q_1 = q_2 = p$. Let us set $q := q_1 - q_2$ and $f_{\pm\varepsilon} := p \pm \varepsilon q$. The representation $f_{\pm\varepsilon} = (\lambda \pm \varepsilon) q_1 + (1 - \lambda \mp \varepsilon) q_2$ shows that $f_{\pm\varepsilon} \in B_1(\Delta)$ for every sufficiently small $\varepsilon > 0$. If $X \in \{O,A,B\}$ then the inequality $|f_{\pm\varepsilon}(X)| \leqslant 1$ leads to $q(X) = 0$, and hence $q \equiv 0$, which finishes the proof of the assertion.

Using (8) we find that $E_1(\Delta) = \{\pm p_0, \pm p_1, \pm p_2, \pm p_3\}$, where $p_0(x,y) := 1$, $p_1(x,y) := 1 - 2x$, $p_2(x,y) := 1 - 2y$, $p_3(x,y) := 1 - 2(x+y)$. Note that $E_1(\Delta) \cap S = \{\pm p_1, \pm p_2, \pm p_3\}$. Then, by Theorem 1 we conclude that

$$BS_1(\Delta;x,y) = \max_{i=1,2,3} F_1(p_i) = \max\left\{ \frac{1}{\sqrt{x(1-x)}}, \frac{1}{\sqrt{y(1-y)}}, \frac{\sqrt{2}}{\sqrt{(x+y)(1-x-y)}} \right\},$$

for every $(x,y) \in \operatorname{int}\Delta$.

Since $F_1(cp) = F_1(p)$, it remains to prove that every extremal polynomial for F_1 from S such that $\|p\| = 1$ belongs to the set $\{\pm p_1, \pm p_2, \pm p_3\}$.

Note first that for $n = 1$ the domain of definition of Φ is $S^1 = B_1(\Delta) \setminus \{\pm 1\}$, because $X \in \operatorname{int}\Delta$. A careful examination of the proof of Lemma 2 shows that, if $n = 1$, then Φ is strictly convex on the segment $[r_1, r_2]$, provided $r_1 \neq r_2$ are nonconstant polynomials from S^1. In fact, the assertion of Lemma 2 remains true for $S^1 \supset \operatorname{int} B_n(K)$.

On the contrary, let us suppose that there is a polynomial $p \in (S \cap \partial B_1(\Delta)) \setminus \{\pm p_i\}_1^3$ such that $\Phi(p) = BS_1^2(\Delta; x, y)$. Since $p_0 \notin S$ it follows that $p \notin E_1(\Delta)$. Then p can be represented in the form $p = \lambda r_1 + (1-\lambda) r_2$, where $r_1, r_2 \in B_1(\Delta)$, $r_1 \neq r_2$, and $\lambda \in (0,1)$.

Case 1. r_1 and r_2 are nonconstant polynomials. The strict convexity of Φ implies $\Phi(p) < \lambda\Phi(r_1) + (1-\lambda)\Phi(r_2) \leqslant BS_1^2(\Delta; x, y)$, which is a contradiction.

Case 2. $r_i \equiv c_i$, for $i = 1, 2$, where $c_1 < c_2$ are constants. Then $p \equiv c \in (c_1, c_2)$ and thus p cannot be extremal.

Case 3. $r_1 \equiv c_1$, while r_2 is not a constant. Let us consider the polynomial $\tilde{r}_1 := (1-\varepsilon) r_1 + \varepsilon r_2$, provided $\varepsilon > 0$ is sufficiently small. Since $p \in (\tilde{r}_1, r_2)$ and $D\tilde{r}_1(X) \neq \mathbf{0}$, we conclude, by Case 1, that p is not extremal, a contradiction.

Theorem 2 is proved. □

Proof of Corollary 1. Let $(x,y) \in \operatorname{int}\Delta$. It is easily seen that $F_1(p_1) < F_1(p_2)$ if and only if $(x,y) \in \operatorname{int}\triangle OAO_1$, where O_1 is the midpoint of the side $[AB]$. Next we compare $F_1(p_2)$ and $F_1(p_3)$ on $\triangle OAO_1$. The condition $F_1(p_2) \leqslant F_1(p_3)$ is equivalent to $y^2 - (2x+1)y + x - x^2 \leqslant 0$, i.e., $y \in [y_1, y_2]$, where $y_1 := \varphi(x)$, $y_2 := \frac{1}{2} + x + \sqrt{2x^2 + \frac{1}{4}}$. Since $0 \leqslant \varphi(x) \leqslant \min\{x, 1-x\} \leqslant y_2(x)$, for every $x \in [0,1]$, we conclude that the graph of φ lies inside $\triangle OAO_1$, while the graph of y_2 does not intersect it. Therefore, p_2 is extremal for $(x,y) \in D_2$ and p_3 – for $(x,y) \in \operatorname{int}\triangle OAO_1 \setminus D_2$. The proof for $\triangle OBO_1$ is similar. □

Proof of Theorem 3. Let us fix the point $(x^*, y^*) \in \operatorname{int}\Delta$. If a polynomial $p \in \mathscr{R}_n$ satisfies $|p(x^*, y^*)| = \|p\|$ then $Dp(x^*, y^*) = \mathbf{0}$, hence (4) holds true. In what follows, we assume that $|p(x^*, y^*)| < \|p\|$, so that (4) is equivalent to

$$F_n(p) \leqslant M(x^*, y^*). \tag{9}$$

Since $F_n(cp) = F_n(p)$, we shall suppose that $\|p\| = 1$. By definition of $\mathscr{R}_n$, let $p(x,y) = P(\alpha x + \beta y + \gamma)$, where $P = P(t) \in \pi_n^1$. If

$$a := \min_{(x,y)\in\Delta} \{\alpha x + \beta y + \gamma\}, \quad b := \max_{(x,y)\in\Delta} \{\alpha x + \beta y + \gamma\},$$

then $\|P\|_{C[a,b]} = \|p\| = 1$. Clearly, $|Dp(x,y)|_2 = \sqrt{\alpha^2 + \beta^2}\,|P'(\alpha x + \beta y + \gamma)|$. Applying (1), we get

$$|Dp(x^*, y^*)|_2 \leqslant \frac{n\sqrt{\alpha^2 + \beta^2}\sqrt{1 - P^2(t^*)}}{\sqrt{(b - t^*)(t^* - a)}}, \tag{10}$$

where $t^* := \alpha x^* + \beta y^* + \gamma$. Note that $t^* \in (a,b)$ since a linear function attains its maximum and minimum only on the boundary of Δ. Dividing the both sides of (10) by $n\sqrt{1-P^2(t^*)}$ we obtain the equivalent inequality

$$F_n(p) \leqslant \frac{\sqrt{\alpha^2+\beta^2}}{\sqrt{(b-t^*)(t^*-a)}}. \tag{11}$$

Recall that $\overline{T}_n$ gives equality in the inequality of Bernstein-Szegő for the interval (a,b). In particular, for $n=1$ this implies

$$\frac{1}{\sqrt{(b-t^*)(t^*-a)}} = \frac{|\overline{T}_1'(t^*)|}{\sqrt{1-\overline{T}_1^2(t^*)}}. \tag{12}$$

It follows from (11) and (12) that

$$F_n(p) \leqslant \frac{\sqrt{\alpha^2+\beta^2}\,|\overline{T}_1'(t^*)|}{\sqrt{1-\overline{T}_1^2(t^*)}} = F_1(q),$$

where $q(x,y) := \overline{T}_1(\alpha x+\beta y+\gamma) \in \pi_1^2$. By Theorem 2,

$$F_1(q) \leqslant BS_1(\Delta; x^*, y^*) = M(x^*, y^*), \tag{13}$$

which completes the proof of (9).

It remains to clarify the cases of equality in (4). Every polynomial p such that $|p(x^*,y^*)| = \|p\|$ satisfies (4) as an equality. Therefore, without loss of generality we can suppose that $p \in S$ and $\|p\| = 1$. Let $F_n(p) = M(x^*,y^*)$. It follows from the proof of (9) that (10) and (13) hold true as equalities. We shall consider only the case $(x^*,y^*) \in \operatorname{int} D_1$, since the remaining cases for the point (x^*,y^*) are similar. Then Remark 1 implies $\overline{T}_1(\alpha x+\beta y+\gamma) = \sigma p_1(x,y) = \sigma(2x-1)$, where $\sigma \in \{-1,1\}$. Noticing that $\overline{T}_1(t) = \frac{2}{b-a}t - \frac{a+b}{b-a}$ we determine the coefficients as follows: $\alpha = \sigma(b-a)$, $\beta = 0$, and $\gamma = \frac{a+b}{2} - \sigma\frac{b-a}{2}$. The equality in (10) is attained only if $P(t) = \pm\overline{T}_n(t) = \pm T_n(\frac{2}{b-a}t - \frac{a+b}{b-a})$. Consequently,

$$\begin{aligned} p(x,y) &= P(\alpha x+\beta y+\gamma) = \pm\overline{T}_n\Big(\sigma\frac{b-a}{2}(2x-1) + \frac{a+b}{2}\Big) \\ &= \pm T_n(\sigma(2x-1)) = \tau q_1(x,y), \end{aligned}$$

where $\tau \in \{-1,1\}$. Conversely, if $p = \pm q_1$, then it can be directly checked that $F_n(p) = M(x^*,y^*)$. Theorem 3 is proved. $\square$

Acknowledgement. The authors thank the referee for the valuable suggestions regarding this paper.

REFERENCES

[1] M. BARAN, *Bernstein type theorems for compact sets in $\mathbb{R}^n$ revisited*, J. Approx. Theory **79** (1994), 190–198.
[2] B. BOJANOV, *Markov-type Inequalities for Polynomials and Splines*, in: "Approximation Theory X: Abstract and Classical Analysis" (C. K. Chui, L. L. Schumaker, J. Stöckler, Eds.), pp. 31–90, Vanderbilt University Press, Nashville, TN, 2002.
[3] P. BORWEIN AND T. ERDÉLYI, *Polynomials and Polynomial Inequalities*, Graduate Texts in Mathematics, **161**, Springer-Verlag, New York, 1995.
[4] A. KROÓ AND SZ. GY. RÉVÉSZ, *On Bernstein and Markov-type inequalities for multivariate polynomials on convex bodies*, J. Approx. Theory **99** (1999), 134–152.
[5] L. MILEV, N. NAIDENOV AND SZ. RÉVÉSZ, *Estimates for the Derivatives of Polynomials on the Standard Simplex*, in: "Jubilee Conference 125 Years Mathematics and Natural Sciences at Sofia University St. Kliment Ohridski", Book of Abstracts, pp. 53–55, St. Kliment Ohridski University Press, Sofia, 2014 [in Bulgarian].
[6] L. B. MILEV AND SZ. GY. RÉVÉSZ, *Bernstein's inequality for multivariate polynomials on the standard simplex*, J. Inequal. Appl. **2005:2** (2005), 145–163.
[7] R. T. ROCKAFELLAR, *Convex Analysis*, Princeton University Press, Princeton, New Jersey, 1970.
[8] Y. SARANTOPOULOS, *Bounds on the derivatives of polynomials on Banach spaces*, Math. Proc. Camb. Phil. Soc. **110** (1991), 307–312.

(Received May 31, 2016)

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