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SINGULAR MOSER–TRUDINGER INEQUALITY WITH THE EXACT GROWTH CONDITION IN $\mathbb{R}^n$

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Abstract. In this paper, we establish the singular Moser-Trudinger inequality with exact growth condition. We prove that there exists a positive constant C_n such that

$$\int_{\mathbb{R}^n} \frac{\Phi(\alpha_n(1-\frac{t}{n})|u|^{\frac{n}{n-1}})}{|x|^t(1+|u|)^{\frac{n}{n-1}}}dx \leqslant C_n \int_{\mathbb{R}^n} \frac{|u(x)|^n}{|x|^t}dx$$

for all $u \in W^{1,n}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} |\nabla u|^n dx \leqslant 1$, where $\Phi(t) := e^t - \sum_{i=0}^{n-2} \frac{t^i}{i!}$. In order to avoid using symmetry and rearrangement, we employ the change of variables developed by Dong and Lu in [14] (see also Lam and Lu in [16]) to transform the singular Moser-Trudinger inequality with the exact growth condition to the corresponding non-singular case.

1. Introduction

Let $W_0^{1,p}(\Omega)$, $\Omega \subseteq \mathbb{R}^n$, be the usual Sobolev space, i.e, the completion of $C_c^\infty(\Omega)$ with the norm

$$\|u\|_{W_0^{1,p}(\Omega)} = \int_\Omega |\nabla u|^p + |u|^p dx.$$

It is well-known that

$$W_0^{1,p}(\Omega) \subset L^{\frac{np}{n-p}}(\Omega),\ 1 \leqslant p < n,$$
$$W_0^{1,p}(\Omega) \subset L^\infty(\Omega), \quad p > n.$$

The case $p = n$ is the limit case of these Sobolev embeddings and one can assert that $W_0^{1,n}(\Omega) \subset L^q(\Omega)$ for any $q \geqslant 1$. However, many examples have showed that $W_0^{1,n}(\Omega) \not\subseteq L^\infty(\Omega)$. In fact, it was established independently by Yudovich [36], Pohozaev [33] and Trudinger [35]. In 1971, Moser [32] proved the following inequality.

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THEOREM A. ([32]) *Let* Ω *be a domain with finite measure in* $\mathbb{R}^n$, $n \geqslant 2$. *Then there exists a positive constant* C_n *and a sharp constant* $\alpha_n = n\omega_{n-1}^{\frac{1}{n-1}}$ *such that*

$$\frac{1}{|\Omega|}\int_\Omega \exp(\alpha |u|^{\frac{n}{n-1}})dx \leqslant C_n$$

for any $\alpha \leqslant \alpha_n$ *and* $u \in C_0^\infty(\Omega)$ *with* $\int_\Omega |\nabla u|^n dx \leqslant 1$, *where* ω_{n-1} *is the area of the surface of the unit ball.*

This result has been generalized in many directions. For instance, the singular Moser- Trudinger inequality which is an interpolation of Hardy inequality and Moser-Trudinger inequality was studied by Adimurthi and Sandeep [2]. The sharp constants for Moser-Trudinger inequalities on domains of finite measure on the Heisenberg group were established by Cohn and Lu [8] and Lam, Lu and Tang [21]. Lam, Lu and Tang [17, 20] further proved the singular Moser-Trudinger inequalities on the entire Heisenberg group. There has also been substantial progress for the Moser-Trudinger inequalities on spheres, CR spheres, or compact Riemannian manifolds, hyperbolic spaces, in Lorentz-Sobolev spaces, etc. We refer the interested reader to [5, 6, 9, 15, 23, 24, 27, 28] and the references therein. Moser-Trudinger inequalities also have many applications in geometric analysis and PDEs. Please see, for example, [10, 19, 25, 32, 34, 37] and the references therein.

When Ω has infinite volume, some versions of Moser-Trudinger type inequalities for unbounded domains were first proposed by Cao [7] when $n = 2$ and J. M. do Ó [11] for the general case $n \geqslant 2$. However, those inequalities are not sharp. Sharp versions of Moser-Trudinger type inequalities with best constants on unbounded domains were obtained by Adachi and Tanaka [3]. They proved that

THEOREM B. *Let* $0 < \alpha < \alpha_n$, *there exists a positive constant* C_n *such that*

$$\sup_{u \in W^{1,n}(\mathbb{R}^n), \int_{\mathbb{R}^n} |\nabla u|^n dx \leqslant 1} \int_{\mathbb{R}^n} \Phi(\alpha |u(x)|^{\frac{n}{n-1}})dx \leqslant C_n \int_{\mathbb{R}^n} |u(x)|^n dx,$$

where $\Phi(t) := e^t - \sum_{i=0}^{n-2} \frac{t^i}{i!}$. *Moreover, the constant* α_n *is sharp in the sense that if* $\alpha \geqslant \alpha_n$, *the supremum will become infinite.*

Ruf [34] (for the case $n = 2$), Li and Ruf [25] (for the general case $n \geqslant 2$) established a critical Moser-Trudinger type inequality for unbounded domains in Euclidean spaces. They obtained the following result.

THEOREM C. *There exists a positive constant* C_n *such that*

$$\sup_{u \in W^{1,n}(\mathbb{R}^n),\ \|u\|_{W^{1,n}(\mathbb{R}^n)} \leqslant 1} \int_{\mathbb{R}^n} \Phi(\alpha_n |u(x)|^{\frac{n}{n-1}})dx \leqslant C_n.$$

Moreover, the constant α_n *is sharp in the sense that if* α_n *is replaced by any* $\alpha > \alpha_n$, *the supremum will become infinite.*

It has been recently shown that the critical Moser-Trudinger inequality in Theorem C and the subcritical Moser-Trudinger inequality are actually equivalent as shown by Lam, Lu and Zhang [22].

M. de Souza and J. M. Do Ó [12] studied the singular Moser-Trudinger inequality with the Dirichlet norm. They proved that

THEOREM D. *Let* $0 < \beta < \alpha_n(1-\frac{t}{n})$*, there exists a positive constant* C_n *such that*

$$\int_{\mathbb{R}^n} \frac{\Phi(\beta|u|^{\frac{n}{n-1}})}{|x|^t}dx \leqslant C_n \int_{\mathbb{R}^n} \frac{|u(x)|^n}{|x|^t}dx,\ \forall\, u \in W^{1,n}(\mathbb{R}^n)\ with\ \int_{\mathbb{R}^n} |\nabla u|^n dx \leqslant 1.$$

As we all know it, the above inequality does not hold if we replace β with $\alpha_n(1-\frac{t}{n})$. So, it is natural to consider the singular Moser-Trudinger inequality with the exact growth condition. The exact growth condition for the Moser-Trudinger inequality was introduced by Masmoudi and Sani in [31] and the Moser-Trudinger inequality with the exact growth condition on hyperbolic space was established by Lu and Tang [26]. For Adams inequalities ([1]) with exact growth, we refer the reader to the works by Masmoudi and Sani [30] in dimension four and by Lu, Tang and Zhu in all dimensions $n \geqslant 3$ [29]. Recently, in an elegant paper of Dong and Lu in [14], they used very nicely the change of variables for non-radial functions to establish the singular Moser-Trudinger inequality. This new method has also been used by Lam and Lu in [16] to obtain sharp constants and maximizers for the Caffarelli-Kohn-Nirenberg type inequalities in a wide range of parameters. This change of variables could transform the singular inequality to the corresponding non-singular inequality. Motivated by their work, we establish the following theorem.

THEOREM 1. *For* $n \geqslant 2$ *and* $0 \leqslant t < n$*, there exists a positive constant* C_n *such that*

$$\int_{\mathbb{R}^n} \frac{\Phi(\alpha_n(1-\frac{t}{n})|u|^{\frac{n}{n-1}})}{|x|^t(1+|u|)^{\frac{n}{n-1}}}dx \leqslant C_n \int_{\mathbb{R}^n} \frac{|u(x)|^n}{|x|^t}dx,\ \forall\, u \in W^{1,n}(\mathbb{R}^n),\ \int_{\mathbb{R}^n} |\nabla u|^n dx \leqslant 1. \quad (1)$$

We remark that the power $\frac{n}{n-1}$ is optimal. This can be justified by the following theorem.

THEOREM 2. *If the power* $\frac{n}{n-1}$ *in the denominator is replaced by any* $p < \frac{n}{n-1}$*, there exists a sequence of functions* $\{u_k\}$ *such that* $\int_{\mathbb{R}^n} |\nabla u|^n dx \leqslant 1$*, but*

$$\left(\int_{\mathbb{R}^n} \frac{|u_k(x)|^n}{|x|^t}dx\right)^{-1} \int_{\mathbb{R}^n} \frac{\Phi(\alpha_n(1-\frac{t}{n})|u_k|^{\frac{n}{n-1}})}{|x|^t(1+|u_k|)^{\frac{n}{n-1}}} \to +\infty.$$

Adimurthi and Yang [4] proved that

THEOREM E. *There exists a positive constant* C_n *such that*

$$\int_{\mathbb{R}^n} \frac{\Phi(\alpha_n(1-\frac{t}{n})|u|^{\frac{n}{n-1}})}{|x|^t}dx \leqslant C_n,\ \forall\, u \in W^{1,n}(\mathbb{R}^n)$$

with

$$\int_{\mathbb{R}^n} |\nabla u|^n + |u|^n dx \leqslant 1.$$

As an application of Theorem 1, we also give some relation between Theorem E and Theorem 1.

COROLLARY 1. *Singular Moser-Trudinger inequality with the exact growth condition (Theorem 1) implies singular Moser-Trudinger inequality (Theorem E).*

This paper is organized as follows. In section 2, we use a way of change of variables to prove the singular Moser-Trudinger inequality with the exact growth condition. In section 3, we give the proof of the sharpness of the singular Moser-Trudinger inequality with the exact growth condition. In section 4, we prove some relation between singular Moser-Trudinger inequality and the singular Moser-Trudinger inequality with the exact growth condition.

2. The proof of Theorem 1

In this section, we will prove that singular Moser-Trudinger inequality with the exact growth condition. We need the following lemma.

LEMMA 1. ([31]) *Let $n \geqslant 2$, there exists a positive constant C_n such that*

$$\int_{\mathbb{R}^n} \frac{\Phi(\alpha_n |u|^{\frac{n}{n-1}})}{(1+|u|)^{\frac{n}{n-1}}} dx \leqslant C_n \int_{\mathbb{R}^n} |u(x)|^n dx, \ \forall\, u \in W^{1,n}(\mathbb{R}^n) \text{ with } \|\nabla u\|_n \leqslant 1.$$

We start to prove Theorem 1. It is hard to transform the inequality (1) to the corresponding radial inequality by symmetry and rearrangement. In fact, It is very difficult to prove that

$$\frac{\int_{\mathbb{R}^n} \frac{\Phi(\alpha_n(1-\frac{t}{n})|u|^{\frac{n}{n-1}})}{|x|^t(1+|u|)^{\frac{n}{n-1}}} dx}{\int_{\mathbb{R}^n} \frac{|u|^n}{|x|^t} dx} \leqslant \frac{\int_{\mathbb{R}^n} \frac{\Phi(\alpha_n(1-\frac{t}{n})|u^*|^{\frac{n}{n-1}})}{|x|^t(1+|u^*|)^{\frac{n}{n-1}}} dx}{\int_{\mathbb{R}^n} \frac{|u^*|^n}{|x|^t} dx},$$

where u^* denotes the rearrangement of u.

In the spirit of the work of Dong and Lu in [14] and Lam and Lu [16], we define a new function $v(x)$ corresponding to $u(x)$ which could keep the gradient norm less than 1, and eliminate the weights of integral at the same time.

Let $u(x) \in W^{1,n}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} |\nabla u|^n dx \leqslant 1$, define the new function $v(x) \in W^{1,n}(\mathbb{R}^n)$ by the formula below,

$$v(x) = \left(\frac{n-t}{n}\right)^{\frac{n-1}{n}} u(|x|^{\frac{t}{n-t}} x).$$

Consider the vector-valued function $F: \mathbb{R}^n \to \mathbb{R}^n$ defined by $F(x) = |x|^{\frac{t}{n-t}} x$.

Direct calculations show that $\det(J_F) = \frac{n}{n-t}|x|^{\frac{nt}{n-t}}$. Then by the change of variables $y_i = |x|^{\frac{t}{n-t}} x_i$ $(1 \leqslant i \leqslant n)$, we can obtain

$$\begin{aligned} \int_{\mathbb{R}^n} |v(x)|^n dx &= \left(\frac{n-t}{n}\right)^{n-1} \int_{\mathbb{R}^n} |u(|x|^{\frac{t}{n-t}} x)|^n dx \\ &= \left(\frac{n-t}{n}\right)^n \int_{\mathbb{R}^n} \frac{|u(y)|^n}{|y|^t} dy \end{aligned}$$

and

$$\int_{\mathbb{R}^n} |\nabla v(x)|^n dx \leqslant \left(\frac{n}{n-t}\right) \int_{\mathbb{R}^n} |x|^{\frac{t}{n-t}} \left|\nabla u\left(|x|^{\frac{t}{n-t}} x\right)\right|^n dx$$
$$= \int_{\mathbb{R}^n} |\nabla u(y)|^n dy.$$

Then we apply Lemma 1 to derive that

$$\begin{aligned}
&\int_{\mathbb{R}^n} \frac{\Phi(\alpha_n(1-\frac{t}{n})|u(y)|^{\frac{n}{n-1}})}{|y|^t(1+|u(y)|)^{\frac{n}{n-1}}} dy \\
&= \sum_{j=n-1}^{\infty} \int_{\mathbb{R}^n} \frac{(\alpha_n(1-\frac{t}{n})|u(y)|^{\frac{n}{n-1}})^j}{j!} \frac{1}{|y|^t(1+|u(y)|)^{\frac{n}{n-1}}} dy \\
&= \sum_{j=n-1}^{\infty} \int_{\mathbb{R}^n} \frac{n}{n-t} \frac{(\alpha_n(1-\frac{t}{n}))^j}{j!} |u(|x|^{\frac{t}{n-t}} x)|^{\frac{nj}{n-1}} \frac{1}{(1+|u(|x|^{\frac{t}{n-t}} x)|)^{\frac{n}{n-1}}} dx \\
&\leqslant \sum_{j=n-1}^{\infty} \int_{\mathbb{R}^n} \left(\frac{n}{n-t}\right)^{j+1} \frac{(\alpha_n(1-\frac{t}{n}))^j}{j!} |v(x)|^{\frac{nj}{n-1}} \frac{1}{(1+|v(x)|)^{\frac{n}{n-1}}} dx \qquad (2) \\
&= \frac{n}{n-t} \int_{\mathbb{R}^n} \frac{\Phi(\alpha_n |v(x)|^{\frac{n}{n-1}})}{(1+|v(x)|)^{\frac{n}{n-1}}} dx \\
&\leqslant \frac{nC_n}{n-t} \int_{\mathbb{R}^n} |v(x)|^n dx \\
&= C_n \left(\frac{n-t}{n}\right)^{n-1} \int_{\mathbb{R}^n} \frac{|u(y)|^n}{|y|^t} dy.
\end{aligned}$$

Then we accomplish the proof of Theorem 1.

3. Sharpness of Theorem 1

In this section, we will prove Theorem 2. Namely, we will give the proof of the sharpness of Theorem 1. We will show that the inequality in Theorem 1 does not hold if the power $\frac{n}{n-1}$ in the denominator is replaced by any $p < \frac{n}{n-1}$.

Let $\{b_k\}_{k\geqslant 1} \subset \mathbb{R}^+$ and $b_k \to +\infty$ as $k \to +\infty$ and define

$$r_k := e^{-\omega_{n-1}^{\frac{1}{n-1}} b_k^{\frac{n}{n-1}}}.$$

For fixed $p < \frac{n}{n-1}$. we choose $\{u_k\}_{k=1}^{\infty}$ as follows:

$$u_k(x) := \begin{cases} b_k, & \text{if } |x| \leqslant r_k, \\ b_k \frac{|\log|x||}{|\log r_n|}, & \text{if } r_k < |x| \leqslant 1, \\ 0, & \text{if } |x| \geqslant 1. \end{cases}$$

The choice of this sequence is inspired by a similar sequence in [31].

For any $q > 0$, define

$$\tilde{u}_k(x) := u_k\Big(\frac{x}{b_k^q}\Big).$$

Then by calculation, we have

$$\int_{\mathbb{R}^n} \frac{|\tilde{u}_k(x)|^n}{|x|^t}dx \sim b_k^{(n-t)q-\frac{n}{n-1}}, \quad \int_{\mathbb{R}^n} |\nabla \tilde{u}_k|^n dx \leqslant 1.$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^n} \frac{\Phi(\alpha_n(1-\frac{t}{n})|\tilde{u}_k|^{\frac{n}{n-1}})}{|x|^t(1+|\tilde{u}_k|)^p}dx \\
&= b_k^{(n-t)q} \int_{\mathbb{R}^n} \frac{\Phi(\alpha_n(1-\frac{t}{n})|u_k|^{\frac{n}{n-1}})}{|x|^t(1+|u_k|)^p}dx \\
&\geqslant b_k^{(n-t)q} \frac{\Phi(\alpha_n(1-\frac{t}{n})b_k^{\frac{n}{n-1}})}{(1+b_k)^p} \int_{|x|\leqslant r_k} \frac{1}{|x|^t}dx \\
&\geqslant b_k^{(n-t)q} \frac{\Phi(\alpha_n(1-\frac{t}{n})b_k^{\frac{n}{n-1}})}{(1+b_k)^p} e^{-\alpha_n(1-\frac{t}{n})b_k^{\frac{n}{n-1}}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\Big(\int_{\mathbb{R}^n} \frac{|\tilde{u}_k(x)|^n}{|x|^t}dx\Big)^{-1} \int_{\mathbb{R}^n} \frac{\Phi(\alpha_n(1-\frac{t}{n})|\tilde{u}_k|^{\frac{n}{n-1}})}{|x|^t(1+|\tilde{u}_k|)^p}dx \\
&\gtrsim b_k^{\frac{n}{n-1}-p} \Phi\Big(\alpha_n\Big(1-\frac{t}{n}\Big)b_k^{\frac{n}{n-1}}\Big) e^{-\alpha_n(1-\frac{t}{n})b_k^{\frac{n}{n-1}}} \to \infty
\end{aligned}$$

as $k \to +\infty$. So the power $\frac{n}{n-1}$ in the denominator can not be replaced by any $p < \frac{n}{n-1}$. This completes the proof of Theorem 2.

4. The proof of Corollary 1

In this section, we will show the singular Moser-Trudinger inequality with the exact growth condition implies the singular Moser-Trudinger inequality.

We will adapt an idea of using the level set of functions to derive the global inequality from the local one developed by Lam and Lu [17, 18] to our situation. We need to do some careful analysis and estimates to make this work here.

Let

$$A := \{x \in \mathbb{R}^n \mid |u(x)| \geqslant 1\}.$$

Since $u \in W^{1,n}(\mathbb{R}^n)$, it is obvious that A is a bounded domain. Now, we split the integral into two parts

$$\begin{aligned}
&\int_{\mathbb{R}^n} \frac{\Phi(\alpha_n(1-\frac{t}{n})|u|^{\frac{n}{n-1}})}{|x|^t}dx \\
&= \int_A \frac{\Phi(\alpha_n(1-\frac{t}{n})|u|^{\frac{n}{n-1}})}{|x|^t}dx + \int_{\mathbb{R}^n\setminus A} \frac{\Phi(\alpha_n(1-\frac{t}{n})|u|^{\frac{n}{n-1}})}{|x|^t}dx.
\end{aligned}$$

For $x \in \mathbb{R}^n \setminus A$, there exists a positive constant C_n such that

$$\Phi\Big(\alpha_n\Big(1-\frac{t}{n}\Big)|u(x)|^{\frac{n}{n-1}}\Big) \leqslant C_n|u(x)|^n.$$

Then we have

$$\begin{aligned}
&\int_{\mathbb{R}^n\setminus A} \frac{\Phi(\alpha_n(1-\frac{t}{n})|u|^{\frac{n}{n-1}})}{|x|^t}dx \\
&\leqslant C_n \int_{\mathbb{R}^n\setminus A} \frac{|u(x)|^n}{|x|^t}dx \\
&\leqslant C_n \int_{\{|u|\leqslant 1\}\cap\{|x|\leqslant 1\}} \frac{|u(x)|^n}{|x|^t}dx + C_n \int_{\{|u|\leqslant 1\}\cap\{|x|\geqslant 1\}} \frac{|u(x)|^n}{|x|^t}dx \\
&\leqslant C_n'.
\end{aligned}$$

Since

$$\Phi^p(x) \leqslant \Phi(px), \ \ \forall\, x \geqslant 0, \ \forall\, p \geqslant 1,$$

then one can apply the Hölder inequality to obtain

$$\begin{aligned}
&\int_A \frac{\Phi(\alpha_n(1-\frac{t}{n})|u|^{\frac{n}{n-1}})}{|x|^t}dx \\
&= \int_A \frac{\Phi(\alpha_n(1-\frac{t}{n})|u|^{\frac{n}{n-1}})(1+|u|)^{\frac{n}{p(n-1)}}}{|x|^{\frac{t}{p}}(1+|u|)^{\frac{n}{p(n-1)}}|x|^{\frac{t(p-1)}{p}}}dx \\
&\leqslant \Big(\int_A \frac{\Phi(p\alpha_n(1-\frac{t}{n})|u|^{\frac{n}{n-1}})}{|x|^t(1+|u|)^{\frac{n}{n-1}}}dx\Big)^{\frac{1}{p}} \Big(\int_A \frac{(1+|u|)^{\frac{n}{(n-1)(p-1)}}}{|x|^t}dx\Big)^{\frac{p-1}{p}} \\
&\leqslant \Big(\int_A \frac{\Phi(p\alpha_n(1-\frac{t}{n})|u|^{\frac{n}{n-1}})}{|x|^t(1+|u|)^{\frac{n}{n-1}}}dx\Big)^{\frac{1}{p}} \Big(\Big(\int_A (1+|u|)^{\frac{ns'}{(n-1)(p-1)}}dx\Big)^{\frac{1}{s'}}\Big(\int_A \frac{1}{|x|^{st}}dx\Big)^{\frac{1}{s}}\Big)^{\frac{p-1}{p}} \\
&=: I_1 \cdot I_2.
\end{aligned}$$

For I_2, we have

$$I_2 \leqslant \Big(\int_A (1+|u(x)|)^{\frac{ns'}{(n-1)(p-1)}}dx\Big)^{\frac{(p-1)}{s'p}} \Big(\int_A \frac{1}{|x|^{ts}}dx\Big)^{\frac{p-1}{sp}}.$$

One can choose s sufficiently close to 1 to obtain that $I_2 \leqslant C$.

For I_1, let $\theta \in (0,1)$ be such that $\|u\|_n^n = \theta$, then $\|\nabla u\|_n^n = 1-\theta$. Let $\tilde{u} = p^{\frac{n-1}{n}}u$, we consider the following two cases:

Case 1: $\theta \geqslant \frac{n-1}{n}$,

Case 2: $\theta < \frac{n-1}{n}$.

For *Case* 1 and $1 < p < n^{\frac{1}{n-1}}$, we have

$$\|\nabla \tilde{u}\|_n^n = p^{n-1}\|\nabla u\|_n^n \leqslant p^{n-1}(1-\theta) \leqslant \frac{p^{n-1}}{n} \leqslant 1.$$

For *Case* 2, let $p = \frac{n-1}{n-1-\theta}$, then $1 < p < \frac{n}{n-1} \leqslant n^{\frac{1}{n-1}}$. By construction, we have

$$\begin{aligned}\|\nabla \tilde{u}\|_n^n = p^{n-1}\|\nabla u\|_n^n &\leqslant \Big(\frac{n-1}{n-1-\theta}\Big)^{n-1}(1-\theta)\\ &= \left(\frac{(1-\theta)^{\frac{1}{n-1}}}{1-\frac{\theta}{n-1}}\right)^{n-1} \leqslant 1\end{aligned}$$

since $(1-x)^q \leqslant 1-qx$ for any $x,q \in [0,1]$. Then, we can apply singular Moser-Trudinger inequality with the exact growth condition to obtain

$$\begin{aligned} I_1 =& \Big(\int_A \frac{\Phi(p\alpha_n(1-\frac{t}{n})|u|^{\frac{n}{n-1}})}{|x|^t(1+|u|)^{\frac{n}{n-1}}}dx\Big)^{\frac{1}{p}}\\ &\leqslant \Big(p\int_{\mathbb{R}^n} \frac{\Phi(\alpha_n(1-\frac{t}{n})|\tilde{u}|^{\frac{n}{n-1}})}{|x|^t(1+|\tilde{u}|)^{\frac{n}{n-1}}}dx\Big)^{\frac{1}{p}}\\ &\leqslant C_n\big(p\int_{\mathbb{R}^n} \frac{|u(x)|^n}{|x|^t}dx\big)^{\frac{1}{p}}\\ &\leqslant C_n'.\end{aligned}$$

This completes the proof of Corollary 1.

We have recently learned that the authors of [13] have independently proved, among many other results, the following theorem.

THEOREM 3. *Let* $0 \leqslant t \leqslant s < n$ *and* $0 \leqslant \alpha < \alpha_n$:

$$SSTM(\alpha,s,t) = \sup_{\|\nabla u\|_n \leqslant 1} \frac{1}{\left(\int\limits_{\mathbb{R}^n} \frac{|u|^n}{|x|^t}dx\right)^{\frac{n-s}{n-t}}} \int\limits_{\mathbb{R}^n} \frac{\Phi\left(\alpha\left(1-\frac{s}{n}\right)|u|^{\frac{n}{n-1}}\right)}{|x|^s}dx < \infty.$$

$$STME(s,t) = \sup_{\|\nabla u\|_n \leqslant 1} \frac{1}{\left(\int\limits_{\mathbb{R}^n} \frac{|u|^n}{|x|^t}dx\right)^{\frac{n-s}{n-t}}} \int\limits_{\mathbb{R}^n} \frac{\phi_n\left(\alpha_n\left(1-\frac{s}{n}\right)|u|^{\frac{n}{n-1}}\right)}{\left(1+\left(1-\frac{s}{n}\right)^{\frac{n-s}{n-t}}|u|^{\frac{n}{n-1}\frac{n-s}{n-t}}\right)|x|^s}dx < \infty. \tag{3}$$

The constant α_n *is sharp. Moreover, the power* $\frac{n}{n-1}$ *in the denominator of (3) cannot be replaced by any* $p < \frac{n}{n-1}$.

Moreover, $SSTM(\alpha,s,t)$, $0 < t \leqslant s < n$, *can be attained and its extremizers are radially symmetric.*

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