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ON THE CLASS OF BANACH SPACES WITH JAMES CONSTANT $\sqrt{2}$, III

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Abstract. We present a new characterization of two-dimensional Banach spaces with James constant $\sqrt{2}$. As an application, we give an example of a two-dimensional Banach space with James constant $\sqrt{2}$ that is not isometrically isomorphic to any absolute, or symmetric, or $\pi/2$-rotation invariant normed space. It is shown that this gives a counterexample to Lassak's conjecture.

1. Introduction

This paper is a continuation of [9, 10]. For a real Banach space X with $\dim X \geqslant 2$, the James constant $J(X)$ of a Banach space X is introduced by Gao and Lau [5] as follows:

$$J(X) = \sup\{\min\{\|x+y\|, \|x-y\|\} : x, y \in S(X)\}.$$

In [9, 10], we studied the class of Banach spaces with James constant $\sqrt{2}$; and gave the following characterizations:

(I) if $\dim X \geqslant 3$, then $J(X) = \sqrt{2}$ if and only if X is a Hilbert space ([9, Theorem 2.3]), and

(II) if $\|\cdot\|$ is a $\pi/2$-rotation invariant norm on $\mathbb{R}^2$, then $J((\mathbb{R}^2, \|\cdot\|)) = \sqrt{2}$ if and only if $\|\cdot\|$ is $\pi/4$-rotation invariant ([10, Theorem 3.10]),

where a norm $\|\cdot\|$ on $\mathbb{R}^2$ is said to be θ*-rotation invariant*, where $\theta \in \mathbb{R}$, if the θ-rotation matrix

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

is an isometry on $(\mathbb{R}^2, \|\cdot\|)$.

The purpose of the present paper is to give a characterization of Banach spaces X satisfying $J(X) = \sqrt{2}$. By (I), the problem reduces to the case of $\dim X = 2$; so we assume, throughout this paper, that X is the space $\mathbb{R}^2$ endowed with the norm

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$\|\cdot\|$. Let $x(\theta) = \|(\cos\theta, \sin\theta)\|^{-1}(\cos\theta, \sin\theta)$ for each θ. If we define a function by $r(\theta) = \|x(\theta)\|_2$, where $\|\cdot\|_2$ denotes the Euclidean norm, the unit ball B_X of X can be expressed as

$$B_X = \{r(\cos\theta, \sin\theta) : 0 \leqslant r \leqslant r(\theta),\ \theta \in \mathbb{R}\}$$

since $r(\theta)\|(\cos\theta, \sin\theta)\| = 1$ and $x(\theta) = r(\theta)(\cos\theta, \sin\theta)$.

In this paper, we present a new characterization of two-dimensional Banach spaces X with $J(X) = \sqrt{2}$, in terms of the curve $x(\theta)$ and function $r(\theta)$ associated with the norm on X. This provides a general form of two-dimensional Banach spaces with James constant $\sqrt{2}$. In particular, we have (II) as a corollary of the main result. Moreover, as an application, we give an example of a two-dimensional Banach space with James constant $\sqrt{2}$ that is not isometrically isomorphic to any absolute, or symmetric, or $\pi/2$-rotation invariant normed space. It is shown that this gives a counterexample to Lassak's conjecture [11].

2. Characterizations in terms of the polar coordinates

If x, y are two elements of a normed space, then x is said to be *isosceles orthogonal* to y, denoted by $x \perp_I y$, if $\|x+y\| = \|x-y\|$. It is clear from the definition that $x \perp_I y$ implies $x \perp_I -y$, $y \perp_I x$ and $\alpha x \perp_I \alpha y$ for any $\alpha \in \mathbb{R}$. Moreover, if $\|x\| = \|y\|$, then $x \perp_I y$ implies $x+y \perp_I x-y$. The following is an important property of the isosceles orthogonality.

LEMMA 2.1. (Gao and Lau [5]; Alonso [1]; Ji et al. [8]) *Let X be a two-dimensional normed space. Suppose that $x \in S_X$. Then there exists a unique (up to the sign) element $y \in S_X$ such that $x \perp_I y$.*

The survey [2] contains many other properties of isosceles orthogonality.

In [9], the following auxiliary result was proved.

LEMMA 2.2. ([9]) *Let $\theta_0 \in \mathbb{R}$, and let φ_0 be a unique real number such that $\varphi_0 \in (\theta_0, \theta_0 + \pi)$ and $x(\theta_0) \perp_I x(\varphi_0)$. Then, for any $\theta \in [\theta_0, \varphi_0]$, there exists a unique $\varphi \in [\varphi_0, \theta_0 + \pi]$ such that $x(\theta) \perp_I x(\varphi)$.*

A self-homeomorphism α on $\mathbb{R}$ is said to be *rotation* if $|\alpha(t+2\pi) - \alpha(t)| = 2\pi$ for each $t \in \mathbb{R}$, or equivalently, if $|s-t| = 2\pi$ implies that $|\alpha(s) - \alpha(t)| = 2\pi$.

As an application of Lemmas 2.1 and 2.2, we have the following result. We note that $x(\theta + 2n\pi) = x(\theta)$ for each $n \in \mathbb{Z}$ and each $\theta \in \mathbb{R}$.

PROPOSITION 2.3. *There exist a pair of increasing rotations ω and η on $\mathbb{R}$ satisfying the following properties:*

(i) $\theta < \eta(\theta) < \omega(\theta) < \theta + \pi$ *for each* $\theta \in \mathbb{R}$;

(ii) $x(\theta) \perp_I x(\omega(\theta))$ *for each* $\theta \in \mathbb{R}$;

(iii) $\omega^2(\theta) = \theta + \pi$ *for each* $\theta \in \mathbb{R}$;

(iv) *the equation*

$$x(\eta(\theta)) = \frac{x(\theta)+x(\omega(\theta))}{\|x(\theta)+x(\omega(\theta))\|}$$

holds for each $\theta \in \mathbb{R}$*; and*

(v) $\eta^2 = \omega$. *In particular,* $\omega \circ \eta = \eta \circ \omega$.

Moreover, ω *and* η *are uniquely determined by* (i), (ii) and (iv).

Proof. For each $\theta \in \mathbb{R}$, we have, by Lemma 2.2, there exists a unique real number $\omega(\theta)$ satisfying $\omega(\theta) \in (\theta, \theta+\pi)$ and $x(\theta) \perp_I x(\omega(\theta))$. It is apparent from the definition that $\omega(\theta+2\pi) = \omega(\theta)+2\pi$ for each $\theta \in \mathbb{R}$. We first show that ω is continuous on $\mathbb{R}$. Suppose that (θ_n) be a sequence in $\mathbb{R}$ that converges to some θ_0. Let (θ_{n_k}) be a subsequence of (θ_n) such that $(\omega(\theta_{n_k}))$ converges to φ. Then $\theta_0 \leqslant \varphi \leqslant \theta_0+\pi$ and

$$\begin{aligned}\|x(\varphi)+x(\theta_0)\| &= \lim_k \|x(\omega(\theta_{n_k}))+x(\theta_{n_k})\| \\ &= \lim_k \|x(\omega(\theta_{n_k}))-x(\theta_{n_k})\| \\ &= \|x(\varphi)-x(\theta_0)\|,\end{aligned}$$

that is $x(\theta_0) \perp_I x(\varphi)$. Hence, by the uniqueness clause of Lemma 2.2, it follows that $\varphi = \omega(\theta_0)$; which in turn implies that the entire sequence $(\omega(\theta_n))$ converges to $\omega(\theta_0)$. Thus ω is continuous on $\mathbb{R}$.

We next show (iii). For this, we note that $x(\omega^2(\theta)) = \pm x(\theta)$, since, by (ii), $x(\omega(\theta)) \perp_I x(\omega^2(\theta))$ as well as $x(\omega(\theta)) \perp_I x(\theta)$. However, it follows from $\theta < \omega(\theta) < \theta+\pi$ for each $\theta \in \mathbb{R}$ that

$$\theta < \omega(\theta) < \omega^2(\theta) < \omega(\theta)+\pi < \theta+2\pi;$$

and so $x(\omega^2(\theta)) \neq x(\theta)$. Thus one has $x(\omega^2(\theta)) = -x(\theta)$, which together with $\theta < \omega^2(\theta) < \theta+2\pi$ show that $\omega^2(\theta) = \theta+\pi$. This proves (iii).

Now we put $\mu(\theta) = \omega^3(\theta)-2\pi$ for each $\theta \in \mathbb{R}$. Then, by (iii), one has that

$$\begin{aligned}(\omega\circ\mu)(\theta) &= \omega(\omega^3(\theta)-2\pi) \\ &= \omega^4(\theta)-2\pi \\ &= \theta;\end{aligned}$$

and similarly that $(\mu\circ\omega)(\theta) = \theta$. Therefore we obtain $\mu = \omega^{-1}$. Since μ is continuous on $\mathbb{R}$, it follows that ω is a self-homeomorphism on $\mathbb{R}$. This proves that ω is a rotation on $\mathbb{R}$. We note that each continuous injection from $\mathbb{R}$ into $\mathbb{R}$ is strictly monotone (by the intermediate value theorem). This together with

$$\omega(0) < \omega(0)+2\pi = \omega(2\pi)$$

show that ω is increasing on $\mathbb{R}$.

Define a mapping η on $\mathbb{R}$ as follows: for each θ, let $\eta(\theta)$ be a unique element in $(\theta,\omega(\theta))$ satisfying

$$x(\eta(\theta)) = \frac{x(\theta)+x(\omega(\theta))}{\|x(\theta)+x(\omega(\theta))\|}.$$

It is apparent that $x(\eta(\theta+2\pi)) = x(\eta(\theta)) = x(\eta(\theta)+2\pi)$; and since

$$\theta+2\pi < \eta(\theta+2\pi), \eta(\theta)+2\pi < \omega(\theta+2\pi) = \omega(\theta)+2\pi,$$

we obtain $\eta(\theta+2\pi) = \eta(\theta)+2\pi$. On the other hand, it follows from (iii) that

$$x(\eta(\omega(\theta))) = \frac{x(\omega(\theta))+x(\omega^2(\theta))}{\|x(\omega(\theta))+x(\omega^2(\theta))\|} = \frac{x(\omega(\theta))-x(\theta)}{\|x(\omega(\theta))-x(\theta)\|} \tag{1}$$

for each $\theta \in \mathbb{R}$. Since $x(\theta) \perp_I x(\omega(\theta))$ by (ii), we have $\|x(\omega(\theta))-x(\theta)\| = \|x(\theta)+x(\omega(\theta))\|$; and hence

$$x(\eta(\omega(\theta))) = \frac{x(\omega(\theta))-x(\theta)}{\|x(\theta)+x(\omega(\theta))\|}.$$

From this we obtain

$$\|x(\eta(\theta))+x(\eta(\omega(\theta)))\| = \frac{2}{\|x(\theta)+x(\omega(\theta))\|} = \|x(\eta(\theta))-x(\eta(\omega(\theta)))\|,$$

that is, $x(\eta(\theta)) \perp_I x(\eta(\omega(\theta)))$. By the (essential) uniqueness of the isosceles orthogonality, either $x(\eta(\omega(\theta))) = x(\omega(\eta(\theta)))$ or $x(\eta(\omega(\theta))) = -x(\omega(\eta(\theta)))$ holds. We show that $x(\eta(\omega(\theta))) = -x(\omega(\eta(\theta)))$ can not occur.

Let $x = x(\theta)$ and $y = x(\omega(\theta))$. Then $\{x,y\}$ forms a basis for $\mathbb{R}^2$ since $x \perp_I y$. Consider, as usual, the four quadrants Q_j $(j=1,2,3,4)$ with respect to the basis $\{x,y\}$. Then it follows that $\omega(\theta) < \omega(\eta(\theta)) < \omega^2(\theta) = \theta+\pi$, which together with $\omega([\theta,\theta+2\pi]) = [\omega(\theta),\omega(\theta)+2\pi]$ and $x(\omega^2(\theta))) = -x$ imply that $x(\omega(\eta(\theta)))$ is in the second quadrant Q_2 (the cone generated by $\{-x,y\}$). On the other hand, by (1), we have

$$x(\eta(\omega(\theta)) = \frac{y-x}{\|y-x\|} \in Q_2.$$

Thus $x(\omega(\eta(\theta)))$ and $x(\eta(\omega(\theta)))$ are in the same quadrant, while $-x(\omega(\eta(\theta)))$ is in the opposite quadrant. This proves that $x(\eta(\omega(\theta))) = x(\omega(\eta(\theta)))$ for each $\theta \in \mathbb{R}$.

Finally, we show that η is a self-homeomorphism on $\mathbb{R}$ satisfying $\eta^2 = \omega$. Then η will be an increasing rotation since $\eta(\theta+2\pi) = \eta(\theta)+2\pi$ for each $\theta \in \mathbb{R}$. For this, we first note that

$$x(\eta(\theta))+x((\eta\circ\omega)(\theta)) = \frac{2x(\omega(\theta))}{\|x(\theta)+x(\omega(\theta))\|},$$

which and $x(\eta(\omega(\theta))) = x(\omega(\eta(\theta)))$ together imply that

$$\begin{aligned} x(\eta^2(\theta)) &= \frac{x(\eta(\theta))+x((\omega\circ\eta)(\theta))}{\|x(\eta(\theta))+x((\omega\circ\eta)(\theta))\|} \\ &= \frac{x(\eta(\theta))+x((\eta\circ\omega)(\theta))}{\|x(\eta(\theta))+x((\eta\circ\omega)(\theta))\|} = x(\omega(\theta)) \end{aligned}$$

for each θ. On the other hand, since $\theta < \eta(\theta) < \omega(\theta)(< \theta+\pi)$ for each θ, one has that

$$\theta < \eta(\theta) < \eta^2(\theta) < \omega(\eta(\theta)) < \omega^2(\theta) = \theta+\pi.$$

It follows that $\eta^2(\theta) = \omega(\theta)$ for each $\theta \in \mathbb{R}$. Thus (v) holds. We note that η is a bijection since $\omega(=\eta^2)$ is a homeomorphism on $\mathbb{R}$. Now let (θ_n) be a sequence in $\mathbb{R}$ that converges to $\theta \in \mathbb{R}$. Take an arbitrary convergent subsequence $(\eta(\theta_{n_k}))$ of $(\eta(\theta_n))$ that converges to $\varphi \in \mathbb{R}$. Then it follows that $\theta \leqslant \varphi \leqslant \omega(\theta)$ (since ω is continuous), and that

$$\begin{aligned} x(\varphi) = \lim_k x(\eta(\theta_{n_k})) &= \lim_k \frac{x(\theta_{n_k}) + x(\omega(\theta_{n_k}))}{\|x(\theta_{n_k}) + x(\omega(\theta_{n_k}))\|} \\ &= \frac{x(\theta)+x(\omega(\theta))}{\|x(\theta)+x(\omega(\theta))\|} = x(\eta(\theta)), \end{aligned}$$

which implies that $\varphi = \eta(\theta)$. This shows that the entire sequence $(\eta(\theta_n))$ converges to $\eta(\theta)$. Therefore η is a continuous bijection on $\mathbb{R}$; and thus it is a homeomorphism.

The uniqueness readily follows from (i), (ii) and (iv). □

We now give the main result, a necessary and sufficient condition for $J(X) = \sqrt{2}$, in terms of the function $r(\theta) = \|x(\theta)\|_2$.

THEOREM 2.4. *$J(X) = \sqrt{2}$ if and only if*

$$r(\theta)^2 + r(\omega(\theta))^2 = r(\eta(\theta))^2 + r((\omega\circ\eta)(\theta))^2$$

for each $\theta \in [0, 2\pi]$.

Proof. We first note that

$$\|x(\theta)+x(\omega(\theta))\| = \|x(\theta)-x(\omega(\theta))\|$$

for each $\theta \in [0,2\pi]$ since $x(\theta) \perp_I x(\omega(\theta))$. From this and $\eta\circ\omega = \omega\circ\eta$ (Proposition 2.3 (v)), it follows that

$$\begin{aligned} r(\eta(\theta))^2 + r((\omega\circ\eta)(\theta))^2 &= \|x(\eta(\theta))\|_2^2 + \|x((\omega\circ\eta)(\theta))\|_2^2 \\ &= \frac{\|x(\theta)+x(\omega(\theta))\|_2^2}{\|x(\theta)+x(\omega(\theta))\|^2} + \frac{\|x(\theta)-x(\omega(\theta))\|_2^2}{\|x(\theta)-x(\omega(\theta))\|^2} \\ &= \frac{\|x(\theta)+x(\omega(\theta))\|_2^2 + \|x(\theta)-x(\omega(\theta))\|_2^2}{\|x(\theta)+x(\omega(\theta))\|^2} \\ &= \frac{2(\|x(\theta)\|_2^2 + \|x(\omega(\theta))\|_2^2)}{\|x(\theta)+x(\omega(\theta))\|^2} \\ &= \frac{2(r(\theta)^2 + r(\omega(\theta))^2)}{\|x(\theta)+x(\omega(\theta))\|^2} \end{aligned}$$

for each θ. Now $J(X) = \sqrt{2}$ holds if and only if $\|x(\theta)+x(\omega(\theta))\| = \sqrt{2}$ for each θ, which happens if and only if

$$r(\theta)^2 + r(\omega(\theta))^2 = r(\eta(\theta))^2 + r((\omega\circ\eta)(\theta))^2$$

for each $\theta \in [0,2\pi]$. The proof is complete. □

We shall give an improvement of the preceding theorem. For each $x \in S_X$, let $\beta(x) = \sup\{\|x+y\| : y \in S_X\}$. Then we have

$$\beta(x(\theta)) = \|x(\theta)+x(\omega(\theta))\| = \|x(\theta)-x(\omega(\theta))\|.$$

On the other hand, as in the proof of Proposition 2.3, we have shown that

$$\|x(\eta(\theta))+x((\eta\circ\omega)(\theta))\| = \frac{2}{\|x(\theta)+x(\omega(\theta))\|},$$

which together with $\eta\circ\omega = \omega\circ\eta$ imply that $\beta(x(\theta))\beta(x(\eta(\theta))) = 2$ for each θ. From this, we have the following result.

COROLLARY 2.5. *$J(X) = \sqrt{2}$ if and only if*

$$r(\theta)^2 + r(\omega(\theta))^2 = r(\eta(\theta))^2 + r((\omega\circ\eta)(\theta))^2$$

for each $\theta \in [0,\eta(0)]$.

Proof. Suppose that

$$r(\theta)^2 + r(\omega(\theta))^2 = r(\eta(\theta))^2 + r((\omega\circ\eta)(\theta))^2$$

for each $\theta \in [0,\eta(0)]$. Then, as in the proof of Theorem 2.4, we have $\beta(x(\theta)) = \sqrt{2}$ for each $\theta \in [0,\eta(0)]$. On the other hand, by Proposition 2.3 (iii) and (v), one obtains

$$\eta^8(\theta) = \omega^4(\theta) = \omega^2(\theta+\pi) = \theta+2\pi$$

for each $\theta \in \mathbb{R}$. In particular, $\eta^8(0) = 2\pi$. Since η is an increasing rotation, it follows that

$$[0,2\pi] = \bigcup_{i=0}^{7}[\eta^i(0),\eta^{i+1}(0)] = \bigcup_{i=0}^{7}\eta^i([0,\eta(0)]).$$

Now let $\theta \in [0,\eta(0)]$. Then $\beta(x(\theta))\beta(x(\eta(\theta))) = 2$; and hence $\beta(x(\eta(\theta))) = \sqrt{2}$ since $\beta(x(\theta)) = \sqrt{2}$. From this and $\beta(x(\eta(\theta)))\beta(x(\eta^2(\theta))) = 2$, we similarly have $\beta(x(\eta^2(\theta))) = \sqrt{2}$. Continuing this process yields $\beta(x(\eta^i(\theta))) = \sqrt{2}$ for $i = 0,1,\ldots,7$. Thus it follows that $\beta(x(\theta)) = \sqrt{2}$ for each $\theta \in [0,2\pi]$; and Theorem 2.4 shows that $J(X) = \sqrt{2}$. □

A norm $\|\cdot\|$ on $\mathbb{R}^2$ is θ_0-rotation invariant if and only if $r(\theta+\theta_0) = r(\theta)$ for each $\theta \in [0,2\pi]$, where $r(\theta) = \|x(\theta)\|_2 = 1/\|(\cos\theta,\sin\theta)\|$. As was mentioned in Section 1 (II), if $\|\cdot\|$ is $\pi/2$-rotation invariant, then $J((\mathbb{R}^2,\|\cdot\|)) = \sqrt{2}$ if and only if $\|\cdot\|$ is $\pi/4$-rotation invariant. The proof was essentially based on a calculation formula given by using absolute normalized norms on $\mathbb{R}^2$. Now, as an application of Theorem 2.3, we can directly prove this result, in terms of the general form given in that theorem. For this, we only need the following simple lemma.

LEMMA 2.6. ([10, Theorem 3.1]) *Let* $\|\cdot\|$ *be a* $\pi/2$*-rotation invariant norm. Suppose that* ω *and* η *are rotations constructed in Proposition 2.3. Then* $\omega(\theta) = \theta + \pi/2$ *and* $\eta(\theta) = \theta + \pi/4$ *for each* $\theta \in \mathbb{R}$.

COROLLARY 2.7. ([10, Thoerem 3.10]) *Let* $\|\cdot\|$ *be a* $\pi/2$*-rotation invariant norm. Then* $J((\mathbb{R}^2, \|\cdot\|)) = \sqrt{2}$ *if and only if* $\|\cdot\|$ *is* $\pi/4$*-rotation invariant.*

Proof. By Theorem 2.4, we have $J((\mathbb{R}^2, \|\cdot\|)) = \sqrt{2}$ if and only if

$$r(\theta)^2 + r(\omega(\theta))^2 = r(\eta(\theta))^2 + r((\omega \circ \eta)(\theta))^2$$

for each $\theta \in [0, 2\pi]$, which is equivalent to

$$\begin{aligned} 2r(\theta)^2 &= r(\theta)^2 + r(\theta + \pi/2)^2 \\ &= r(\theta + \pi/4)^2 + r(\theta + 3\pi/4)^2 \\ &= 2r(\theta + \pi/4)^2 \end{aligned}$$

for each $\theta \in [0, 2\pi]$ by the preceding lemma. Thus $J((\mathbb{R}^2, \|\cdot\|)) = \sqrt{2}$ if and only if $r(\theta) = r(\theta + \pi/4)$ for each θ, that is, $\|\cdot\|$ is $\pi/4$-rotation invariant. $\square$

3. Examples

In this section, we apply our result (mainly Corollary 2.5) to present some new examples of two-dimensional Banach spaces with James constant $\sqrt{2}$. We first give a simple class of such spaces; and then construct an explicit example.

PROPOSITION 3.1. *Let* $(\mathbb{R}^2, \|\cdot\|)$ *be a normed space. Then* $J((\mathbb{R}^2, \|\cdot\|)) = \sqrt{2}$ *if the function* $\theta \to r(\theta)(= 1/\|(\cos\theta, \sin\theta)\|)$ *satisfies the following conditions:*

(i) $r(\theta)^2 + r(\theta + \pi/2)^2 = 2$ *for each* $\theta \in [0, \pi/4]$*; and*

(ii) $r(\theta) = 1$ *for each* $\theta \in [\pi/4, \pi/2] \cup [3\pi/4, \pi]$.

Proof. We first note that $x(\theta) = r(\theta)(\cos\theta, \sin\theta)$ is a parametrization of the unit sphere of $(\mathbb{R}^2, \|\cdot\|)(= X)$. By (i) and (ii), one has $r(0) = r(\pi/4) = r(\pi/2) = r(\pi/3) = 1$; and hence $x(0) = (1, 0)$, $x(\pi/4) = (1/\sqrt{2}, 1/\sqrt{2})$, $x(\pi/2) = (0, 1)$ and $x(3\pi/4) = (-1/\sqrt{2}, 1/\sqrt{2})$. In particular, it follows from

$$\|x(0) + x(\pi/2)\| = \sqrt{2} = \|x(0) - x(\pi/2)\|$$

that $\omega(0) = \pi/2$ by Proposition 2.3, which together with

$$\frac{x(0) + x(\pi/2)}{\|x(0) + x(\pi/2)\|} = x(\pi/4)$$

implies that $\eta(0)=\pi/4$. Hence, by Corollary 2.5, it is enough to show that

$$r(\theta)^2+r(\omega(\theta))^2=r(\eta(\theta))^2+r((\omega\circ\eta)(\theta))^2$$

for each $\theta\in[0,\pi/4]$.

Now we note that

$$\|x(\pi/4)+x(3\pi/4)\|=\sqrt{2}=\|x(\pi/4)-x(3\pi/4)\|$$

and that

$$\frac{x(\pi/4)+x(3\pi/4)}{\|x(\pi/4)+x(3\pi/4)\|}=x(\pi/2).$$

From these we have $\omega(\pi/4)=3\pi/4$ and $\eta(\pi/4)=\pi/2$. Of course, $\omega(\pi/2)=\pi$ by $x(\pi/2)\perp_I x(\pi)(=-x(0))$ and the definition of ω. Since ω and η are both increasing rotations by Proposition 2.3, the following hold:

(a) $\omega([0,\pi/4])=[\omega(0),\omega(\pi/4)]=[\pi/2,3\pi/4]$;

(b) $\eta([0,\pi/4])=[\eta(0),\eta(\pi/4)]=[\pi/4,\pi/2]$; and

(c) $(\omega\circ\eta)([0,\pi/4])=\omega([\eta(0),\eta(\pi/4)])=[\omega(\pi/4),\omega(\pi/2)]=[3\pi/4,\pi]$.

Thus, once it has been proved that $\omega(\theta)=\theta+\pi/2$ for each $\theta\in[0,\pi/4]$, we obtain $r(\theta)^2+r(\omega(\theta))^2=2=r(\eta(\theta))^2+r((\omega\circ\eta)(\theta))^2$ from the assumption (ii).

Let $\theta\in[0,\pi/4]$. Put $x=x(\theta)$ and $y=x(\omega(\theta))$. Since $\eta(\theta)\in[\pi/4,\pi/2]$, one obtains $\|x+y\|=\|x+y\|_2$. On the other hand, as in the proof of Proposition 2.3 (see equation (1)), it follows that

$$\frac{y-x}{\|y-x\|}=x((\omega\circ\eta)(\theta)).$$

Since $\|x((\omega\circ\eta)(\theta))\|_2=r((\omega\circ\eta)(\theta))=1$ from (c) and the assumption (ii), we have $\|x-y\|=\|x-y\|_2$. Now, by Proposition 2.3 (ii), the equation $\|x+y\|_2=\|x-y\|_2$ holds, which implies that $\langle x,y\rangle=0$. In other words, x and y are orthogonal to each other in the usual sense. This and the definition of ω together show that $\omega(\theta)=\theta+\pi/2$, as desired. The proof is complete. □

The following example was appeared in [9, Section 4] and [10, Subsection 5.1]. By Proposition 3.1, we can treat it rather easily.

EXAMPLE 3.2. ([9]) For each $1<a\leqslant 1/34$, let

$$r_a(\theta)=\begin{cases}\sqrt{1+a(1-\cos 8\theta)} & (\theta\in[0,\pi/4])\\ 1 & (\theta\in[\pi/4,\pi/2])\\ \sqrt{1-a(1-\cos 8\theta)} & (\theta\in[\pi/2,3\pi/4])\\ 1 & (\theta\in[3\pi/4,\pi])\end{cases}.$$

We extend this curve to $[0,2\pi]$ by letting $r_a(\theta)=r_a(\theta-\pi)$ for each $\theta\in[\pi,2\pi]$. Put

$$B_a=\{r(\cos\theta,\sin\theta):0\leqslant r\leqslant r_a(\theta),\theta\in[0,2\pi]\}.$$

Then, by [9, Proposition 4.5], the set B_a is absorbing, closed and convex; and hence its Minkowski functional $\|\cdot\|_a$ is a norm on $\mathbb{R}^2$.

Of course, we have $r_a(\theta) = 1/\|(\cos\theta,\sin\theta)\|_a$ for each θ. It is easy to check that the function $\theta \to r_a(\theta)$ satisfies the conditions (i) and (ii) set out in Proposition 3.1. Hence $J((\mathbb{R}^2,\|\cdot\|_a)) = \sqrt{2}$.

We next construct a new explicit example of two-dimensional Banach space with James constant $\sqrt{2}$. The basic idea comes from the preceding example.

For each $0 < b < 1/2$, let

$$s_b(\theta) = \begin{cases} \sqrt{1+b(1-\cos 16\theta)} & (\theta \in [0,\pi/8]) \\ \sqrt{1-b(1-\cos 16\theta)} & (\theta \in [\pi/8,\pi/4]) \\ 1 & (\theta \in [\pi/4,\pi/2]) \end{cases} .$$

Extending this curve to $[0,\pi]$ by putting $s_b(\theta) := \sqrt{2-s_b(\theta-\pi/2)^2}$ for each $\theta \in [\pi/2,\pi]$, and then to $[0,2\pi]$ by putting $s_b(\theta) := s_b(\theta-\pi)$ for each $\theta \in [\pi,2\pi]$. The resulting curve $\theta \to s_b(\theta)(\cos\theta,\sin\theta) : [0,2\pi] \to \mathbb{R}^2$ is a (continuous) simple closed curve; and the set

$$C_b = \{s(\cos\theta,\sin\theta) : 0 \leqslant s \leqslant s_b(\theta), \theta \in [0,2\pi]\}$$

is absorbing, balanced and closed. Let $\|\cdot\|^{(b)}$ be the Minkowski functional of C_b. Then it is a norm on $\mathbb{R}^2$ if C_b is convex, and in that case, $s_b(\theta) = 1/\|(\cos\theta,\sin\theta)\|^{(b)}$ for each θ. Consequently, if C_b is convex, then $J((\mathbb{R}^2,\|\cdot\|^{(b)})) = \sqrt{2}$ by the construction of s_b and Proposition 3.1.

Recall that the curvature of a curve $r(\theta)$ in polar coordinate is given by

$$\kappa(\theta) = \frac{r(\theta)^2 + 2r'(\theta)^2 - r(\theta)r''(\theta)}{(r(\theta)^2 + r'(\theta)^2)^{3/2}}.$$

To examine the convexity of C_b, we make use of the fact that a simple closed curve of positive curvature is convex.

Below we shall show that C_b is convex if $0 < b \leqslant 1/130$.

LEMMA 3.3. *Let $s_b^{(1)}(\theta) = \sqrt{1+b(1-\cos 16\theta)}$ for $\theta \in [0,2\pi]$, and $\kappa_b^{(1)}$ be the curvature of $s_b^{(1)}$. If $0 < b \leqslant 1/128$, then $\kappa_b^{(1)} \geqslant 0$.*

Proof. The first and second derivatives of $s_b^{(1)}$ are given by

$$(s_b^{(1)})'(\theta) = \frac{8b\sin 16\theta}{s_b^{(1)}(\theta)},$$
$$(s_b^{(1)})''(\theta) = \frac{-64b^2\sin^2 16\theta - 128b^2\cos^2 16\theta + 128b(1+b)\cos 16\theta}{s_b^{(1)}(\theta)^3}.$$

It follows that

$$s_b^{(1)}(\theta)^2+2(s_b^{(1)})'(\theta)^2-s_b^{(1)}(\theta)(s_b^{(1)})''(\theta)$$
$$=\frac{-63b^2\cos^2 16\theta-130b(1+b)\cos 16\theta+(1+b)^2+192b^2}{s_b^{(1)}(\theta)^2}.$$

Put $t=\cos 16\theta$ and $f(t)=-63b^2t^2-130b(1+b)t+(1+b)^2+192b^2$. Then $t\in[-1,1]$, f is concave and

$$f(-1)=260b^2+132b+1>0,$$
$$f(1)=1-128b.$$

Thus $\kappa_b^{(1)}\geqslant 0$ if and only if $f(t)\geqslant 0$ for each $t\in[-1,1]$, which happens if and only if $0<b\leqslant 1/128$. □

LEMMA 3.4. *Let* $s_b^{(2)}(\theta)=\sqrt{1-b(1-\cos 16\theta)}$ *for* $\theta\in[0,2\pi]$, *and* $\kappa_b^{(2)}$ *be the curvature of* $s_b^{(2)}$. *If* $0<b\leqslant 1/130$, *then* $\kappa_b^{(2)}\geqslant 0$.

Proof. The first and second derivatives of $s_b^{(2)}$ are as follows:

$$(s_b^{(2)})'(\theta)=-\frac{8b\sin 16\theta}{s_b^{(2)}(\theta)},$$
$$(s_b^{(2)})''(\theta)=\frac{-64b^2\sin^2 16\theta-128b^2\cos^2 16\theta-128b(1-b)\cos 16\theta}{s_b^{(2)}(\theta)^3}.$$

From these, one has that

$$s_b^{(2)}(\theta)^2+2(s_b^{(2)})'(\theta)^2-s_b^{(2)}(\theta)(s_b^{(2)})''(\theta)$$
$$=\frac{-63b^2\cos^2 16\theta+130b(1-b)\cos 16\theta+(1-b)^2+192b^2}{s_b^{(1)}(\theta)^2}.$$

Put $t=\cos 16\theta$ and $g(t)=-63b^2t^2+130b(1-b)t+(1-b)^2+192b^2$. Then g is a concave function. Since $t\in[-1,1]$ and

$$g(-1)=260b^2-132b+1=(130b-1)(2b-1),$$
$$g(1)=1+128b>0,$$

It follows that $\kappa_b^{(2)}\geqslant 0$ if and only if $0<b\leqslant 1/130$. □

By Lemmas 3.3 and 3.4, if $0<b\leqslant 1/130$, the curve s_b consists of parts of a circle and convex curves $s_b^{(1)}$ and $s_b^{(2)}$. It remains to show that the curve s_b is convex at the joints $\theta=0,\pi/8,\pi/4,\pi/2,5\pi/8,3\pi/4$. To see this, it is enough to prove that the set C_b has tangent lines at these points.

Let $x \in \mathbb{R}^2$. For each θ, we note that $x \in R(\theta)(B_{\ell_\infty^2})$ if and only if $\|R(-\theta)x\|_\infty \leqslant 1$, where $B_{\ell_\infty^2}$ is the unit ball of the space $\mathbb{R}^2$ endowed with the norm given by $\|(a,b)\|_\infty = \max\{|a|,|b|\}$. For our purpose, it suffices to show that $C_b \subset R(\theta)(B_{\ell_\infty^2})$ for $\theta = 0, \pi/8, \pi/4$ (since $r(\theta) = 1$ at the joints and ℓ_∞^2 is $\pi/2$-rotation invariant). Before arguing this, we determine the values of

$$\min_{0 \leqslant \theta \leqslant \pi/4} \frac{\tan^2\theta}{1-\cos 16\theta} \quad \text{and} \quad \min_{-\pi/8 \leqslant \theta \leqslant \pi/8} \frac{\tan^2\theta}{1-\cos 16\theta}.$$

Let $\theta \in (-\pi/2, \pi/2)$. We first note that

$$\tan^2\theta = \frac{1-\cos 2\theta}{1+\cos 2\theta},$$

and that

$$\begin{aligned} \cos 16\theta = 2\cos^2 8\theta - 1 &= 2(2\cos^2 4\theta - 1)^2 - 1 \\ &= 8\cos^4 4\theta - 8\cos^2 4\theta + 1. \end{aligned}$$

From this one has that $1 - \cos 16\theta = 8\cos^2 4\theta(1-\cos^2 4\theta)$. Since $\cos 4\theta = 2\cos^2 2\theta - 1$, it follows that

$$\frac{\tan^2\theta}{1-\cos 16\theta} = \frac{1}{32\cos^2 2\theta(1+\cos 2\theta)^2(2\cos^2 2\theta - 1)^2}.$$

Now put $t = \cos 2\theta$. Then $t \in [0,1]$ if $\theta \in [0, \pi/4]$, and $t \in [1/\sqrt{2}, 1]$ if $\theta \in [-\pi/8, \pi/8]$. It is easy to verify that $\max_{0 \leqslant t \leqslant 1} h(t) = \max_{1/\sqrt{2} \leqslant t \leqslant 1} h(t) = h(1) = 128$, where

$$h(t) = 32t^2(1+t)^2(2t^2-1)^2.$$

Thus we obtain

$$\min_{0 \leqslant \theta \leqslant \pi/4} \frac{\tan^2\theta}{1-\cos 16\theta} = \min_{-\pi/8 \leqslant \theta \leqslant \pi/8} \frac{\tan^2\theta}{1-\cos 16\theta} = \frac{1}{128}.$$

We now ready to prove that C_b is convex for $0 < b \leqslant 1/130$.

PROPOSITION 3.5. *Let $0 < b \leqslant 1/130$. Then C_b is convex. Consequently, $\|\cdot\|^{(b)}$ is a norm on $\mathbb{R}^2$.*

Proof. As was mentioned above, it is enough to show that $C_b \subset R(\theta_0)(B_{\ell_\infty^2})$ for $\theta_0 = 0, \pi/8, \pi/4$. For this, we have to show that $R(-\theta_0)x(\theta) \in B_{\ell_\infty^2}$ for each $\theta \in [0, \pi]$ and $\theta_0 = 0, \pi/8, \pi/4$.

Let $\theta_0 \in \mathbb{R}$. We note that if $s_b(\theta) \leqslant 1$, then $R(-\theta_0)x(\theta) \in B_{\ell_2^2} \subset B_{\ell_\infty^2}$, where ℓ_2^2 is the Euclidean 2-space. By the definition of s_b, we have $s_b(\theta)^2 + s_b(\theta + \pi/2)^2 = 2$

for each $\theta \in [0,\pi/2]$. It follows that $s_b(\theta) \leqslant 1$ for each $\theta \in [\pi/8, 5\pi/8] \cup [3\pi/4, \pi]$. Moreover, for each $\theta \in [5\pi/8, 3\pi/4]$, we obtain $\theta - \pi/2 \in [\pi/8, \pi/4]$ and

$$s_b(\theta) = \sqrt{2 - s_b(\theta - \pi/2)^2} = \sqrt{1 + b(1 - \cos 16(\theta - \pi/2))} = s_b^{(1)}(\theta - \pi/2).$$

Since $\|\cdot\|_\infty$ is $\pi/2$-rotation invariant, the equation

$$\begin{aligned} \|R(-\theta_0)x(\theta)\|_\infty &= \|R(-\pi/2)R(-\theta_0)x(\theta)\|_\infty \\ &= s_b(\theta)\|R(-\theta_0)(\cos(\theta-\pi/2), \sin(\theta-\pi/2))\|_\infty \\ &= s_b^{(1)}(\theta-\pi/2)\|R(-\theta_0)(\cos(\theta-\pi/2), \sin(\theta-\pi/2))\|_\infty \end{aligned}$$

holds. From this and the fact that $s_b(\theta) = s_b^{(1)}(\theta)$ for each $\theta \in [0,\pi/8]$, to prove $C_b \subset R(\theta_0)(B_{\ell^2_\infty})$, it is enough to show that

$$s_b^{(1)}(\theta)\|R(-\theta_0)(\cos\theta, \sin\theta)\|_\infty \leqslant 1 \tag{2}$$

for each $\theta \in [0,\pi/4]$.

Now let $\theta_0 \in \{0, \pi/8, \pi/4\}$ and $\theta \in [0,\pi/4]$. Then we have $\theta - \theta_0 \in [-\pi/4, \pi/4]$, which implies that

$$\|R(-\theta_0)(\cos\theta, \sin\theta)\|_\infty = \|(\cos(\theta-\theta_0), \sin(\theta-\theta_0))\|_\infty = \cos(\theta-\theta_0).$$

Hence the inequality (2) is equivalent to

$$(1 + b(1 - \cos 16\theta))\cos^2(\theta - \theta_0) \leqslant 1,$$

which happens if and only if

$$b \leqslant \frac{\tan^2(\theta-\theta_0)}{1-\cos 16\theta}.$$

(I) $\theta_0 = 0$. In this case

$$\frac{\tan^2(\theta-\theta_0)}{1-\cos 16\theta} = \frac{\tan^2\theta}{1-\cos 16\theta}.$$

It follows from

$$b \leqslant \frac{1}{130} < \frac{1}{128} = \min_{0\leqslant\theta\leqslant\pi/4} \frac{\tan^2\theta}{1-\cos 16\theta}$$

that $C_b \subset B_{\ell^2_\infty}$.

(II) $\theta_0 = \pi/8$. We note that

$$\frac{\tan^2(\theta-\theta_0)}{1-\cos 16\theta} = \frac{\tan^2(\theta-\pi/8)}{1-\cos 16(\theta-\pi/8)},$$

which implies that

$$\min_{0\leqslant\theta\leqslant\pi/4} \frac{\tan^2(\theta-\theta_0)}{1-\cos 16\theta} = \min_{-\pi/8\leqslant\theta\leqslant\pi/8} \frac{\tan^2\theta}{1-\cos 16\theta} = \frac{1}{128}.$$

Hence, $b \leqslant 1/130$ guarantees that $C_b \subset R(\pi/8)(B_{\ell_\infty^2})$.

(III) $\theta_0 = \pi/4$. We have

$$\frac{\tan^2(\theta-\theta_0)}{1-\cos 16\theta} = \frac{\tan^2(\theta-\pi/4)}{1-\cos 16(\theta-\pi/4)},$$

This, together with $\tan^2(-\theta) = \tan^2\theta$ and $\cos(-\theta) = \cos\theta$, assures that

$$\min_{0\leqslant\theta\leqslant\pi/4} \frac{\tan^2(\theta-\theta_0)}{1-\cos 16\theta} = \min_{-\pi/4\leqslant\theta\leqslant 0} \frac{\tan^2\theta}{1-\cos 16\theta} = \min_{0\leqslant\theta\leqslant\pi/4} \frac{\tan^2\theta}{1-\cos 16\theta} = \frac{1}{128}.$$

Thus it follows that $C_b \subset R(\pi/4)(B_{\ell_\infty^2})$ by $b \leqslant 1/130$.

From these, the set C_b is convex if $b \leqslant 1/130$. The proof is complete. □

We conclude this section with the following result which is a consequence of Propositions 3.1 and 3.5.

THEOREM 3.6. *Let* $0 < b \leqslant 1/130$. *Then* $J((\mathbb{R}^2, \|\cdot\|^{(b)})) = \sqrt{2}$.

4. Some special properties of $\|\cdot\|^{(b)}$

Throughout this section, we assume that $0 < b \leqslant 1/130$. We here show that $(\mathbb{R}^2, \|\cdot\|^{(b)})$ is not isometrically isomorphic to any absolute, or symmetric, or $\pi/2$-rotation invarinat normed space. Recall that, by [9, Lemma 5.6], a two-dimensional normed space is isometrically isomorphic to some absolute normed space if and only if it is isometrically isomorphic to some symmetric normed space. Hence, it is enough to consider absolute or $\pi/2$-rotation invariant norms. The following are useful for our purpose.

LEMMA 4.1. ([10]) *A normed space* $(\mathbb{R}^2, \|\cdot\|)$ *is isometrically isomorphic to some absolute normed space if and only if there exists a unit vector basis* $\{x,y\}$ *such that* $\|x+\alpha y\| = \|x-\alpha y\|$ *for each* $\alpha \in \mathbb{R}$.

LEMMA 4.2. *A normed space* $(\mathbb{R}^2, \|\cdot\|)$ *is isometrically isomorphic to some* $\pi/2$*-rotation invariant normed space if and only if there exists a unit vector basis* $\{x,y\}$ *such that* $\|x+\alpha y\| = \|-\alpha x+y\|$ *for each* $\alpha \in \mathbb{R}$.

Proof. If $(\mathbb{R}^2, \|\cdot\|)$ is isometrically isomorphic to a $\pi/2$-rotation invariant normed space $(\mathbb{R}^2, \|\cdot\|_0)$. Let $T : (\mathbb{R}^2, \|\cdot\|_0) \to (\mathbb{R}^2, \|\cdot\|)$ be an isometric isomorphism, and let $x = T(1,0)$ and $y = T(0,1)$. Then we have $\|x\| = \|y\| = k > 0$ and

$$\|x+\alpha y\| = \|(1,\alpha)\|_0 = \|R(\pi/2)(1,\alpha)\|_0 = \|(-\alpha,1)\|_0 = \|-\alpha x+y\|$$

for each $\alpha \in \mathbb{R}$. Replacing $\{x,y\}$ with $\{k^{-1}x, k^{-1}y\}$ if necessary, we obtain a unit vector basis with the desired property.

Conversely, if there exists a unit vector basis $\{x,y\}$ satisfying the property set out in the statement of this lemma, we obtain $\|\alpha x+\beta y\| = \|-\beta x+\alpha y\|$ for each $\alpha,\beta \in \mathbb{R}$. Putting $\|(\alpha,\beta)\|' = \|\alpha x+\beta y\|$ yields another norm on $\mathbb{R}^2$ which is isometric to $\|\cdot\|$; and it is obvious that $\|\cdot\|'$ is $\pi/2$-rotation invariant. This completes the proof. □

We introduce some notions about generalized orthogonality types in normed spaces. Let x,y be elements of a normed space. Then x and y are *Roberts orthogonal* to each other, denoted by $x \perp_R y$, if $\|x+\alpha y\| = \|x-\alpha y\|$ for each $\alpha \in \mathbb{R}$ (Roberts [12]); are *twisted orthogonal* to each other, denoted by $x \perp_t y$, if $\|x+\alpha y\| = \|-\alpha x+y\|$ for each $\alpha \in \mathbb{R}$; and x is *Birkhoff orthogonal* to y, denoted by $x \perp_B y$, if $\|x+\alpha y\| \geqslant \|x\|$ for each $\alpha \in \mathbb{R}$ (Birkhoff [3]). It is not difficult to check that $x \perp_R y$ implies $y \perp_R x$, $x \perp_B y$ and $x \perp_I y$, and that $x \perp_t y$ implies $y \perp_t x$ and $x \perp_I y$.

From the definitions of Roberts orthogonality and twisted orthogonality, we have to show that there is no unit vector basis $\{x,y\}$ for $(\mathbb{R}^2,\|\cdot\|^{(b)})$ such that $x \perp_R y$ or $x \perp_t y$. For this purpose, the following lemma plays an important role. The proof is essentially based on [10, Theorem 3.2]. Recall that, for each convex function ψ on $[0,1]$ satisfying $\max\{1-t,t\} \leqslant \psi(t) \leqslant 1$ for each t, the equation

$$\|(\alpha,\beta)\|_\psi = \begin{cases} (|\alpha|+|\beta|)\psi\left(\dfrac{|\beta|}{|\alpha|+|\beta|}\right) & ((\alpha,\beta)\neq(0,0)) \\ 0 & ((\alpha,\beta)=(0,0)) \end{cases}$$

defines an absolute (normalized) norm on $\mathbb{R}^2$ (Bonsall and Duncan [4]). For such an norm, we have

$$\max\{|\alpha|,|\beta|\} \leqslant \|(\alpha,\beta)\|_\psi \leqslant |\alpha|+|\beta|$$

for each $\alpha,\beta \in \mathbb{R}$. Now put $\widetilde{\psi}(t) = \psi(1-t)$. Then, by [10, Proposition 3.4], the norm given by

$$\|(\alpha,\beta)\|_{\psi,\widetilde{\psi}} = \begin{cases} \|(\alpha,\beta)\|_\psi \ (\alpha\beta \geqslant 0) \\ \|(\alpha,\beta)\|_{\widetilde{\psi}} \ (\alpha\beta \leqslant 0) \end{cases}$$

is $\pi/2$-rotation invariant. The space $(\mathbb{R}^2,\|\cdot\|_{\psi,\widetilde{\psi}})$ is denoted by $\ell^2_{\psi,\widetilde{\psi}}$.

LEMMA 4.3. *If there exists a unit vector basis $\{x,y\}$ with $x \perp_t y$ in a normed space $(\mathbb{R}^2,\|\cdot\|)$. Then there exists another unit vector basis $\{x_0,y_0\}$ with $x_0 \perp_t y_0$, $x_0 \perp_B y_0$ and $y_0 \perp_B x_0$.*

Proof. From the assumption, $(\mathbb{R}^2,\|\cdot\|)$ is isometrically isomorphic to the $\pi/2$-rotation invariant space $(\mathbb{R}^2,\|\cdot\|_0)$, where $\|(\alpha,\beta)\|_0 = \|\alpha x+\beta y\|$ for each $\alpha,\beta \in \mathbb{R}^2$. By [10, Theorem 3.2], there exists a convex function ψ on $[0,1]$ satisfying $\max\{1-t,t\} \leqslant \psi(t) \leqslant 1$ for each t such that $(\mathbb{R}^2,\|\cdot\|_0)$ is isometrically isomorphic to $\ell^2_{\psi,\widetilde{\psi}}$. We note that $e_1=(1,0)$ and $e_2=(0,1)$ (considered in $\ell^2_{\psi,\widetilde{\psi}}$) satisfy $e_1 \perp_t e_2$, $e_1 \perp_B e_2$ and $e_2 \perp_B e_1$. Let T be an isometric isomorphism from $\ell^2_{\psi,\widetilde{\psi}}$ onto $(\mathbb{R}^2,\|\cdot\|)$, and let $x_0 = Te_1$ and $y_0 = Te_2$. Then $\{x_0,y_0\}$ is a unit vector basis for $(\mathbb{R}^2,\|\cdot\|)$ having the desired properties. □

From this, for our purpose, it is enough to consider the case $x = x(\theta)$ with $\theta \in [0,\pi/2]$ and $y = x(\omega(\theta))$. Indeed, let $\|x\| = \|y\| = 1$, and $x \perp_R y$ or $x \perp_t y$. Then $x \perp_I y$. From the preceding lemma, we may assume that $x \perp_B y$ and $y \perp_B x$. We note that $x \perp_R y$ implies that $\pm x \perp_R \pm y$; $x \perp_B y$ implies that $\pm x \perp_B \pm y$; and $x \perp_t y$ implies that $\pm x \perp_t \pm y$. From these, in any case, we may assume that x, y are in the first or second quadrants. Replacing x with y if necessary, we can write $x = x(\theta)$ and $y = x(\omega(\theta))$. As was mentioned in the proof of Proposition 3.1, one has $\omega(0) = \pi/2$; and so $\omega(-\pi/2) = 0$, $\omega(\pi/2) = \pi$ and $\omega(\pi) = 3\pi/2$, which implies that $\omega([-\pi/2,0]) = [0,\pi/2]$, $\omega([0,\pi/2]) = [\pi/2,\pi]$ and $\omega([\pi/2,\pi]) = [\pi,3\pi/2]$. These, together with the fact that ω is strictly increasing, prove that $\theta \in [0,2\pi]$.

The following auxiliary lemma will be needed.

LEMMA 4.4. *Let $(\mathbb{R}^2, \|\cdot\|)$ be a normed space, and let $r(\theta) = 1/\|(\cos\theta,\sin\theta)\|$ and $x(\theta) = r(\theta)(\cos\theta,\sin\theta)$ for each θ. Suppose that $\theta_0 \in [0,\pi]$. Then $x(\theta_0) \perp_R x(\theta_0+\pi/2)$ if and only if $r(\theta_0+\theta) = r(\theta_0-\theta)$ for each $\theta \in [0,\pi/2]$.*

Proof. We note that

$$\begin{aligned} x(\theta_0) + \alpha x(\theta_0+\pi/2) &= r(\theta_0)(\cos\theta_0,\sin\theta_0) + \alpha r(\theta_0+\pi/2)(-\sin\theta_0,\cos\theta_0) \\ &= s(\alpha)(\cos(\theta_0-\varphi),\sin(\theta_0+\psi)), \end{aligned}$$

where $s(\alpha) = \sqrt{r(\theta_0)^2 + \alpha^2 r(\theta_0+\pi/2)^2}$ and φ, ψ are real numbers satisfying

$$\sin\varphi = -\frac{\alpha r(\theta_0+\pi/2)}{s(\alpha)} \quad \text{and} \quad \cos\varphi = \frac{r(\theta_0)}{s(\alpha)}$$

and

$$\sin\psi = \frac{\alpha r(\theta_0+\pi/2)}{s(\alpha)} \quad \text{and} \quad \cos\psi = \frac{r(\theta_0)}{s(\alpha)},$$

respectively. In particular, we have

$$x(\theta_0) + \alpha x(\theta_0+\pi/2) = s(\alpha)(\cos(\theta_0+\psi),\sin(\theta_0+\psi)).$$

Similarly, we obtain

$$x(\theta_0) - \alpha x(\theta_0+\pi/2) = s(\alpha)(\cos(\theta_0-\psi),\sin(\theta_0-\psi)).$$

Since the function

$$\alpha \to \frac{\alpha r(\theta_0+\pi/2)}{s(\alpha)}$$

maps $\mathbb{R}^+$ onto $[0,1)$ while $r(\theta_0)/s(\alpha)$ is always positive, the parameter ψ takes all the values in $[0,\pi/2)$ when α varies in $\mathbb{R}^+$.

Now we have

$$\|x(\theta_0) \pm \alpha x(\theta_0+\pi/2)\| = \frac{s(\alpha)}{r(\theta_0 \pm \psi)}$$

for each $\alpha \in \mathbb{R}^+$. Hence $x(\theta_0) \perp_R x(\theta_0+\pi/2)$ holds if and only if $r(\theta_0+\theta) = r(\theta_0 - \theta)$ for each $\theta \in [0,\pi/2]$. □

We now proceed to the proof of the absence of a Roberts orthogonal unit vector basis in $(\mathbb{R}^2, \|\cdot\|^{(b)})$. To this end, recall that Birkhoff orthogonality is deeply related to tangent lines. Indeed, if $x, y \in S_X$, then $x \perp_B y$ means $\|x+\alpha y\| \geqslant 1$. Consequently, in two-dimensional case, $x, y \in S_X$ and $x \perp_B y$ imply that $\alpha \to x+\alpha y$ is a tangent line for B_X at x.

PROPOSITION 4.5. *There is no unit vector basis $\{x,y\}$ satisfying $x \perp_R y$ in $(\mathbb{R}^2, \|\cdot\|^{(b)})$. Consequently, $(\mathbb{R}^2, \|\cdot\|^{(b)})$ is not isometrically isomorphic to any absolute normed space.*

Proof. We first show that if $x(\theta) \perp_R x(\omega(\theta))$ (with $\theta \in [0,\pi/2]$), then $\omega(\theta) = \theta+\pi/2$. Indeed, as in the proof of Proposition 3.1, we have $\omega(\theta) = \theta+\pi/2$ for each $\theta \in [0,\pi/4]$; and so $\omega(\pi/2) = \pi$. On the other hand, the curve $x(\theta)$ coincides with the unit circle on $(\pi/4, \pi/2)$, which implies that the tangent line for the unit ball of $(\mathbb{R}^2, \|\cdot\|^{(b)})$ at $\theta \in (\pi/4,\pi/2)$ is only the straight line $\alpha \to x(\theta)+\alpha x(\theta+\pi/2)$. This, together with the fact that$x(\theta) \perp_B x(\omega(\theta))$, proves $\omega(\theta) = \theta+\pi/2$.

Now Lemma 4.4 applies; and if $x(\theta_0) \perp_R x(\omega(\theta_0))$ for some $\theta_0 \in [0,\pi/2]$, then $s_b(\theta_0+\theta) = s_b(\theta_0-\theta)$ for each $\theta \in [0,\pi/2]$. However, s_b satisfies the following conditions:

(i) $s_b(\theta) > 1$ for each $\theta \in (-3\pi/8, -\pi/4) \cup (0,\pi/8) \cup (5\pi/8, 3\pi/4)$;

(ii) $s_b(\theta) = 1$ for each $\theta \in [-\pi/4, 0] \cup [\pi/4, \pi/2] \cup [3\pi/4, \pi] \cup \{-3\pi/8, \pi/8, 5\pi/8\}$; and

(iii) $s_b(\theta) < 1$ for each $\theta \in (-\pi/2, -3\pi/8) \cup (\pi/8, \pi/4) \cup (\pi/2, 5\pi/8)$.

From these, we conclude that $x(\theta_0) \perp_R x(\omega(\theta_0))$ cannot happen. Thus there is no unit vector basis $\{x,y\}$ satisfying $x \perp_R y$ in $(\mathbb{R}^2, \|\cdot\|^{(b)})$. □

We next show that there is no unit vector basis $\{x,y\}$ satisfying $x \perp_t y$ in $(\mathbb{R}^2, \|\cdot\|^{(b)})$. In fact, it is a consequence of a more general result. For this we need the following simple lemma.

LEMMA 4.6. *Let $(\mathbb{R}^2, \|\cdot\|)$ be a normed space, and $r(\theta) = 1/\|(\cos\theta, \sin\theta)\|$ and $x(\theta) = r(\theta)(\cos\theta, \sin\theta)$ for each θ. Suppose that $r(\theta)$ satisfies the conditions (i) and (ii) set out in Proposition 3.1. Then*

$$\frac{2+\sqrt{2}}{4} \leqslant r(\theta)^2 \leqslant 4-2\sqrt{2}$$

for each $\theta \in [0,\pi/4]$.

Proof. From the definition, $r(0) = r(\pi/4) = 1$, that is, $x(0) = (1,0)$ and $x(\pi/4) = (1/\sqrt{2}, 1/\sqrt{2})$. Since the unit ball of $(\mathbb{R}^2, \|\cdot\|)$ is convex and $r(\theta) = \|x(\theta)\|_2$, we have

$$\min_{0\leqslant t\leqslant 1} \|(1-t)x(0)+tx(\pi/4)\|_2 \leqslant r(\theta)$$

for each $\theta \in [0,\pi/4]$. It follows from

$$\min_{0\leqslant t\leqslant 1} \|(1-t)x(0)+tx(\pi/4)\|_2 = \|2^{-1}(x(0)+x(\pi/4))\|_2 = \frac{(2+\sqrt{2})^{1/2}}{2}$$

that

$$\frac{2+\sqrt{2}}{4} \leqslant r(\theta)^2$$

for each $\theta \in [0,\pi/4]$.

On the other hand, since $r(\theta) = 1$ for each $\theta \in [-\pi/4, 0] \cup [\pi/4, \pi/2]$, the straight lines $t \to (1,t)$ and $t \to x(\pi/4) + (t,-t)$ are tangent to the unit ball of $(\mathbb{R}^2, \|\cdot\|)$ at $x(0)$ and $x(\pi/4)$, respectively. In other words, for $0 \leqslant \theta \leqslant \pi/4$, the vector $x(\theta)$ is in the regular octagon that circumscribes the unit circle. Now let

$$\|(\alpha,\beta)\|_0 = \max\{\|(\alpha,\beta)\|_\infty, \|(\alpha,\beta)\|_1/\sqrt{2}\} = \max\{|\alpha|, |\beta|, (|\alpha|+|\beta|)/\sqrt{2}\}$$

for each $\alpha, \beta \in \mathbb{R}$. Then $\|\cdot\|_0$ is a norm on $\mathbb{R}^2$ such that the unit ball with respect to this norm is the regular octagon that circumscribes the unit circle. Thus our argument in above shows that $\|x(\theta)\|_0 \leqslant 1$ for each $\theta \in [0,\pi/4]$, that is, the inequality

$$r(\theta)\max\{\cos\theta, \cos(\pi/4-\theta)\} \leqslant 1$$

holds for each $\theta \in [0,\pi/4]$. Since

$$\min_{0\leqslant\theta\leqslant\pi/4} \max\{\cos\theta, \cos(\pi/4-\theta)\} = \cos(\pi/8) = \frac{\sqrt{2+\sqrt{2}}}{2},$$

it follows that

$$r(\theta) \leqslant \frac{2}{\sqrt{2+\sqrt{2}}}$$

for each $\theta \in [0,\pi/4]$; and hence

$$r(\theta)^2 \leqslant 4-2\sqrt{2}$$

for each $\theta \in [0,\pi/4]$. $\square$

PROPOSITION 4.7. *Let $(\mathbb{R}^2, \|\cdot\|)$ be a normed space, and $r(\theta) = 1/\|(\cos\theta, \sin\theta)\|$ and $x(\theta) = r(\theta)(\cos\theta, \sin\theta)$ for each θ. Suppose that $r(\theta)$ satisfies the conditions (i) and (ii) set out in Proposition 3.1. Then $(\mathbb{R}^2, \|\cdot\|)$ is isometrically isomorphic to some $\pi/2$-rotation invariant normed space if and only if $r(\theta) = 1$ for each $\theta \in [0, 2\pi]$.*

Proof. If $r(\theta)=1$ for each $\theta\in[0,2\pi]$, then $(\mathbb{R}^2,\|\cdot\|)$ is itself a $\pi/2$-rotation invariant normed space. For the converse, suppose that $(\mathbb{R}^2,\|\cdot\|)$ is isometrically isomorphic to some $\pi/2$-rotation invariant normed space. Then, by Lemma 4.3, there exists a unit vector basis $\{x,y\}$ for $(\mathbb{R}^2,\|\cdot\|)$ with the properties that $x\perp_t y$, $x\perp_B y$ and $y\perp_B x$. From the comment following Lemma 4.3, we may assume that $x=x(\theta_0)$ with $\theta_0\in[0,\pi/2]$ and $y=x(\omega(\theta_0))$. Moreover, by an argument similar to that in the beginning of the proof of Proposition 4.5, we have $\omega(\theta_0)=\theta_0+\pi/2$.

We first show that $r(\theta_0)=1$ (which happens, at least, if $\theta_0\in\{0\}\cup[\pi/4,\pi/2]$). To see this, it is enough to consider the case $\theta_0\in(0,\pi/4)$. Then $0<\tan\theta_0<1$. As in the proof of Lemma 4.4, we have

$$\begin{aligned}x(\theta_0)+\alpha x(\theta_0+\pi/2)&=r(\theta_0)(\cos\theta_0,\sin\theta_0)+\alpha r(\theta_0+\pi/2)(-\sin\theta_0,\cos\theta_0)\\&=s_1(\alpha)(\cos(\theta_0+\varphi_1),\sin(\theta_0+\varphi_1)),\end{aligned}$$

where $s_1(\alpha)=\sqrt{r(\theta_0)^2+\alpha^2r(\theta_0+\pi/2)^2}$ and φ_1 is a real number satisfying

$$\sin\varphi_1=\frac{\alpha r(\theta_0+\pi/2)}{s_1(\alpha)}\quad\text{and}\quad\cos\varphi_1=\frac{r(\theta_0)}{s_1(\alpha)}.$$

Similarly, one has

$$\begin{aligned}-\alpha x(\theta_0)+x(\theta_0+\pi/2)&=-\alpha r(\theta_0)(\cos\theta_0,\sin\theta_0)+r(\theta_0+\pi/2)(-\sin\theta_0,\cos\theta_0)\\&=s_2(\alpha)(-\sin(\theta_0+\varphi_2),\cos(\theta_0+\varphi_2)),\end{aligned}$$

where $s_2(\alpha)=\sqrt{\alpha^2r(\theta_0)^2+r(\theta_0+\pi/2)^2}$ and φ_2 satisfies

$$\sin\varphi_2=\frac{\alpha r(\theta_0)}{s_2(\alpha)}\quad\text{and}\quad\cos\varphi_2=\frac{r(\theta_0+\pi/2)}{s_2(\alpha)}.$$

On the other hand, the inequalities

$$0\leqslant r(\theta_0)\cos\theta_0-\alpha r(\theta_0+\pi/2)\sin\theta_0\leqslant r(\theta_0)\sin\theta_0+\alpha r(\theta_0+\pi/2)\cos\theta_0$$

hold if and only if

$$\frac{r(\theta_0)}{r(\theta_0+\pi/2)}\cdot\frac{1-\tan\theta_0}{1+\tan\theta_0}\leqslant\alpha\leqslant\frac{r(\theta_0)}{r(\theta_0+\pi/2)}\cdot\frac{1}{\tan\theta_0}.$$

Moreover, we obtain

$$0\leqslant-\alpha r(\theta_0)\sin\theta_0+r(\theta_0+\pi/2)\cos\theta_0\leqslant\alpha r(\theta_0)\cos\theta_0+r(\theta_0+\pi/2)\sin\theta_0$$

if and only if

$$\frac{r(\theta_0+\pi/2)}{r(\theta_0)}\cdot\frac{1-\tan\theta_0}{1+\tan\theta_0}\leqslant\alpha\leqslant\frac{r(\theta_0+\pi/2)}{r(\theta_0)}\cdot\frac{1}{\tan\theta_0}.$$

From these, for each α satisfying

$$\begin{aligned} &\max\left\{\frac{r(\theta_0)}{r(\theta_0+\pi/2)}, \frac{r(\theta_0+\pi/2)}{r(\theta_0)}\right\}\cdot\frac{1-\tan\theta_0}{1+\tan\theta_0} \\ &\leqslant \alpha \\ &\leqslant \min\left\{\frac{r(\theta_0)}{r(\theta_0+\pi/2)}, \frac{r(\theta_0+\pi/2)}{r(\theta_0)}\right\}\frac{1}{\tan\theta_0}, \end{aligned} \tag{3}$$

it follows that

$$\|x(\theta_0)+\alpha x(\theta_0+\pi/2)\| = \|x(\theta_0)+\alpha x(\theta_0+\pi/2)\|_2 = s_1(\alpha)$$

and

$$\|-\alpha x(\theta_0)+x(\theta_0+\pi/2)\| = \|-\alpha x(\theta_0)+x(\theta_0+\pi/2)\|_2 = s_2(\alpha).$$

However, since

$$s_1(\alpha)^2 - s_2(\alpha)^2 = (1-\alpha^2)(r(\theta_0)^2 - r(\theta_0+\pi/2)^2),$$

if there exists an $\alpha(\neq 1)$ satisfying the inequalities (3), then $x(\theta_0) \perp_t x(\theta_0+\pi/2)$ implies that $r(\theta_0) = r(\theta_0+\pi/2)$. This, together with the condition $r(\theta_0)^2 + r(\theta_0+\pi/2)^2 = 2$, guarantees that $r(\theta_0) = 1$.

Now we aim to prove the strict inequality

$$\begin{aligned} &\max\left\{\frac{r(\theta_0)}{r(\theta_0+\pi/2)}, \frac{r(\theta_0+\pi/2)}{r(\theta_0)}\right\}\cdot\frac{1-\tan\theta_0}{1+\tan\theta_0} \\ &< \min\left\{\frac{r(\theta_0)}{r(\theta_0+\pi/2)}, \frac{r(\theta_0+\pi/2)}{r(\theta_0)}\right\}\frac{1}{\tan\theta_0}. \end{aligned}$$

Once it has been proved, we have at least two distinct α satisfying (3). Our argument is divided into two cases.

(I) $r(\theta_0) \geqslant r(\theta_0+\pi/2)$. In this case, we have to show that

$$\frac{\tan\theta_0(1-\tan\theta_0)}{1+\tan\theta_0} < \frac{r(\theta_0+\pi/2)^2}{r(\theta_0)^2} = \frac{2-r(\theta_0)^2}{r(\theta_0)^2}.$$

Since $\tan\theta_0 \in (0,1)$, one has

$$\frac{\tan\theta_0(1-\tan\theta_0)}{1+\tan\theta_0} \leqslant \max_{0\leqslant t\leqslant 1}\frac{t(1-t)}{1+t} = (\sqrt{2}-1)^2.$$

Therefore, it suffices to show that

$$(\sqrt{2}-1)^2 < \frac{2-r(\theta_0)^2}{r(\theta_0)^2},$$

which is equivalent to

$$r(\theta_0)^2 < \frac{2}{(\sqrt{2}-1)^2+1} = \frac{1}{2-\sqrt{2}} = \frac{2+\sqrt{2}}{2}.$$

On the other hand, by Lemma 4.6, we already have

$$r(\theta_0)^2 \leqslant 4-2\sqrt{2} < \frac{2+\sqrt{2}}{2}.$$

(II) $r(\theta_0) \leqslant r(\theta_0+\pi/2)$. Our aim is to show that

$$\frac{\tan\theta_0(1-\tan\theta_0)}{1+\tan\theta_0} < \frac{r(\theta_0)^2}{r(\theta_0+\pi/2)^2} = \frac{r(\theta_0)^2}{2-r(\theta_0)^2}.$$

As in the preceding paragraph, it is enough to prove that

$$(\sqrt{2}-1)^2 < \frac{r(\theta_0)^2}{2-r(\theta_0)^2},$$

or equivalently, that

$$\frac{2(\sqrt{2}-1)^2}{(\sqrt{2}-1)^2+1} < r(\theta_0)^2.$$

We here note that

$$\frac{2(\sqrt{2}-1)^2}{(\sqrt{2}-1)^2+1} = \frac{2-\sqrt{2}}{2} < \frac{2+\sqrt{2}}{4},$$

which, together with Lemma 4.6, assures that

$$\frac{2(\sqrt{2}-1)^2}{(\sqrt{2}-1)^2+1} < \frac{2+\sqrt{2}}{4} \leqslant r(\theta_0)^2.$$

Thus, in any case, we have the strict inequality

$$\begin{aligned}\max&\left\{\frac{r(\theta_0)}{r(\theta_0+\pi/2)}, \frac{r(\theta_0+\pi/2)}{r(\theta_0)}\right\}\cdot\frac{1-\tan\theta_0}{1+\tan\theta_0}\\ &< \min\left\{\frac{r(\theta_0)}{r(\theta_0+\pi/2)}, \frac{r(\theta_0+\pi/2)}{r(\theta_0)}\right\}\frac{1}{\tan\theta_0}.\end{aligned}$$

Consequently, we obtain $r(\theta_0)=1$, as desired.

Now it follows from $x(\theta_0) \perp_t x(\theta_0+\pi/2)$ and $r(\theta_0)^2+r(\theta_0+\pi/2)^2=2$ that $r(\theta_0+\pi/2)=\sqrt{2-r(\theta_0)^2}=1=r(\theta_0)$ and

$$\|\alpha x(\theta_0)+\beta x(\theta_0+\pi/2)\| = \|-\beta x(\theta_0)+\alpha x(\theta_0+\pi/2)\|$$

for each $\alpha,\beta\in\mathbb{R}$. From these, one has that $x(\theta_0)=(\cos\theta_0,\sin\theta_0)$ and $x(\theta_0+\pi/2)=(\cos(\theta_0+\pi/2),\sin(\theta_0+\pi/2))$, and that

$$\begin{aligned}\|R(\pi/2)(\alpha x(\theta_0)+\beta x(\theta_0+\pi/2))\| &= \|\alpha x(\theta_0+\pi/2)+\beta x(\theta_0+\pi)\|\\ &= \|\alpha x(\theta_0+\pi/2)-\beta x(\theta_0)\|\\ &= \|\alpha x(\theta_0)+\beta x(\theta_0+\pi/2)\|\end{aligned}$$

for each $\alpha,\beta \in \mathbb{R}$; that is, $\|\cdot\|$ is itself $\pi/2$-rotation invariant. Hence it follows that $r(\theta+\pi/2) = r(\theta)$ for each θ, which together with the condition (i) set out in Proposition 3.1 implies that

$$2r(\theta)^2 = 2r(\theta+\pi/2)^2 = r(\theta)^2 + r(\theta+\pi/2)^2 = 2$$

for each $\theta \in [0,\pi/4]$, that is, $r(\theta) = 1$ for each $\theta \in [0,\pi/4] \cup [\pi/2,3\pi/4]$. This and the condition (ii) in Proposition 3.1, together with the fact that $r(\theta+\pi) = r(\theta)$ for all θ, guarantee that $r(\theta) = 1$ for each $\theta \in [0,2\pi]$. The proof is complete. □

COROLLARY 4.8. *The normed space $(\mathbb{R}^2, \|\cdot\|^{(b)})$ is not isometrically isomorphic to any $\pi/2$-rotation invariant normed space.*

REMARK 4.9. Proposition 4.7 guarantees that the normed space $(\mathbb{R}^2, \|\cdot\|_a)$ that appeared in Example 3.2 is not isometrically isomorphic to any $\pi/2$-rotation invariant normed space. This result was already mentioned in [10, Theorem 5.2]. However, in that theorem, the proof was based on the existence of a unit vector basis $\{x,y\}$ for $(\mathbb{R}^2, \|\cdot\|_a)$ with the property that $x \perp_R y$. On the contrary, for the type of normed spaces considered in Proposition 3.1, Proposition 4.7 provides a more general and direct criterion for the existence of unit vector bases that are twisted orthogonal to each other.

We conclude this section with the following result that summarizes Theorem 3.6, Proposition 4.5 and Corollary 4.8.

THEOREM 4.10. *Let $0 < b \leqslant 1/130$. Then the normed space $(\mathbb{R}^2, \|\cdot\|^{(b)})$ satisfies the following properties:*

(i) $J((\mathbb{R}^2, \|\cdot\|^{(b)})) = \sqrt{2}$;

(ii) *$(\mathbb{R}^2, \|\cdot\|^{(b)})$ is not isometrically isomorphic to any absolute normed space;*

(iii) *$(\mathbb{R}^2, \|\cdot\|^{(b)})$ is not isometrically isomorphic to any symmetric normed space; and*

(iv) *$(\mathbb{R}^2, \|\cdot\|^{(b)})$ is not isometrically isomorphic to any $\pi/2$-rotation invariant normed space.*

5. Remarks

Finally, we point out the relationship between our results and Hadwiger's unsettled problem on covering convex bodies by smaller homothets [6]. Let K be a convex body in $\mathbb{R}^n$, that is, a compact convex subset with nonempty interior. If $x \in \mathbb{R}^n$ and $\lambda \in (0,1)$, the set

$$x + \lambda K := \{x + \lambda y : y \in K\}$$

is called a *smaller homothetic copy of* K. Now let $c(K)$ be the smallest number of smaller homothetic copies of K needed to cover K. Then it is equal to the smallest integer m such that there exist m points $x_1,\dots,x_m$ and a $\lambda \in (0,1)$ satisfying

$$K \subset \bigcup_{j=1}^{m} (x_j + \lambda K).$$

Hadwiger's conjecture is as follows: Let K be a convex body of $\mathbb{R}^n$. Then $c(K) \leqslant 2^n$, and the equality holds if and only if K is a parallelotope. This conjecture is unsettled except for $n = 2$.

In connection with Hadwiger's conjecture, the notion of the *m-covering number* of a convex body was considered in [11]. For a convex body K in $\mathbb{R}^n$ and an $m \in \mathbb{N}$, let $h_m(K)$ be the smallest possible positive ratio of m homothetical copies of K whose union covers K. In [11], Lassak conjectured that every convex body K in $\mathbb{R}^2$ with $h_m(K) = 1/\sqrt{2}$ is an affine image of a convex body whose boundary $r(\theta)$ satisfies $r(\theta + \pi/4) = r(\theta)$.

Very recently, He et al. [7] showed the connection between m-covering numbers and the Schäffer constant of normed spaces. More precisely, they proved that if $K \subset \mathbb{R}^2$ is centrally symmetric, that is, if it is the unit ball B_X of some normed space $X = (\mathbb{R}^2, \|\cdot\|)$, then

$$h_4(K) = 2^{-1} S(X) = 2^{-1} \inf\{\|x+y\| : x, y \in S_X,\ x \perp_I y\}.$$

On the other hand, the James constant and Schäffer constant are closely related to each other by the equality $J(X)S(X) = 2$; see Gao and Lau [5]. From this, it follows that

$$h_4(K) = 1/J(X).$$

Now we consider Lassak's conjecture. Let K be the unit ball of the normed space $(\mathbb{R}^2, \|\cdot\|^{(b)})$ studied in the preceding two sections. Then we have

$$h_4(K) = 1/J((\mathbb{R}^2, \|\cdot\|^{(b)})) = 1/\sqrt{2}.$$

However, we showed that the space $(\mathbb{R}^2, \|\cdot\|^{(b)})$ is not isometrically isomorphic to any $\pi/2$-rotation invariant normed space, which means that K can not be an affine image of a convex body whose boundary $r(\theta)$ satisfies $r(\theta + \pi/4) = r(\theta)$. Hence we see that the answer to Lassak's conjecture is negative.

We here remark that Lassak's conjecture is true for the partial case that K is the unit ball of a $\pi/2$-rotation invariant normed space. In that case, we can make use of Corollary 2.7.

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