Mathematical
Inequalities
& Applications
Volume 20, Number 4 (2017), 929–947 doi:10.7153/mia-2017-20-58

CHARACTERIZATIONS OF BOUNDEDNESS FOR GENERALIZED FRACTIONAL INTEGRALS ON MARTINGALE MORREY SPACES

EIICHI NAKAI AND GAKU SADASUE

(*Communicated by J. Soria*)

Abstract. On generalized martingale Morrey spaces we give necessary and sufficient conditions for the boundedness of generalized fractional integrals as martingale transforms.

1. Introduction

It is well known as the Hardy-Littlewood-Sobolev theorem that the fractional integral operators I_α on the Euclidean space $\mathbb{R}^n$ is bounded from L_p to L_q for $1<p<q<\infty$, $0<\alpha<n$ and $-n/p+\alpha=-n/q$. The fractional integrals are very useful tools to analyze function spaces in harmonic analysis and the Hardy-Littlewood-Sobolev theorem was generalized and extended to various function spaces. In martingale theory, based on the result by Watari [16, Theorem 1.1], Chao and Ombe [3] proved the boundedness of the fractional integrals for H_p, L_p, BMO and Lipschitz spaces of the dyadic martingales. These fractional integrals were defined for more general martingales in [14]. See also Hao and Jiao [8]. On the other hand, martingale Morrey spaces and their generalization were introduced by [12] and [13], respectively, and the boundedness of fractional integrals as martingale transforms were established. In this paper we give necessary and sufficient conditions for the boundedness of fractional integrals on generalized martingale Morrey spaces which are improvement of the results in [13].

Let $(\Omega,\mathscr{F},P)$ be a probability space and let $\{\mathscr{F}_n\}_{n\geqslant 0}$ be a nondecreasing sequence of sub-σ-algebras of $\mathscr{F}$ such that $\mathscr{F}=\sigma(\bigcup_n \mathscr{F}_n)$. We suppose that every σ-algebra $\mathscr{F}_n$ is generated by countable atoms, where $B\in\mathscr{F}_n$ is called an atom (more precisely a $(\mathscr{F}_n,P)$-atom), if any $A\subset B$ with $A\in\mathscr{F}_n$ satisfies $P(A)=P(B)$ or $P(A)=0$. Denote by $A(\mathscr{F}_n)$ the set of all atoms in $\mathscr{F}_n$. The expectation operator is denoted by E. Let $L_{p,\mathrm{loc}}$ be the set of all measurable functions such that $|f|^p\chi_B$ is integrable for all $B\in A(\mathscr{F}_0)$. If $\mathscr{F}_0=\{\Omega,\emptyset\}$, then $L_{p,\mathrm{loc}}=L_p$. An $\mathscr{F}_n$-measurable function $g\in L_{1,\mathrm{loc}}$ is called the conditional expectation of $f\in L_{1,\mathrm{loc}}$ relative to $\mathscr{F}_n$ if

$$E[g\chi_B\chi_G]=E[f\chi_B\chi_G] \quad \text{for all} \quad B\in A(\mathscr{F}_0) \quad \text{and} \quad G\in\mathscr{F}_n.$$

We denote by E_nf the conditional expectation of f relative to $\mathscr{F}_n$. We say a sequence $(f_n)_{n\geqslant 0}$ in $L_{1,\mathrm{loc}}$ is a martingale relative to $\{\mathscr{F}_n\}_{n\geqslant 0}$ if it is adapted to $\{\mathscr{F}_n\}_{n\geqslant 0}$ and satisfies $E_n[f_m]=f_n$ for every $n\leqslant m$.

We first recall the definition of generalized fractional integrals of martingales.

Mathematics subject classification (2010): 60G46, 60G42, 46E30, 42B35.
Keywords and phrases: Martingale, Morrey space, fractional integral.

DEFINITION 1. ([13]) Let $(\gamma_n)_{n\geqslant 0}$ be a non-increasing sequence of non-negative bounded functions adapted to $\{\mathscr{F}_n\}_{n\geqslant 0}$. For a martingale $(f_n)_{n\geqslant 0}$, its generalized fractional integral $I_\gamma f = ((I_\gamma f)_n)_{n\geqslant 0}$ is defined as a martingale by

$$(I_\gamma f)_n = \sum_{k=0}^{n} \gamma_{k-1}(f_k - f_{k-1})$$

with convention $\gamma_{-1} = \gamma_0$ and $f_{-1} = 0$.

Our definition of I_γ is based on the notion of martingale transform in the sense of Burkholder [2]. For quasi-normed spaces M_1 and M_2 of functions, we denote by $B(M_1, M_2)$ the set of all bounded martingale transforms from M_1 to M_2, that is, $I_\gamma \in B(M_1, M_2)$ means that

$$\sup_{n\geqslant 0} \|(I_\gamma f)_n\|_{M_2} \leqslant C \sup_{n\geqslant 0} \|f_n\|_{M_1},$$

for all M_1-bounded martingales $f = (f_n)_{n\geqslant 0}$.

Next we state the definition of generalized Morrey spaces.

DEFINITION 2. ([13]) For $p \in [1,\infty)$ and $\phi : (0,1] \to (0,\infty)$, let

$$L_{p,\phi} = \{f \in L_{p,\mathrm{loc}} : \|f\|_{L_{p,\phi}} < \infty\},$$

where

$$\|f\|_{L_{p,\phi}} = \sup_{n\geqslant 0} \sup_{B\in A(\mathscr{F}_n)} \frac{1}{\phi(P(B))} \left(\frac{1}{P(B)} \int_B |f|^p \, dP\right)^{1/p}.$$

Then $\|\cdot\|_{L_{p,\phi}}$ is a norm on $L_{p,\phi}$. If $\lambda \in \mathbb{R}$ and $\phi(t) = t^\lambda$, then $L_{p,\phi} = L_{p,\lambda}$ which was introduced in [12]. We also define weak Morrey spaces. Let L_0 be the set of all measurable functions.

DEFINITION 3. For $p \in [1,\infty)$ and $\phi : (0,1] \to (0,\infty)$, let

$$\mathrm{w}L_{p,\phi} = \{f \in L_0 : \|f\|_{\mathrm{w}L_{p,\phi}} < \infty\},$$

where

$$\|f\|_{\mathrm{w}L_{p,\phi}} = \sup_{n\geqslant 0} \sup_{B\in A(\mathscr{F}_n)} \frac{1}{\phi(P(B))} \left(\frac{\sup_{t>0} t^p P(B\cap\{|f|>t\})}{P(B)}\right)^{1/p}.$$

Then $\|\cdot\|_{\mathrm{w}L_{p,\phi}}$ is a quasi-norm on $\mathrm{w}L_{p,\phi}$. It is easy to see that

$$\|f\|_{\mathrm{w}L_{p,\phi}} \leqslant \|f\|_{L_{p,\phi}}, \quad f \in L_{p,\phi}. \tag{1}$$

REMARK 1. If $(\Omega, \mathscr{F}, P)$ is a σ-finite measure space and each atom in $A(\mathscr{F}_0)$ has a finite measure, then we also define generalized Morrey spaces $L_{p,\phi}$ and weak Morrey spaces $\mathrm{w}L_{p,\phi}$ by using $\phi : (0,\infty) \to (0,\infty)$ instead of $\phi : (0,1] \to (0,\infty)$. See [15] for martingales on σ-finite measure spaces.

We denote by $\mathscr{M}_{L_{p,\phi}}$ the set of all $L_{p,\phi}$-bounded martingales $f=(f_n)_{n\geqslant 0}$. In this paper we give necessary and sufficient conditions on p,q,ϕ,ψ and $\gamma=(\gamma_n)_{n\geqslant 0}$ for

$$\sup_{n\geqslant 0}\|(I_\gamma f)_n\|_{L_{q,\psi}} \leqslant C\sup_{n\geqslant 0}\|f_n\|_{L_{p,\phi}}, \quad f\in \mathscr{M}_{L_{p,\phi}},$$

or

$$\sup_{n\geqslant 0}\|(I_\gamma f)_n\|_{\mathrm{w}L_{q,\psi}} \leqslant C\sup_{n\geqslant 0}\|f_n\|_{L_{p,\phi}}, \quad f\in \mathscr{M}_{L_{p,\phi}}.$$

It is known that, if there exists a positive constant C_0 such that $\phi(t)\leqslant C_0\phi(s)$ for $0<s\leqslant t\leqslant 1$, then, for $f\in L_{1,\mathrm{loc}}$ and its corresponding martingale $(f_n)_{n\geqslant 0}$; $f_n=E_nf$, we have

$$\|f\|_{L_{p,\phi}} \leqslant \sup_{n\geqslant 0}\|f_n\|_{L_{p,\phi}} \leqslant C_0\|f\|_{L_{p,\phi}}, \tag{2}$$

see [12, 13].

Next, for a function $\rho:(0,1]\to(0,\infty)$ such that $\int_0^1 \frac{\rho(t)}{t}\,dt<\infty$, let

$$\gamma_n=\int_0^{b_n}\frac{\rho(t)}{t}\,dt, \quad b_n=\sum_{B\in A(\mathscr{F}_n)}P(B)\chi_B, \quad n=0,1,2,\cdots. \tag{3}$$

In this case we denote I_γ by I_ρ, namely, for a martingale $f=(f_n)_{n\geqslant 0}$,

$$I_\rho f=((I_\rho f)_n)_{n\geqslant 0}, \quad (I_\rho f)_n=\sum_{n=0}^{n}\left(\int_0^{b_{k-1}}\frac{\rho(t)}{t}\,dt\right)(f_k-f_{k-1}). \tag{4}$$

If $\rho(t)=\alpha t^\alpha$ and $\alpha>0$, then $\int_0^{b_{k-1}}\frac{\rho(t)}{t}\,dt=(b_{k-1})^\alpha$ and I_ρ is the fractional integrals introduced by [12] as a generalization of I_α on dyadic martingales investigated in [3]. As corollaries of boundedness of I_γ we get necessary and sufficient conditions for the boundedness of I_ρ. See [9, 10, 4] for I_ρ on $\mathbb{R}^n$.

We state main results in the next section. To prove them we establish Doob's inequality on martingale Morrey spaces and show several lemmas in Sections 3 and 4, respectively. Then we prove main results in Section 5.

At the end of this section, we make some conventions. Throughout this paper, we always use C to denote a positive constant that is independent of the main parameters involved but whose value may differ from line to line. Constants with subscripts, such as C_p, is dependent on the subscripts. If $f\leqslant Cg$, we then write $f\lesssim g$ or $g\gtrsim f$; and if $f\lesssim g\lesssim f$, we then write $f\sim g$.

2. Main results

In this section we state our main results. Theorem 1 gives a sufficient condition for the boundedness of I_γ which is improvement of [13], and Theorems 2–4 give three kinds of necessary and sufficient condition under some different restriction which are independent each other. As corollaries we state results on the boundedness of I_ρ. At the end of this section we give examples to show the independence of Theorems 2–4.

First we state some notion on functions defined on an interval. Let I be an interval in $(0,\infty)$. A function $\theta : I \to (0,\infty)$ is said to be almost increasing (almost decreasing) if there exists a positive constant C such that

$$\theta(r) \leqslant C\theta(s) \quad (\theta(r) \geqslant C\theta(s)) \quad \text{for} \quad r \leqslant s. \tag{5}$$

A function $\theta : I \to (0,\infty)$ is said to satisfy the doubling condition if there exists a positive constant C such that

$$C^{-1} \leqslant \frac{\theta(r)}{\theta(s)} \leqslant C \quad \text{for} \quad \frac{1}{2} \leqslant \frac{r}{s} \leqslant 2. \tag{6}$$

For functions $\theta, \kappa : I \to (0,\infty)$, we denote $\theta(r) \sim \kappa(r)$ if there exists a positive constant C such that

$$C^{-1}\theta(r) \leqslant \kappa(r) \leqslant C\theta(r) \quad \text{for} \quad r \in I.$$

Let $\phi_i : (0,1] \to (0,\infty)$, $i = 1,2$. If $\phi_1 \sim \phi_2$, then $L_{p,\phi_1} = L_{p,\phi_2}$ with equivalent norms.

Recall that

$$b_n = \sum_{B \in A(\mathscr{F}_n)} P(B)\chi_B.$$

The first result is a sufficient condition for the boundedness of I_γ.

THEOREM 1. *Let $1 \leqslant p < q < \infty$ and $\phi : (0,1] \to (0,\infty)$. Assume that ϕ is almost decreasing. If there exists a positive constant C such that*

$$\sum_{k=0}^{n} (\gamma_{k-1} - \gamma_k)\phi(b_k) + \gamma_n \phi(b_n) \leqslant C\phi(b_n)^{p/q} \quad \text{for all } n \geqslant 0 \tag{7}$$

with convention $\gamma_{-1} = \gamma_0$, then $I_\gamma \in B(L_{p,\phi}, \mathrm{w}L_{q,\phi^{p/q}})$. Moreover, if $1 < p < q < \infty$, then $I_\gamma \in B(L_{p,\phi}, L_{q,\phi^{p/q}})$.

REMARK 2. Let $\sup_{B\in A(\mathscr{F}_n)} P(B) \to 0$ $(n \to \infty)$. For example the dyadic martingales on the interval $[0,1]$. If $\phi(r) \to 0$ $(r \to 0)$, then $L_{p,\phi} = \{0\}$. Actually, for $f \in L_{p,\phi}$ and $B_0 \in A(\mathscr{F}_0)$,

$$\begin{aligned}
\int_{B_0} |f|^p \, dP &= \sum_{B\in A(\mathscr{F}_n)} \int_{B\cap B_0} |f|^p \, dP \leqslant \sum_{B\in A(\mathscr{F}_n)} \phi(P(B))^p P(B \cap B_0) \, \|f\|_{L_{p,\phi}} \\
&\leqslant \sup_{B\in A(\mathscr{F}_n)} \phi(P(B))^p \sum_{B\in A(\mathscr{F}_n)} P(B\cap B_0) \, \|f\|_{L_{p,\phi}} \\
&= \sup_{B\in A(\mathscr{F}_n)} \phi(P(B))^p P(B_0) \, \|f\|_{L_{p,\phi}} \to 0 \quad (n \to \infty).
\end{aligned}$$

Therefore, the almost decreasingness of ϕ is not a strong assumption.

REMARK 3. For an almost decreasing function ϕ, let $\psi(r) = \inf_{0<t\leqslant r} \phi(t)$. Then ψ is non-increasing and satisfies the relation $\phi \sim \psi$.

Now we state necessary and sufficient conditions for the boundedness of I_γ. Theorems 2–4 below are independent each other, see Examples 1–3.

THEOREM 2. *Let* $1 \leqslant p < q < \infty$ *and* $\phi : (0,1] \to (0,\infty)$. *Assume that* ϕ *is almost decreasing and that there exists a positive constant* $C_{p,\phi}$ *such that*

$$\int_0^r \frac{\phi(t)t^{1/p}}{t}\,dt \leqslant C_{p,\phi}\,\phi(r)r^{1/p} \quad \text{for all } r \in (0,1]. \tag{8}$$

Then $I_\gamma \in B(L_{p,\phi}, \mathrm{w}L_{q,\phi^{p/q}})$ *if and only if* (7) *holds for some constant* C. *Moreover, if* $1 < p < q < \infty$, *then* $I_\gamma \in B(L_{p,\phi}, \mathrm{w}L_{q,\phi^{p/q}})$ *is equivalent to* $I_\gamma \in B(L_{p,\phi}, L_{q,\phi^{p/q}})$.

REMARK 4. If ϕ is almost decreasing and (8) (or (11) bellow) holds for some positive constant $C_{p,\phi}$, then the function $t \mapsto \phi(t)t^{1/p}$ (or $t \to \phi(t)t$) is almost increasing, see the proof of Lemma 5. Hence ϕ satisfies the doubling condition (6).

REMARK 5. If the function $t \mapsto \phi(t)t^{1/p}$ is almost decreasing, then

$$\inf_{t\in(0,1]} \phi(t)t^{1/p} > 0.$$

This implies $L_{p,\phi}(B_0) = L_p(B_0)$ for all $B_0 \in A(\mathscr{F}_0)$. Actually, for $f \in L_{p,\mathrm{loc}}$, $B_0 \in A(\mathscr{F}_0)$ and $B \in \cup_{n=0}^{\infty} A(\mathscr{F}_n)$ with $B \subset B_0$, we have

$$\frac{1}{\phi(P(B))}\left(\frac{1}{P(B)}\int_B |f|^p\,dP\right)^{1/p} \leqslant \frac{\|f\|_{L_p(B_0)}}{\phi(P(B))P(B)^{1/p}} \leqslant \frac{\|f\|_{L_p(B_0)}}{\inf_{t\in(0,1]}\phi(t)t^{1/p}}.$$

Therefore, the almost increasingness of $\phi(t)t^{1/p}$ is not a strong assumption.

THEOREM 3. *Let* $1 \leqslant p < q < \infty$ *and* $\phi : (0,1] \to (0,\infty)$. *Assume that* ϕ *is almost decreasing, that* $t \mapsto \phi(t)t^{1/p}$ *is almost increasing and that there exists a positive constant* $C_{\gamma,\phi}$ *such that*

$$\sum_{k=0}^{n} (\gamma_{k-1} - \gamma_k)\phi(b_k) \leqslant C_{\gamma,\phi}\,\gamma_n\phi(b_n) \quad \text{for all } n \geqslant 0 \tag{9}$$

with convention $\gamma_{-1} = \gamma_0$. *Then* $I_\gamma \in B(L_{p,\phi}, \mathrm{w}L_{q,\phi^{p/q}})$ *if and only if there exists a positive constant* C *such that*

$$\gamma_n \leqslant C\phi(b_n)^{p/q-1} \quad \text{for all } n \geqslant 0. \tag{10}$$

Moreover, if $1 < p < q < \infty$, *then* $I_\gamma \in B(L_{p,\phi}, \mathrm{w}L_{q,\phi^{p/q}})$ *is equivalent to* $I_\gamma \in B(L_{p,\phi}, L_{q,\phi^{p/q}})$.

THEOREM 4. *Let $\phi,\psi:(0,1]\to(0,\infty)$. Assume that ϕ and ψ are almost decreasing and that there exists a positive constant C_ϕ such that*

$$\int_0^r \phi(t)\,dt \leqslant C_\phi\,\phi(r)r \quad \textit{for all}\ \ r\in(0,1]. \tag{11}$$

Then the following are equivalent:

(i) *There exists a positive constant C such that*

$$\sum_{k=0}^{n}(\gamma_{k-1}-\gamma_k)\phi(b_k)+\gamma_n\phi(b_n)\leqslant C\psi(b_n) \quad \textit{for all}\ \ n\geqslant 0 \tag{12}$$

with convention $\gamma_{-1}=\gamma_0$.

(ii) $I_\gamma\in B(L_{1,\phi},L_{1,\psi})$.

(iii) $I_\gamma\in B(L_{1,\phi},\mathrm{w}L_{1,\psi})$.

REMARK 6. In Theorem 4, to prove $(\mathrm{ii})\Rightarrow(\mathrm{iii})$, there is no need to assume (11).

REMARK 7. If $(\Omega,\mathscr{F},P)$ is a σ-finite measure space and each atom in $A(\mathscr{F}_0)$ has a finite measure, then we also have the same results as Theorems 1–4 by using $\phi:(0,\infty)\to(0,\infty)$ instead of $\phi:(0,1]\to(0,\infty)$.

Next, for a function $\rho:(0,1]\to(0,\infty)$ such that $\int_0^1\frac{\rho(t)}{t}\,dt<\infty$, let I_ρ be the generalized fractional integral defined by (4). If $\{\mathscr{F}_n\}_{n\geqslant 0}$ is regular, that is, there exists $R\geqslant 2$ such that

$$E_nf\leqslant RE_{n-1}f \tag{13}$$

for all non-negative integrable function f, then the inequality $b_n\leqslant b_{n-1}\leqslant Rb_n$ holds, see [12, Lemma 3.1]. Hence, if ϕ satisfies the doubling condition (6), then

$$\begin{aligned}\sum_{k=0}^{n}(\gamma_{k-1}-\gamma_k)\phi(b_k)&=\sum_{k=1}^{n}\phi(b_k)\int_{b_k}^{b_{k-1}}\frac{\rho(t)}{t}\,dt\\&\sim\sum_{k=1}^{n}\int_{b_k}^{b_{k-1}}\frac{\phi(t)\rho(t)}{t}\,dt\\&=\int_{b_n}^{b_0}\frac{\phi(t)\rho(t)}{t}\,dt.\end{aligned}$$

That is, (7) is equivalent to

$$\phi(b_n)\int_0^{b_n}\frac{\rho(t)}{t}\,dt+\int_{b_n}^{b_0}\frac{\phi(t)\rho(t)}{t}\,dt\leqslant C\phi(b_n)^{p/q}\quad \text{for all}\quad n\geqslant 0. \tag{14}$$

This type inequality was first introduced by Gunawan [6] to extend the result of Adams [1] on Morrey spaces defined on $\mathbb{R}^n$ to generalized Morrey spaces.

Recall that the almost decreasingness of ϕ and the condition (8) (or (11)) imply the doubling condition (6) of ϕ, see Remark 4. For the boundedness of I_ρ, we have the following corollaries:

COROLLARY 1. *Let* $\{\mathscr{F}_n\}_{n\geqslant 0}$ *be regular,* $1 \leqslant p < q < \infty$ *and* $\phi : (0,1] \to (0,\infty)$. *Assume that* ϕ *is almost decreasing and satisfies the doubling condition* (6). *If* (14) *holds for some positive constant* C, *then* $I_\rho \in B(L_{p,\phi}, \mathrm{w}L_{q,\phi^{p/q}})$. *Moreover, if* $1 < p < q < \infty$, *then* $I_\rho \in B(L_{p,\phi}, L_{q,\phi^{p/q}})$.

COROLLARY 2. *Let* $\{\mathscr{F}_n\}_{n\geqslant 0}$ *be regular,* $1 \leqslant p < q < \infty$ *and* $\phi : (0,1] \to (0,\infty)$. *Assume that* ϕ *is almost decreasing and that* (8) *holds for some positive constant* $C_{p,\phi}$. *Then* $I_\rho \in B(L_{p,\phi}, \mathrm{w}L_{q,\phi^{p/q}})$, *if and only if* (14) *holds for some positive constant* C. *Moreover, if* $1 < p < q < \infty$, *then* $I_\rho \in B(L_{p,\phi}, \mathrm{w}L_{q,\phi^{p/q}})$ *is equivalent to* $I_\rho \in B(L_{p,\phi}, L_{q,\phi^{p/q}})$.

COROLLARY 3. *Let* $\{\mathscr{F}_n\}_{n\geqslant 0}$ *be regular,* $1 \leqslant p < q < \infty$ *and* $\phi : (0,1] \to (0,\infty)$. *Assume that* ϕ *is almost decreasing and satisfies the doubling condition* (6) *and that there exists a positive constant* $C_{\rho,\phi}$ *such that*

$$\int_r^1 \frac{\phi(t)\rho(t)}{t}\,dt \leqslant C_{\rho,\phi}\,\phi(r)\rho(r) \quad \text{for all} \quad r \in (0,1]. \tag{15}$$

Then $I_\rho \in B(L_{p,\phi}, \mathrm{w}L_{q,\phi^{p/q}})$ *if and only if there exists a positive constant* C *such that*

$$\int_0^{b_n} \frac{\rho(t)}{t}\,dt \leqslant C\phi(b_n)^{p/q-1} \quad \text{for all} \quad n \geqslant 0. \tag{16}$$

Moreover, if $1 < p < q < \infty$, *then* $I_\rho \in B(L_{p,\phi}, \mathrm{w}L_{q,\phi^{p/q}})$ *is equivalent to* $I_\gamma \in B(L_{p,\phi}, L_{q,\phi^{p/q}})$.

COROLLARY 4. *Let* $\{\mathscr{F}_n\}_{n\geqslant 0}$ *be regular and* $\phi, \psi : (0,1] \to (0,\infty)$. *Assume that* ϕ *and* ψ *are almost decreasing and that* (11) *holds for some positive constant* C_ϕ. *Then the following are equivalent:*

(i) *There exists a positive constant* C *such that*

$$\phi(b_n)\int_0^{b_n} \frac{\rho(t)}{t}\,dt + \int_{b_n}^{b_0} \frac{\phi(t)\rho(t)}{t}\,dt \leqslant C\psi(b_n) \quad \text{for all} \quad n \geqslant 0, \tag{17}$$

(ii) $I_\rho \in B(L_{1,\phi}, L_{1,\psi})$.

(iii) $I_\rho \in B(L_{1,\phi}, \mathrm{w}L_{1,\psi})$.

REMARK 8. The definition (4) is an improvement of [13]. Hence Corollary 1 is also improvement of the corresponding result in [13]. See [5] for necessary and sufficient conditions for the boundedness of I_ρ on generalized Morrey spaces defined on $\mathbb{R}^n$.

We denote I_ρ by I_α if $\rho(t) = \alpha t^\alpha$ and $\alpha > 0$. If $\phi(t) = t^\lambda$, then $L_{p,\phi} = L_{p,\lambda}$ (see [12]). In this case (14) means $t^{\alpha+\lambda} \leqslant Ct^{\lambda p/q}$ if $\alpha + \lambda < 0$. Then we have the following corollary.

COROLLARY 5. *Let* $\{\mathscr{F}_n\}_{n\geqslant 0}$ *be regular,* $1\leqslant p<q<\infty$ *and* $-1/p\leqslant\lambda<-\alpha<0$*. Then* $I_\alpha\in B(L_{p,\lambda},\mathrm{w}L_{q,\lambda p/q})$ *if and only if* $\lambda p/q\leqslant\alpha+\lambda$*. Moreover, if* $1<p<q<\infty$*, then* $I_\alpha\in B(L_{p,\lambda},\mathrm{w}L_{q,\lambda p/q})$ *is equivalent to* $I_\alpha\in B(L_{p,\lambda},L_{q,\lambda p/q})$*.*

If $\mathscr{F}_0=\{\Omega,\emptyset\}$ and $\phi(t)=t^{-1/p}$, then $L_{p,\phi}=L_p$. In this case (14) means $t^{\alpha-1/p}\leqslant Ct^{-1/q}$ if $\alpha<1/p$. Then, for the boundedness of I_α on Lebesgue spaces, we have the following corollary.

COROLLARY 6. *Let* $\{\mathscr{F}_n\}_{n\geqslant 0}$ *be regular,* $\mathscr{F}_0=\{\Omega,\emptyset\}$*,* $1\leqslant p<q<\infty$ *and* $0<\alpha<1/p$*. Then* $I_\alpha\in B(L_p,\mathrm{w}L_q)$ *if and only if* $-1/q\leqslant\alpha-1/p$*. Moreover, if* $1<p<q<\infty$*, then* $I_\alpha\in B(L_p,\mathrm{w}L_q)$ *is equivalent to* $I_\alpha\in B(L_p,L_q)$*.*

The following examples show the independence of Corollary 2–4, which also show the independence of Theorems 2–4 by setting $\gamma_n=\int_0^{b_n}\frac{\rho(t)}{t}\,dt$.

EXAMPLE 1. Let $1<p<q<\infty$, $\mu<0$, $\beta=(p/q-1)\mu+1$ and

$$\rho(r)=(\log(e/r))^{-\beta},\quad \phi(r)=(\log(e/r))^{-\mu}.$$

Then (8) and (14) hold, but (15) does not hold. More precisely,

$$\int_0^r\frac{\phi(t)t^{1/p}}{t}\,dt\sim\phi(r)r^{1/p}$$

and

$$\phi(r)\int_0^r\frac{\rho(t)}{t}\,dt\sim\int_r^1\frac{\phi(t)\rho(t)}{t}\,dt\sim(\log(e/r))^{1-\beta-\mu}=\phi(r)^{p/q},$$

but

$$\int_r^1\frac{\phi(t)\rho(t)}{t}\,dt\not\lesssim\phi(r)\rho(r).$$

EXAMPLE 2. Let $1<p<q<\infty$, $\mu>0$, $\alpha=1/p-1/q$, $\beta=(p/q-1)\mu$ and

$$\rho(r)=r^{\alpha}(\log(e/r))^{-\beta},\quad \phi(r)=r^{-1/p}(\log(e/r))^{-\mu}.$$

Then (15) and (16) hold, but (8) does not hold. More precisely,

$$\int_r^1\frac{\phi(t)\rho(t)}{t}\,dt\lesssim\phi(r)\rho(r)$$

and

$$\int_0^r\frac{\rho(t)}{t}\,dt\sim\rho(r)=\phi(r)^{p/q-1},$$

but

$$\int_0^r\frac{\phi(t)t^{1/p}}{t}\,dt=\int_0^r\frac{(\log(e/t))^{-\mu}}{t}\,dt\not\lesssim(\log(e/t))^{-\mu}=\phi(r)r^{1/p}.$$

EXAMPLE 3. Let $1 < p < q < \infty$, $0 < \lambda < 1$, $\beta > 1$ and

$$\rho(r) = (\log(e/r))^{-\beta}, \quad \phi(r) = r^{-\lambda}, \quad \psi(r) = r^{-\lambda}(\log(e/r))^{1-\beta}.$$

Then (11) and (17) hold, but (14) does not hold. More precisely,

$$\int_0^r \phi(t)\,dt \sim r^{1-\lambda} = \phi(r)r$$

and

$$\phi(r)\int_0^r \frac{\rho(t)}{t}\,dt + \int_r^1 \frac{\phi(t)\rho(t)}{t}\,dt \sim r^{-\lambda}(\log(e/r))^{1-\beta} = \psi(r),$$

but

$$\phi(r)\int_0^r \frac{\rho(t)}{t}\,dt + \int_r^1 \frac{\phi(t)\rho(t)}{t}\,dt \sim r^{-\lambda}(\log(e/r))^{1-\beta} \not\lesssim r^{-\lambda p/q} = \phi(r)^{p/q}.$$

3. Doob's inequality on martingale Morrey spaces

In this section we establish Doob's inequality on martingale Morrey spaces which we use to prove the boundedness of I_γ.

For a martingale $f = (f_n)_{n\geqslant 0}$ relative to $\{\mathscr{F}_n\}_{n\geqslant 0}$, the maximal functions are defined by

$$M_n f = \sup_{0\leqslant m\leqslant n} |f_m|, \quad Mf = \sup_{n\geqslant 0} |f_n|.$$

For a function $f \in L_{p,\mathrm{loc}}$ with $p \in [1,\infty)$, let $f_n = E_n f$, $n \geqslant 0$. Then $(f_n)_{n\geqslant 0}$ is a martingale and $\lim_{n\to\infty} f_n = f$ in $L_p(B)$ for each $B \in A(\mathscr{F}_0)$. For this reason a function $f \in L_{1,\mathrm{loc}}$ and its corresponding martingale $(f_n)_{n\geqslant 0}$ with $f_n = E_n f$ will be denoted by the same symbol f. In this case

$$M_n f = \sup_{0\leqslant m\leqslant n} |E_m f|, \quad Mf = \sup_{n\geqslant 0} |E_n f| \quad \text{for} \quad f \in L_{1,\mathrm{loc}}.$$

It is known as Doob's inequality that (see for example [17, pages 20–21])

$$\|Mf\|_{L_p} \leqslant \frac{p}{p-1}\|f\|_{L_p}, \quad f \in L_p\ (p > 1), \tag{18}$$

$$\|Mf\|_{\mathrm{w}L_1} \leqslant \|f\|_{L_1}, \quad f \in L_1. \tag{19}$$

In this section we extend (18) and (19) to generalized Morrey norms.

THEOREM 5. *Let $p \in [1,\infty)$ and $\phi : (0,1] \to (0,\infty)$. Assume that ϕ is almost decreasing. Then there exists a positive constant $C_{p,\phi}$ such that, for any $f \in L_{p,\phi}$,*

$$\|Mf\|_{L_{p,\phi}} \leqslant C_{p,\phi}\|f\|_{L_{p,\phi}}, \quad \text{if } p > 1,$$

$$\|Mf\|_{\mathrm{w}L_{1,\phi}} \leqslant C_{1,\phi}\|f\|_{L_{1,\phi}}, \quad \text{if } p = 1.$$

Proof. Case 1: $p > 1$. For any $B \in A(\mathscr{F}_m)$, $m \geqslant 0$, let $f = g + h$ and $g = f\chi_B$. Then, using (18), we have

$$\int_B (Mg)^p \, dP \leqslant \int_\Omega (Mg)^p \, dP \lesssim \int_\Omega |g|^p \, dP = \int_B |f|^p \, dP.$$

Hence

$$\frac{1}{\phi(P(B))} \left(\frac{1}{P(B)} \int_B (Mg)^p \, dP \right)^{1/p} \lesssim \|f\|_{L_{p,\phi}}. \tag{20}$$

Next, take $B_n \in A(\mathscr{F}_n)$, $n = 0, 1, \cdots, m$, such that $B = B_m \subset B_{m-1} \subset \cdots \subset B_0$. Then, for a.s. $\omega \in B$,

$$E_n h(\omega) = \begin{cases} 0 & (n \geqslant m) \\ \dfrac{1}{P(B_n)} \displaystyle\int_{B_n} h \, dP & (n < m). \end{cases}$$

If $n < m$, then

$$|E_n h(\omega)| \leqslant \left(\frac{1}{P(B_n)} \int_{B_n} |h|^p \, dP \right)^{1/p} \leqslant \phi(P(B_n)) \|f\|_{L_{p,\phi}} \lesssim \phi(P(B)) \|f\|_{L_{p,\phi}},$$

since ϕ is almost decreasing. Hence

$$Mh \lesssim \phi(P(B)) \|f\|_{L_{p,\phi}} \quad \text{on } B. \tag{21}$$

Then

$$\frac{1}{\phi(P(B))} \left(\frac{1}{P(B)} \int_B (Mh)^p \, dP \right)^{1/p} \lesssim \|f\|_{L_{p,\phi}}. \tag{22}$$

By (20), (22) and the inequality $Mf \leqslant Mg + Mh$, we have

$$\frac{1}{\phi(P(B))} \left(\frac{1}{P(B)} \int_B (Mf)^p \, dP \right)^{1/p} \lesssim \|f\|_{L_{p,\phi}},$$

which shows the conclusion.

Case 2: $p = 1$. Let g and h be as in Case 1. Then, using (19), we have, for all $t > 0$,

$$tP(B \cap \{Mg > t\}) \leqslant tP(Mg > t) \leqslant \int_\Omega |g| \, dP = \int_B |f| \, dP$$

and then

$$\frac{1}{\phi(P(B))} \frac{tP(B \cap \{Mg > t\})}{P(B)} \leqslant \|f\|_{L_{1,\phi}}.$$

We also have (21) for the case $p = 1$. Then

$$\frac{1}{\phi(P(B))} \frac{tP(B \cap \{Mh > t\})}{P(B)} \leqslant \|f\|_{L_{1,\phi}}.$$

Therefore we have the conclusion. □

From Theorem 5 we have the following corollary for martingales:

COROLLARY 7. *Let* $p \in [1,\infty)$ *and* $\phi : (0,1] \to (0,\infty)$. *Assume that* ϕ *is almost decreasing. Then there exists a positive constant* $C_{p,\phi}$ *such that, for any* $f = (f_n)_{n \geqslant 0} \in \mathscr{M}_{L_{p,\phi}}$,

$$\|Mf\|_{L_{p,\phi}} \leqslant C_{p,\phi} \sup_{n \geqslant 0} \|f_n\|_{L_{p,\phi}}, \quad \text{if } p > 1,$$
$$\|Mf\|_{\mathrm{w}L_{1,\phi}} \leqslant C_{1,\phi} \sup_{n \geqslant 0} \|f_n\|_{L_{1,\phi}}, \quad \text{if } p = 1.$$

4. Lemmas

In this section we show several lemmas to prove the main results.

LEMMA 1. *Let* $p \in [1,\infty)$ *and* $\phi : (0,1] \to (0,\infty)$. *Let* $(\gamma_n)_{n \geqslant 0}$ *be a non-increasing sequence of non-negative bounded functions adapted to* $\{\mathscr{F}_n\}_{n \geqslant 0}$. *Let* $f = (f_n)_{n \geqslant 0} \in \mathscr{M}_{L_{p,\phi}}$. *Suppose that* $\sup_{n \geqslant 0} \|f_n\|_{L_{p,\phi}} = 1$. *Then*

$$|(I_\gamma f)_n| \leqslant \sum_{k=0}^{n} (\gamma_{k-1} - \gamma_k)\phi(b_k) + \gamma_n \phi(b_n) \tag{23}$$

for all $n \geqslant 0$ *with convention* $\gamma_{-1} = \gamma_0$ *and* $f_{-1} = 0$.

Proof. First note that

$$|f_n| \leqslant \phi(b_n). \tag{24}$$

Actually, for any $B \in A(\mathscr{F}_n)$ and for a.s. $\omega \in B$, we have

$$|f_n(\omega)| = \left| \frac{1}{P(B)} \int_B f_n \, dP \right| \leqslant \left(\frac{1}{P(B)} \int_B |f_n|^p \, dP \right)^{1/p} \leqslant \phi(P(B)),$$

since $\|f_n\|_{L_{p,\phi}} \leqslant 1$. Hence, for $n = 0$, (23) is easily verified. For $n \geqslant 1$, by resummation and the assumption on $(\gamma_n)_{n \geqslant 0}$, we have

$$\begin{aligned} |(I_\gamma f)_n| &= \left| \gamma_0 f_0 + \sum_{k=1}^{n} \gamma_{k-1}(f_k - f_{k-1}) \right| \\ &= \left| \sum_{k=0}^{n} \gamma_{k-1} f_k - \sum_{k=0}^{n} \gamma_k f_k + \gamma_n f_n \right| \\ &\leqslant \sum_{k=0}^{n} (\gamma_{k-1} - \gamma_k)|f_k| + \gamma_n |f_n|. \end{aligned}$$

Hence, by (24) we obtain (23). □

LEMMA 2. *Let* $(\gamma_n)_{n \geqslant 0}$ *be a non-increasing sequence of non-negative bounded functions adapted to* $\{\mathscr{F}_n\}_{n \geqslant 0}$. *Let* $f = (f_n)_{n \geqslant 0}$ *be a martingale. Then,*

$$|(I_\gamma f)_m - (I_\gamma f)_{n-1}| \leqslant 2\gamma_{n-1} Mf \tag{25}$$

for all $0 \leqslant n \leqslant m$ *with convention* $(I_\gamma f)_{-1} = f_{-1} = 0$ *and* $\gamma_{-1} = \gamma_0$.

Proof. If $n = m$, then

$$|(I_\gamma f)_m - (I_\gamma f)_{n-1}| = |\gamma_{n-1}(f_n - f_{n-1})| \leqslant 2\gamma_{n-1} Mf.$$

If $n < m$, then, by resummation, we have

$$\begin{aligned}
|(I_\gamma f)_m - (I_\gamma f)_{n-1}| &= \left| \sum_{k=n}^{m} \gamma_{k-1}(f_k - f_{k-1}) \right| \\
&= \left| \sum_{k=n}^{m} \gamma_{k-1} f_k - \sum_{k=n-1}^{m-1} \gamma_k f_k \right| \\
&\leqslant \sum_{k=n}^{m-1} (\gamma_{k-1} - \gamma_k)|f_k| + \gamma_{m-1}|f_m| + \gamma_{n-1}|f_{n-1}| \\
&\leqslant \sum_{k=n}^{m-1} (\gamma_{k-1} - \gamma_k) Mf + \gamma_{m-1} Mf + \gamma_{n-1} Mf \\
&= 2\gamma_{n-1} Mf.
\end{aligned}$$

This is the conclusion. □

LEMMA 3. *Let $(\gamma_n)_{n \geqslant 0}$ be a non-increasing sequence of non-negative bounded functions adapted to $\{\mathscr{F}_n\}_{n \geqslant 0}$. Let $f = (f_n)_{n \geqslant 0} \in \mathscr{M}_{L_{1,\phi}}$ such that* $\sup_{n \geqslant 0} \|f_n\|_{L_{1,\phi}} = 1$. *If $0 \leqslant n \leqslant m$, then*

$$\frac{\chi_B}{P(B)} \int_B |(I_\gamma f)_m - (I_\gamma f)_n| \, dP \leqslant 2\chi_B \gamma_n \phi(P(B)) \quad \textit{for all} \quad B \in A(\mathscr{F}_n).$$

Proof. If $n = m$, then it is clear. If $n + 1 = m$, then

$$|(I_\gamma f)_m - (I_\gamma f)_n| = |\gamma_{m-1}(f_m - f_{m-1})| \leqslant \gamma_n |f_m| + \gamma_n |f_n|.$$

If $n + 2 \leqslant m$, then, by resummation as in the proof of the previous lemma, we have

$$|(I_\gamma f)_m - (I_\gamma f)_n| \leqslant \sum_{k=n+1}^{m-1} (\gamma_{k-1} - \gamma_k)|f_k| + \gamma_{m-1}|f_m| + \gamma_n |f_n|.$$

Using the fact $|f_k| \leqslant E_k |f_m|$, where $n \leqslant k \leqslant m$, for $B \in A(\mathscr{F}_n)$, we have

$$\frac{1}{P(B)} \int_B (\gamma_{k-1} - \gamma_k)|f_k| \, dP \leqslant \frac{1}{P(B)} \int_B (\gamma_{k-1} - \gamma_k)|f_m| \, dP,$$

and

$$\frac{1}{P(B)} \int_B \gamma_n |f_n| \, dP \leqslant \frac{1}{P(B)} \int_B \gamma_n |f_m| \, dP.$$

Then,

$$\begin{aligned}
&\frac{\chi_B}{P(B)}\int_B |(I_\gamma f)_m-(I_\gamma f)_n|\,dP\\
&\leqslant \frac{\chi_B}{P(B)}\int_B \left(\sum_{k=n+1}^{m-1}(\gamma_{k-1}-\gamma_k)|f_m|+\gamma_{m-1}|f_m|+\gamma_n|f_m|\right)dP\\
&=\frac{\chi_B}{P(B)}\int_B 2\gamma_n|f_m|\,dP\\
&\leqslant 2\chi_B\gamma_n\phi(P(B))\|f_m\|_{L_{1,\phi}}.
\end{aligned}$$

Since $\|f_m\|_{L_{1,\phi}}\leqslant 1$, we have the conclusion. □

LEMMA 4. *Let $1\leqslant p<q<\infty$ and $\phi:(0,1]\to(0,\infty)$. Assume that ϕ is almost decreasing and that $t\mapsto\phi(t)t^{1/p}$ is almost increasing. Then, for any atom $B\in\cup_{n=0}^{\infty}A(\mathscr{F}_n)$, its characteristic function χ_B is in $L_{p,\phi}$ and*

$$\|\chi_B\|_{L_{p,\phi}}\leqslant\frac{C}{\phi(P(B))},$$

where the positive constant C is independent of B.

Proof. Let $B'\in\cup_{n=0}^{\infty}A(\mathscr{F}_n)$ such that $P(B'\cap B)>0$. Then, $B'\subset B$ or $B'\supset B$. If $B'\subset B$, then

$$\left(\frac{1}{P(B')}\int_{B'}\chi_B\,dP\right)^{1/p}=\left(\frac{1}{P(B)}\int_B\chi_B\,dP\right)^{1/p}=1$$

and $\phi(P(B))\lesssim\phi(P(B'))$, since ϕ is almost decreasing. Hence,

$$\frac{1}{\phi(P(B'))}\left(\frac{1}{P(B')}\int_{B'}\chi_B\,dP\right)^{1/p}\lesssim\frac{1}{\phi(P(B))}\left(\frac{1}{P(B)}\int_B\chi_B\,dP\right)^{1/p}=\frac{1}{\phi(P(B))}.$$

If $B'\supset B$, then

$$\left(\int_{B'}\chi_B\,dP\right)^{1/p}=\left(\int_B\chi_B\,dP\right)^{1/p}$$

and $\phi(P(B))P(B)^{1/p}\lesssim\phi(P(B'))P(B')^{1/p}$, since $t\mapsto\phi(t)t^{1/p}$ is almost increasing. Hence,

$$\begin{aligned}
\frac{1}{\phi(P(B'))P(B')^{1/p}}\left(\int_{B'}\chi_B\,dP\right)^{1/p}&\lesssim\frac{1}{\phi(P(B))P(B)^{1/p}}\left(\int_B\chi_B\,dP\right)^{1/p}\\
&=\frac{1}{\phi(P(B))}.
\end{aligned}$$

Therefore, we have the conclusion. □

LEMMA 5. *Let $1 \leqslant p < q < \infty$ and $\phi : (0,1] \to (0,\infty)$. Assume that ϕ is almost decreasing and that (8) holds some positive constant $C_{p,\phi}$. For any n and any atom $B \in A(\mathscr{F}_n)$, take a sequence*

$$B = B_n \subset B_{n-1} \subset \cdots \subset B_k \subset \cdots \subset B_0, \quad B_k \in A(\mathscr{F}_k),$$

and let

$$f^B = \sum_{k=0}^{n-1} \phi(P(B_k))(\chi_{B_k} - \chi_{B_{k+1}}) + \phi(P(B_n))\chi_{B_n}. \tag{26}$$

Then f^B is in $L_{p,\phi}$ and

$$\|f^B\|_{L_{p,\phi}} \leqslant C,$$

where the positive constant C is independent of B and n.

Proof. First note that, from the almost decreasingness of ϕ and (8) it follows that

$$\phi(r)r^{1/p} \lesssim \int_0^r \frac{\phi(t)t^{1/p}}{t}\,dt \leqslant \int_0^s \frac{\phi(t)t^{1/p}}{t}\,dt \lesssim \phi(s)s^{1/p} \quad \text{for } r < s.$$

This shows that $t \mapsto \phi(t)t^{1/p}$ is almost increasing. Then we have

$$\begin{aligned}\int_0^r \phi(t)^p\,dt &= \int_0^r \frac{(\phi(t)t^{1/p})^p}{t}\,dt \\ &\lesssim (\phi(r)r^{1/p})^{p-1}\int_0^r \frac{\phi(t)t^{1/p}}{t}\,dt \lesssim (\phi(r)r^{1/p})^p = \phi(r)^p r \quad \text{for } r > 0.\end{aligned} \tag{27}$$

See also Lemma 7.1 in [11] for these type inequalities.

Now, for any atom $B' \in \cup_{\ell=1}^{\infty} A(\mathscr{F}_\ell)$, if $P(B' \cap B_0) = 0$, then

$$\int_{B'} |f^B|^p\,dP = 0.$$

If $P(B' \cap B_0) > 0$, then $B' \subset B_0$. Let $k_0 = \max\{k \leqslant n : B' \subset B_k\}$. Then we have three cases:

$$\begin{aligned} k_0 < n \quad &\text{and} \quad B' = B_{k_0}, \\ k_0 < n \quad &\text{and} \quad B' \subset B_{k_0} \setminus B_{k_0+1}, \\ k_0 = n \quad &\text{and} \quad B' \subset B_{k_0} = B_n. \end{aligned}$$

In all cases we have

$$\left(\frac{1}{P(B')}\int_{B'} |f^B|^p\,dP\right)^{1/p} \lesssim \left(\frac{1}{P(B_{k_0})}\int_{B_{k_0}} |f^B|^p\,dP\right)^{1/p},$$

since ϕ is almost decreasing. Then we have

$$\|f^B\|_{L_{p,\phi}} \lesssim \sup_{0\leqslant k\leqslant n} \frac{1}{\phi(P(B_k))}\left(\frac{1}{P(B_k)}\int_{B_k}|f^B|^p\,dP\right)^{1/p}.$$

By the definition of f^B and (27) we have

$$\begin{aligned}\int_{B_k}|f^B|^p\,dP &= \sum_{j=k}^{n-1}\phi(P(B_j))^p(P(B_j)-P(B_{j+1}))+\phi(P(B_n))^pP(B_n)\\ &= \sum_{j=k}^{n-1}\int_{P(B_{j+1})}^{P(B_j)}\phi(P(B_j))^p\,dt+\int_0^{P(B_n)}\phi(P(B_n))^p\,dt\\ &\lesssim \int_0^{P(B_k)}\phi(t)^p\,dt \lesssim \phi(P(B_k))^pP(B_k).\end{aligned}$$

This shows that

$$\frac{1}{\phi(P(B_k))}\left(\frac{1}{P(B_k)}\int_{B_k}|f_n|^p\,dP\right)^{1/p}\lesssim 1 \quad \text{for} \quad 0\leqslant k\leqslant n,$$

and the desired conclusion. □

5. Proofs of main results

In this section we prove Theorems 1–4. First we show the following pointwise estimate by the method of Hedberg [7].

PROPOSITION 1. *Under the assumption in Theorem 1, there exists a positive constant C such that, for all $f=(f_n)_{n\geqslant 0}\in\mathcal{M}L_{p,\phi}$ satisfying $\sup_{n\geqslant 0}\|f_n\|_{L_{p,\phi}}=1$,*

$$M(I_\gamma f)\leqslant C(Mf)^{p/q}. \tag{28}$$

Proof. We may assume that ϕ is non-increasing by Remark 3. Let $f=(f_n)_{n\geqslant 0}\in\mathcal{M}L_{p,\phi}$ such that $\sup_{n\geqslant 0}\|f_n\|_{L_{p,\phi}}=1$. Combining (7) and Lemmas 1 and 2, we have, for all $n\geqslant 0$,

$$|(I_\gamma f)_n|\leqslant C\phi(b_n)^{p/q}, \tag{29}$$

and, if $0\leqslant n\leqslant m$,

$$|(I_\gamma f)_m-(I_\gamma f)_{n-1}|\leqslant C\phi(b_n)^{p/q-1}Mf, \tag{30}$$

with convention $(I_\gamma f)_{-1}=0$, since (7) implies $\gamma_{n-1}\leqslant\phi(b_n)^{p/q-1}$. Let

$$N=\sum_{n=0}^{\infty}\chi_{\{\phi(b_n)\leqslant Mf\}},$$

and define measurable subsets Ω_1, Ω_2 and Ω_3 by

$$\Omega_1=\{N=\infty\},\quad \Omega_2=\{N=0\},\quad \Omega_3=\{0<N<\infty\}.$$

Case 1: Let $\omega \in \Omega_1$. Since $\phi(b_n)$ in non-decreasing with respect to n, we have $\phi(b_n(\omega)) \leqslant (Mf)(\omega)$ for all $n \geqslant 0$. Hence, we have

$$|(I_\gamma f)_n(\omega)| \leqslant C\phi(b_n(\omega))^{p/q} \leqslant C(Mf)(\omega)^{p/q}$$

by (29), that is, (28) holds on Ω_1.

Case 2: Let $\omega \in \Omega_2$. Combining the fact $Mf(\omega) \leqslant \phi(b_0(\omega))$ and (30) with $n = 0$, we have

$$|(I_\gamma f)_m(\omega)| \lesssim \phi(b_0(\omega))^{p/q-1} Mf(\omega) \lesssim (Mf(\omega))^{p/q-1} Mf(\omega) = (Mf(\omega))^{p/q},$$

that is, (28) holds on Ω_2.

Case 3: Let $\omega \in \Omega_3$. Then, we can take an integer n such that

$$\phi(b_{n-1}(\omega)) \leqslant Mf(\omega) \quad \text{and} \quad \phi(b_n(\omega)) > Mf(\omega).$$

If $m \leqslant n-1$, then we have $|(I_\gamma f)_m(\omega)| \leqslant CMf(\omega)^{p/q}$ by (29). If $m \geqslant n$, then, using (29) and (30), we have

$$\begin{aligned}
|(I_\gamma f)_m(\omega)| &\leqslant |(I_\gamma f)_{n-1}(\omega)| + |(I_\gamma f)_m(\omega) - (I_\gamma f)_{n-1}(\omega)| \\
&\lesssim \phi(b_{n-1}(\omega))^{p/q} + \phi(b_n(\omega))^{p/q-1} Mf(\omega) \\
&\leqslant Mf(\omega)^{p/q} + \{Mf(\omega)\}^{p/q-1} Mf(\omega) \\
&\lesssim Mf(\omega)^{p/q}.
\end{aligned}$$

That is, (28) holds on Ω_3 and we have the conclusion. □

Now, using the above pointwise estimate, we prove Theorem 1.

Proof of Theorem 1. Let $f = (f_n)_{n\geqslant 0} \in \mathscr{M}_{L_{p,\phi}}$ such that $\sup_{n\geqslant 0} \|f_n\|_{L_{p,\phi}} = 1$. Then, using (28) and Corollary 7, we have

$$\begin{aligned}
\sup_{n\geqslant 0} \|(I_\gamma f)_n\|_{\mathrm{w}L_{q,\phi^{p/q}}} &\leqslant \|M(I_\gamma f)\|_{\mathrm{w}L_{q,\phi^{p/q}}} \lesssim \|(Mf)^{p/q}\|_{\mathrm{w}L_{q,\phi^{p/q}}} \\
&= (\|Mf\|_{\mathrm{w}L_{p,\phi}})^{p/q} \lesssim \left(\sup_{n\geqslant 0} \|f_n\|_{L_{p,\phi}}\right)^{p/q} = 1.
\end{aligned}$$

Moreover, $1 < p < q < \infty$, then, using the boundedness of M on $\mathscr{M}_{L_{p,\phi}}$, we have the desired conclusion. □

Proof of Theorem 2. It is enough to prove the necessity by Theorem 1. For $B \in A(\mathscr{F}_n)$, let f^B be the function defined by (26). Then $\|f^B\|_{L_{p,\phi}} \lesssim 1$ by Lemma 5 and

$$E_k f^B \geqslant \chi_B \phi(b_k), \quad 0 \leqslant k \leqslant n. \tag{31}$$

We regard f^B as a martingale $f^B = (E_k f^B)_{k\geqslant 0}$. By resummation and (31), we obtain

$$\begin{aligned}
\chi_B (I_\gamma f^B)_n &= \chi_B \sum_{k=0}^{n} (\gamma_{k-1} - \gamma_k) E_k f^B + \chi_B \gamma_n E_n f^B \\
&\geqslant \chi_B \left(\sum_{k=0}^{n} (\gamma_{k-1} - \gamma_k) \phi(b_k) + \chi_B \gamma_n \phi(b_n) \right).
\end{aligned}$$

Then, noting (2), we have

$$\begin{aligned}\chi_B\left(\sum_{k=0}^{n}(\gamma_{k-1}-\gamma_k)\phi(b_k)+\gamma_n\phi(b_n)\right) &\leqslant \chi_B(I_\gamma f^B)_n\\ &= \chi_B\sup_{t>0} t\left(\frac{P(B\cap\{|(I_\gamma f^B)_n|>t\})}{P(B)}\right)^{1/q}\\ &\leqslant \chi_B\phi(P(B))^{p/q}\|(I_\gamma f^B)_n\|_{\mathrm{w}L_{q,\phi^{p/q}}}\\ &\leqslant \chi_B\phi(b_n)^{p/q}\sup_{k\geqslant 0}\|(I_\gamma f^B)_k\|_{\mathrm{w}L_{q,\phi^{p/q}}}\\ &\lesssim \chi_B\phi(b_n)^{p/q}\sup_{k\geqslant 0}\|E_k f^B\|_{L_{p,\phi}}\\ &\lesssim \chi_B\phi(b_n)^{p/q}\|f^B\|_{L_{p,\phi}}\\ &\lesssim \chi_B\phi(b_n)^{p/q}.\end{aligned}$$

This shows the conclusion. □

Proof of Theorem 3. If (9) and (10) hold, then (7) holds. Therefore, by Theorem 1, we have $I_\gamma\in B(L_{p,\phi},\mathrm{w}L_{q,\phi^{p/q}})$ if $1\leqslant p<q<\infty$ and $I_\gamma\in B(L_{p,\phi},L_{q,\phi^{p/q}})$ if $1<p<q<\infty$. Conversely, assume that (9) holds and that $I_\gamma\in B(L_{p,\phi},\mathrm{w}L_{q,\phi^{p/q}})$. For the martingale $\chi_B=(E_m\chi_B)_{m\geqslant 0}$ with $B\in A(\mathscr{F}_n)$, we have

$$(I_\gamma\chi_B)_n=\sum_{k=0}^{n}(\gamma_{k-1}-\gamma_k)E_k\chi_B+\gamma_n\chi_B \tag{32}$$

by resummation. Therefore, noting that $\gamma_{k-1}-\gamma_k\geqslant 0$ and using the fact $\|\chi_B\|_{L_{p,\phi}}\leqslant C/\phi(P(B))$ (Lemma 4) and (2), we have

$$\begin{aligned}\chi_B\gamma_n &\leqslant \chi_B(I_\gamma\chi_B)_n\\ &= \chi_B\sup_{t>0} t\left(\frac{P(B\cap\{|(I_\gamma\chi_B)_n|>t\})}{P(B)}\right)^{1/q}\\ &\leqslant \chi_B\phi(P(B))^{p/q}\|(I_\gamma\chi_B)_n\|_{\mathrm{w}L_{q,\phi^{p/q}}}\\ &\leqslant \chi_B\phi(b_n)^{p/q}\sup_{k\geqslant 0}\|(I_\gamma\chi_B)_k\|_{\mathrm{w}L_{q,\phi^{p/q}}}\\ &\lesssim \chi_B\phi(b_n)^{p/q}\sup_{k\geqslant 0}\|E_k\chi_B\|_{L_{p,\phi}}\\ &\lesssim \chi_B\phi(b_n)^{p/q}\|\chi_B\|_{L_{p,\phi}}\\ &\lesssim \chi_B\phi(b_n)^{p/q-1}.\end{aligned}$$

Then (10) holds. □

Proof of Theorem 4. *Proof of* (i) ⇒ (ii): Let $f=(f_n)_{n\geqslant 0}\in\mathscr{M}_{L_{1,\phi}}$ such that $\sup_{n\geqslant 0}\|f_n\|_{L_{1,\phi}}=1$. Combining (12) and Lemmas 1 and 3, we have

$$|(I_\gamma f)_m|\leqslant C\psi(b_m)\quad\text{for all } m\geqslant 0, \tag{33}$$

and, if $0 \leqslant n \leqslant m$,

$$\frac{1}{P(B)} \int_B |(I_\gamma f)_m - (I_\gamma f)_n| \, dP \leqslant C\psi(P(B)) \quad \text{for all } B \in A(\mathscr{F}_n). \tag{34}$$

Now, for any m,n and any atom $B \in A(\mathscr{F}_n)$, if $m \leqslant n$, then, using (33) and the almost decreasingness of ψ, we have

$$|(I_\gamma f)_m| \lesssim \psi(b_m) \lesssim \psi(b_n) = \psi(P(B)) \quad \text{on } B.$$

Hence

$$\frac{1}{P(B)} \int_B |(I_\gamma f)_m| \, dP \lesssim \psi(P(B)).$$

If $m > n$, then, using (33) and (34), we have

$$\begin{aligned} \frac{1}{P(B)} \int_B |(I_\gamma f)_m| \, dP &\leqslant \frac{1}{P(B)} \int_B |(I_\gamma f)_n| \, dP + \frac{1}{P(B)} \int_B |(I_\gamma f)_m - (I_\gamma f)_n| \, dP \\ &\lesssim \psi(P(B)). \end{aligned}$$

From these two cases, we have

$$\sup_{m \geqslant 0} \|(I_\gamma f)_m\|_{L_{1,\psi}} \lesssim 1,$$

which shows $I_\gamma \in B(L_{1,\phi}, L_{1,\psi})$.

Proof of (ii) $\Rightarrow$ (iii): It is clear.

Proof of (iii) $\Rightarrow$ (i): Taking $p = q = 1$ and replacing $\phi^{p/q}$ by ψ in the proof of Theorem 2 above, we have the conclusion. □

Acknowledgement. The authors would like to thank the referee for her/his careful reading and useful comments. The first author was supported by Grant-in-Aid for Scientific Research (B), No. 15H03621, Japan Society for the Promotion of Science. The second author was supported by Grant-in-Aid for Scientific Research (C), No. 16K05203, Japan Society for the Promotion of Science.

REFERENCES

[1] D. R. ADAMS, *A note on Riesz potentials*, Duke Math. J., **42**, 4 (1975), 765–778.
[2] D. L. BURKHOLDER, *Martingale transforms*, Ann. Math. Stat., **37** (1966), 1494–1504.
[3] J.-A. CHAO AND H. OMBE, *Commutators on dyadic martingales*, Proc. Japan Acad., **61**, Ser. A (1985), 35–38.
[4] ERIDANI, H. GUNAWAN AND E. NAKAI, *On generalized fractional integral operators*, Sci. Math. Jpn., **60** (2004), 539–550.
[5] ERIDANI, H. GUNAWAN, E. NAKAI AND Y. SAWANO, *Characterizations for the generalized fractional integral operators on Morrey spaces*, Math. Inequal. Appl., **17**, 2 (2014), 761–777.
[6] H. GUNAWAN, *A note on the generalized fractional integral operators*, J. Indones. Math. Soc., **9**, 1 (2003), 39–43.
[7] L. I. HEDBERG, *On certain convolution inequalities*, Proc. Amer. Math. Soc., **36** (1972), 505–510.

[8] Z. HAO AND Y. JIAO, *Fractional integral on martingale Hardy spaces with variable exponents*, Fract. Calc. Appl. Anal. **18**, 5 (2015), 1128–1145.
[9] E. NAKAI, *On generalized fractional integrals*, Taiwanese J. Math., **5**, 3 (2001), 587–602.
[10] E. NAKAI, *On generalized fractional integrals on the weak Orlicz spaces,* BMO_ϕ *, the Morrey spaces and the Campanato spaces*, Function spaces, interpolation theory and related topics, 389–401, de Gruyter, Berlin, 2002.
[11] E. NAKAI, *A generalization of Hardy spaces* H^p *by using atoms*, Acta Math. Sinica **24** (2008), 1243–1268.
[12] E. NAKAI AND G. SADASUE, *Martingale Morrey-Campanato spaces and fractional integrals*, J. Funct. Spaces Appl., **2012** (2012), Article ID 673929, 29 pages.
[13] E. NAKAI, G. SADASUE AND Y. SAWANO, *Martingale Morrey-Hardy and Campanato-Hardy Spaces*, J. Funct. Spaces Appl., **2013** (2013), Article ID 690258, 14 pages. DOI:10.1155/2013/690258
[14] G. SADASUE, *Fractional integrals on martingale Hardy spaces for* $0 < p \leqslant 1$, Mem. Osaka Kyoiku Univ. Ser. III Nat. Sci. Appl. Sci., **60**, 1 (2011), 1–7.
[15] H. TANAKA AND Y. TERASAWA, *Positive operators and maximal operators in a filtered measure space*, J. Funct. Anal. **264**, 4 (2013), 920–946.
[16] C. WATARI, *Multipliers for Walsh Fourier series*, Tohoku Math. J. **16** (1964), 239–251.
[17] F. WEISZ, *Martingale Hardy spaces and their applications in Fourier analysis*, Lecture Notes in Mathematics, 1568, Springer-Verlag, Berlin, 1994.

(Received September 14, 2016)

Eiichi Nakai
Department of Mathematics
Ibaraki University
Mito, Ibaraki 310-8512, Japan
e-mail: eiichi.nakai.math@vc.ibaraki.ac.jp

Gaku Sadasue
Department of Mathematics
Osaka Kyoiku University
Kashiwara, Osaka 582-8582, Japan
e-mail: sadasue@cc.osaka-kyoiku.ac.jp