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A REMARK ON WEIGHTED INTEGRABILITY

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Abstract. In this paper, we will generalize the result in weighted integrability to include all positive non-integers γ connecting with derivatives of the sum-functions.

1. Introduction

A real sequence $A = \{a_n\}$ is said to satisfy the *mean value bounded variation condition* (in real sense) if there is a $\lambda \geqslant 2$ and a positive constant M_0 depending upon the sequence A and λ only such that for all n we have

$$\sum_{k=n}^{2n} |\Delta a_k| := \sum_{k=n}^{2n} |a_k - a_{k+1}| \leqslant \frac{M_0}{n} \sum_{k=n/\lambda}^{\lambda n} |a_k|, \tag{1}$$

where $\sum\limits_{k=n/\lambda}^{\lambda n}$ means $\sum\limits_{n/\lambda \leqslant k \leqslant \lambda n}$, and we may assume that $M_0 > 1$ without loss of generality.

We denote the set of real sequences satisfying (1) as MVBVS (Mean Value Bounded Variation Sequences)

The MVBV concept is generalized from positive sense (see [9]) to real sense in [2].

In Fourier analysis, in many important classical results which play a fundamental role in the field, positivity and monotonicity are two key conditions.

Mean value bounded variation concept is considered not only as the ultimate generalization to monotonicity ([9]) but also as the natural replacement of positivity ([2]).

Let $L_{2\pi}$ be the space of integrable functions of period 2π. In weighted integrability case, our work [8] proved the following theorem:

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THEOREM 1. *Suppose that a real sequence* $\{a_n\}$ *satisfies condition* (1), *and consider the trigonometric series*

$$S(x) \equiv \sum_{n=1}^{\infty} a_n \sin nx$$

or

$$S(x) \equiv \sum_{n=1}^{\infty} a_n \cos nx,$$

and its sum function is denoted by $f(x)$. *Let* $0 < \gamma < 1$. *Then* $x^{-\gamma} f(x) \in L_{2\pi}$ *and* $\{a_n\}$ *is the Fourier coefficients of* $f(x)$ *if and only if*

$$\sum_{n=1}^{\infty} n^{\gamma-1} |a_n| < \infty.$$

The claim for nonnegative sequences is in [6] which generalizes a classical result of Boas [1] and Heywood [4].

In this paper, we will generalize the above result (Theorem 1) to include all positive non-integers γ connecting with derivatives of the sum-functions.

Throughout the paper, we always use M to stand for a positive constant that may not be necessarily the same at each occurrence. Sometimes, also use $O(1)$ to indicate the same meaning.

2. Main result

We establish the following main result.

THEOREM 2. *Suppose that a real sequence* $\{a_n\}$ *satisfies condition* (1), *and consider the trigonometric series*

$$S(x) \equiv \sum_{n=1}^{\infty} a_n \sin nx \tag{2}$$

or

$$S(x) \equiv \sum_{n=1}^{\infty} a_n \cos nx, \tag{3}$$

and its sum function is denoted by $f(x)$. *Let* $\gamma > 0$, $\gamma \neq 1, 2, \cdots$, *and* $\kappa_\gamma = [\gamma]$. *Then* $x^{-\gamma+\kappa_\gamma} f^{(\kappa_\gamma)}(x) \in L_{2\pi}$ *and* $\{n^{\kappa_\gamma} a_n\}$ *is the Fourier coefficients of* $f^{(\kappa_\gamma)}(x)$ *if and only if*

$$\sum_{n=1}^{\infty} n^{\gamma-1} |a_n| < \infty. \tag{4}$$

We divide the proof into several lemmas.

LEMMA 1. *Let a real sequence* $\{a_n\}$ *satisfy condition* (1), *then, for any natural number* $\kappa \geqslant 1$, $\{n^{\kappa} a_n\}$ *satisfies condition* (1).

Proof. Let $\{a_n\}$ satisfy condition (1), by writing $A_n = n^\kappa a_n$, we check that

$$|\Delta A_j| = \left|a_j \Delta j^\kappa + (j+1)^\kappa \Delta a_j\right| \leqslant M\left(|a_j| j^{\kappa-1} + j^\kappa |\Delta a_j|\right),$$

so that, by (1),

$$\sum_{j=n}^{2n} |\Delta A_j| \leqslant \frac{M}{n}\left(\sum_{j=n}^{2n} j^\kappa |a_j| + n^\kappa \sum_{j=n/\lambda}^{\lambda n} |a_j|\right) \leqslant \frac{M}{n} \sum_{j=n/\lambda}^{\lambda n} |A_j|. \quad \square$$

Lemma 1 was also proved in [5] as a discrete case to sine integrals.

LEMMA 2. *Let a real sequence $\{a_n\}$ satisfy condition (1), then*

$$|a_n| \leqslant \frac{2M_0}{n} \sum_{k=n/\lambda}^{\lambda n} |a_k|.$$

This result can be found, for example, in [2, Lemma 2.2].

LEMMA 3. *Suppose that a real sequence $\{a_n\}$ satisfies condition (1), and consider the trigonometric series (2) or (3), which sum function is denoted by $f(x)$. Let $\gamma > 0$, $\gamma \neq 1, 2, \cdots$. If (4) holds, then $x^{-\gamma+\kappa_\gamma} f^{(\kappa_\gamma)}(x) \in L_{2\pi}$ and $\{n^{\kappa_\gamma} a_n\}$ is the Fourier coefficients of $f^{(\kappa_\gamma)}(x)$.*

Proof. We only prove the case for sine series here, the other case can be treated similarly. Considering the series

$$\sum_{n=1}^{\infty} A_n \sin(nx + \kappa_\gamma \pi/2), \tag{5}$$

where $A_n = n^{\kappa_\gamma} a_n$. From conditions (1) and (4), it is not difficult to see that

$$\lim_{n\to\infty} A_n = 0, \quad \sum_{n=1}^{\infty} |\Delta A_n| < \infty. \tag{6}$$

Indeed, by condition (4), we have $\sum_{k=n/\lambda}^{n\lambda} k^{\gamma-1}|a_k| < \varepsilon$ for arbitrary $\varepsilon > 0$ and sufficiently large n, noticing that $\kappa_\gamma < \gamma$ and combining with Lemma 2, we derive that

$$n^{\kappa_\gamma}|a_n| \leqslant 2M_0 n^{\kappa_\gamma - 1} \sum_{k=n/\lambda}^{\lambda n} |a_k| \leqslant M \sum_{k=n/\lambda}^{n\lambda} k^{\gamma-1}|a_k| < \varepsilon.$$

Since ε is arbitrary, it is obvious that $n^{\kappa_\gamma} a_n \to 0, \ \ n \to \infty$. At the same time, by Lemma 1,

$$\begin{aligned}\sum_{k=1}^{\infty} |\Delta A_k| &= \sum_{j=0}^{\infty} \sum_{k=2^j}^{2^{j+1}} |\Delta A_k| \leqslant \sum_{j=0}^{\infty} \frac{M}{2^j} \sum_{k=2^j/\lambda}^{2^j\lambda} |A_k| \\ &\leqslant M \sum_{j=0}^{\infty} \sum_{k=2^j/\lambda}^{2^j\lambda} k^{\gamma-1}|a_k| \leqslant M \sum_{k=0}^{\infty} k^{\gamma-1}|a_k| < \infty,\end{aligned}$$

this proves the second inequality in (6).

By the classical results (see, e.g., [10] or [7]), the series (5) converges to its sum function $g(x)$ in $(0,\pi]$. Assume that $x\in[\frac{\pi}{n+1},\frac{\pi}{n})$, by using the inequality $|\sin x|\leqslant|x|$ and Abel's transformation, we get

$$|g(x)|\leqslant\sum_{j=1}^{n}|A_j|+\frac{n+1}{\pi}\sum_{j=n}^{\infty}|\Delta A_j|.$$

Therefore,

$$\begin{aligned}\int_0^{\pi}x^{-\gamma+\kappa_\gamma}|g(x)|\mathrm{d}x&\leqslant\sum_{n=1}^{\infty}\left(\frac{n+1}{\pi}\right)^{\gamma-\kappa_\gamma}\int_{\pi/(n+1)}^{\pi/n}|g(x)|\mathrm{d}x\\&\leqslant\sum_{n=1}^{\infty}\left(\frac{n+1}{\pi}\right)^{\gamma-\kappa_\gamma}\frac{\pi}{n(n+1)}\sum_{j=1}^{n}|A_j|\\&\quad+\sum_{n=1}^{\infty}\left(\frac{n+1}{\pi}\right)^{\gamma-\kappa_\gamma+1}\frac{\pi}{n(n+1)}\sum_{j=n}^{\infty}|\Delta A_j|=:I_1+I_2.\end{aligned}$$

In view of $0<\gamma-\kappa_\gamma<1$, a direct calculation leads to that

$$\begin{aligned}I_1&\leqslant M\sum_{n=1}^{\infty}n^{\gamma-\kappa_\gamma-2}\sum_{j=1}^{n}|A_j|\leqslant M\sum_{n=1}^{\infty}|A_n|\sum_{j=n}^{\infty}j^{\gamma-\kappa_\gamma-2}\\&\leqslant M\sum_{n=1}^{\infty}n^{\gamma-\kappa_\gamma-1}|A_n|\leqslant M\sum_{n=1}^{\infty}n^{\gamma-1}|a_n|.\end{aligned}\tag{7}$$

At the same time, since $\{a_n\}$ satisfies (1), by Lemma 1, $\{A_n\}$ satisfies (1). Then, for any sufficiently large n, there is a $\lambda\geqslant 2$ such that

$$\sum_{j=n}^{\infty}|\Delta A_j|\leqslant\sum_{j=0}^{\infty}\sum_{l=2^jn}^{2^{j+1}n}|\Delta A_l|\leqslant M\sum_{j=0}^{\infty}\frac{1}{2^jn}\sum_{l=2^jn/\lambda}^{\lambda 2^jn}|A_l|\leqslant M\sum_{l=n/\lambda}^{\infty}\frac{|A_l|}{l}.$$

It follows that

$$\begin{aligned}\sum_{n=\lambda+1}^{\infty}n^{\gamma-\kappa_\gamma-1}\sum_{j=n}^{\infty}|\Delta A_j|&\leqslant M\sum_{n=\lambda+1}^{\infty}n^{\gamma-\kappa_\gamma-1}\sum_{j=n/\lambda}^{\infty}\frac{|A_j|}{j}\\&\leqslant M\sum_{n=1}^{\infty}n^{\gamma-\kappa_\gamma-1}\sum_{j=n}^{\infty}\frac{|A_j|}{j}\\&\leqslant M\sum_{n=1}^{\infty}\frac{|A_n|}{n}\sum_{j=1}^{n}j^{\gamma-\kappa_\gamma-1}\\&\leqslant M\sum_{n=1}^{\infty}n^{\gamma-\kappa_\gamma-1}|A_n|,\end{aligned}$$

that is,

$$I_2\leqslant M\sum_{n=1}^{\infty}n^{\gamma-\kappa_\gamma-1}|A_n|\leqslant M\sum_{n=1}^{\infty}n^{\gamma-1}|a_n|.\tag{8}$$

Combining (7) with (8), we have $x^{-\gamma+\kappa_\gamma}g(x) \in L_{2\pi}$, consequently $g(x) \in L_{2\pi}$. From condition (6) and that the series (5) converges to its sum function $g(x)$ in $(0,\pi]$, we already know that $\{A_n\}$ is the Fourier coefficients of $g(x)$. Also it is easy to see that $g(x) = f^{(\kappa_\gamma)}(x)$ almost everywhere by termwise integration. Lemma 3 is proved. □

LEMMA 4. *Suppose that a real sequence $\{A_n\}$ satisfies condition (1), and consider the trigonometric series*

$$S(x) \equiv \sum_{n=1}^{\infty} A_n \sin nx$$

or

$$S(x) \equiv \sum_{n=1}^{\infty} A_n \cos nx,$$

and its sum function is denoted by $g(x)$. Let $0<\alpha<1$. If $x^{-\alpha}g(x) \in L_{2\pi}$ and $\{A_n\}$ is the Fourier coefficients of $g(x)$, then

$$\limsup_{n\to\infty} n^{\alpha-1} \sum_{k=n/\lambda}^{\lambda n} |A_k| < \infty.$$

This was proved in [8, Thorem 2.6].

LEMMA 5. *Let a real sequence $\{a_n\}$ satisfy condition (1), $0<\alpha<1$. Then, for any n,*

$$\left|\sum_{k=1}^{n} a_k \sin kx\right| = O(x^{-\alpha})$$

holds if and only if

$$n^{1-\alpha} a_n = O(1). \tag{9}$$

This result was established in [3, Theorem 3.1]. It also holds for cosine sums.

Write $I_k = \{2^k, 2^k+1, \cdots, 2^{k+1}-1\}$, and select disjoint subsets $S_1,\ldots,S_{\mu_k}$ of I_k according to the property of sequence $\{A_n\}$ as follows. Set

$$m_1 = \min\{m \in I_k : A_m \neq 0\}.$$

Let $\nu_1 = k_0$ for which $a_{m_1+k_0}$ is the first element with $m_1+k_0 \in I_k$ of opposite sign to a_{m_1}. Define now

$$S_1 = \{m_1,\ m_1+1, \cdots, m_1+\nu_1-1\}.$$

In case otherwise $\{A_n\}$ keeps sign in I_k, simply take $m_1+\nu_1 = 2^{k+1}$, and define S_1 in the same manner.

Next, set $m_2 = \min(I_k \setminus S_1)$ if this latter set is not empty, and using the same procedure we select ν_2 and define

$$S_2 = \{m_2,\ m_2+1, \cdots, m_2+\nu_2-1\}.$$

We continue this procedure until we reach an S_{μ_k} for which $I_k \setminus (S_1 \cup \cdots \cup S_{\mu_k}) = \emptyset$. Set I_k^+ to be the union of all subsets $\{S_j\}$ whose elements A_n keep positive sign, and I_k^- the union of all subsets $\{S_j\}$ whose elements A_n keep negative sign. Also, define

$$J_k^{(1)} = \{\cup S_j : |S_j| \geqslant 2^k/(32\lambda^2 M_0), 1 \leqslant j \leqslant \mu_k\},$$

$$J_k^{(2)} = \{\cup S_j : |S_j| < 2^k/(32\lambda^2 M_0), 1 \leqslant j \leqslant \mu_k\},$$

where M_0 is the positive constant appearing in (1). With these symbols, we have

LEMMA 6. *Let $0<\alpha<1$, $\{A_n\}$ satisfy condition (1). Then for sufficiently large k_0 and arbitrary N we have*

$$\sum_{k=k_0}^{N} \sum_{m\in J_k^{(2)}} m^{\alpha-1}|A_m| \leqslant 2\left(\sum_{k=k_0}^{N} \sum_{m\in J_k^{(1)}} m^{\alpha-1}|A_m| + \sum_{n=2^{k_0}/\lambda}^{2^{k_0}-1} n^{\alpha-1}|A_n| + \sum_{n=2^{N+1}}^{\lambda 2^N} n^{\alpha-1}|A_n|\right).$$

See [8, Corollary 2.8].

Also by using the above symbols, let $0<\alpha<1$, for sufficiently large k_0 and $k=k_0, k_0+1, \cdots$, set

$$d_m = \begin{cases} m^{\alpha-1}, & m \in J_k^{(1)} \cap I_k^+, \\ -m^{\alpha-1}, & m \in J_k^{(1)} \cap I_k^-, \\ 0, & m \in J_k^{(2)}. \end{cases}$$

LEMMA 7. *Under the above symbols, $\{d_m\}$ satisfies condition* (1).

See [8, Lemma 2.9].

Proof of Theorem 2. We need only prove the conclusion for sine series, the other case can be treated in the same manner. Write $g(x) = f^{(\kappa_\gamma)}(x)$, then $g(x) \in L_{2\pi}$, and $\{A_n\}$ is the Fourier coefficients of $g(x)$. We clearly see that $A_n = n^{\kappa_\gamma} a_n$. Using $\alpha = \gamma - \kappa_\gamma$, we know that $0<\alpha<1$. Also $\{A_n\}$ satisfies (1) by Lemma 1. Hence

$$\sum_{k=k_0}^{N} \sum_{m\in J_k^{(1)}} m^{\gamma-1}|a_m| = \sum_{k=k_0}^{N}\left(\sum_{m\in J_k^{(1)}\cap I_k^+} m^{\alpha-1}A_m + \sum_{m\in J_k^{(1)}\cap I_k^-} m^{\alpha-1}(-A_m)\right)$$

$$= \frac{2}{\pi}\sum_{k=k_0}^{N}\left(\sum_{m\in J_k^{(1)}\cap I_k^+} m^{\alpha-1}\int_0^\pi g(x)\sin mx\,\mathrm{d}x + \sum_{m\in J_k^{(1)}\cap I_k^-} (-m^{\alpha-1})\int_0^\pi g(x)\sin mx\,\mathrm{d}x\right)$$

$$= \frac{2}{\pi}\int_0^\pi g(x)\left(\sum_{m=2^{k_0}}^{2^N-1} d_m \sin mx\right)\mathrm{d}x,$$

or

$$\sum_{k=k_0}^{N} \sum_{m\in J_k^{(1)}} m^{\gamma-1}|a_m| \leqslant \frac{2}{\pi}\int_0^\pi |g(x)| \left|\sum_{m=2^{k_0}}^{2^N-1} d_m \sin mx\right| \mathrm{d}x.$$

By Lemma 5 and Lemma 7, we immediately deduce that

$$\left|\sum_{m=2^{k_0}}^{2^N-1} d_m \sin mx\right| = O(x^{-\alpha}),$$

so that

$$\sum_{k=k_0}^{N} \sum_{m\in J_k^{(1)}} m^{\gamma-1}|a_m| \leqslant M\int_0^\pi x^{-\gamma+\kappa_\gamma}|f^{(\kappa_\gamma)}(x)|\mathrm{d}x.$$

Then it follows from Lemma 6 that

$$\sum_{m=2^{k_0}}^{2^N} m^{\gamma-1}|a_m| \leqslant M\left(\int_0^\pi x^{-\gamma+\kappa_\gamma}|f^{(\kappa_\gamma)}(x)|\mathrm{d}x + \sum_{n=2^{k_0}/\lambda}^{2^{k_0}-1} n^{\gamma-1}|a_n| + \sum_{n=2^{N+1}}^{\lambda 2^N} n^{\gamma-\kappa_\gamma-1}|A_n|\right),$$

in connecting with Lemma 4 we have

$$\sum_{m=2^{k_0}}^{2^N} m^{\gamma-1}|a_m| \leqslant M\int_0^\pi x^{-\gamma+\kappa_\gamma}|f^{(\kappa_\gamma)}(x)|\mathrm{d}x + O(1),$$

that already completes the proof of necessity.

Sufficiency can be derived from Lemma 3. □

3. Remark

In Theorem 2, we assume that $\gamma \neq 1,2,\cdots$, and it is natural to ask what happens for these positive integers? In this section, we give an answer for nonnegative coefficients, while leave an open problem in general case.

THEOREM 3. *Suppose that a nonnegative sequence $\{a_n\}$ satisfies condition (1), and consider the trigonometric series*

$$S(x) \equiv \sum_{n=1}^{\infty} a_n \sin nx,$$

and its sum function is denoted by $f(x)$.

(i) Let $\gamma = 1,3,5,\cdots$. Then $x^{-1}f^{(\gamma-1)}(x) \in L_{2\pi}$ and $\{n^{\gamma-1}a_n\}$ is the Fourier coefficients of $f^{(\gamma-1)}(x)$ if and only if

$$\sum_{n=1}^{\infty} n^{\gamma-1}a_n < \infty.$$

(ii) Let $\gamma = 2,4,6,\cdots$. *Then if* $0 < \sum_{n=1}^{\infty} n^{\gamma-1} a_n < \infty$, *we have*

$$\int_0^{\pi} x^{-1} |f^{(\gamma-1)}(x)| \mathrm{d}x = \infty.$$

THEOREM 4. *Suppose that a nonnegative sequence* $\{a_n\}$ *satisfies condition* (1), *and consider the trigonometric series*

$$S(x) \equiv \sum_{n=1}^{\infty} a_n \cos nx,$$

and its sum function is denoted by $f(x)$.

(i) Let $\gamma = 2,4,6,\cdots$. *Then,* $x^{-1} f^{(\gamma-1)}(x) \in L_{2\pi}$ *and* $\{n^{\gamma-1} a_n\}$ *is the Fourier coefficients of* $f^{(\gamma-1)}(x)$ *if and only if*

$$\sum_{n=1}^{\infty} n^{\gamma-1} a_n < \infty.$$

(ii) Let $\gamma = 1,3,5,\cdots$. *Then if* $0 < \sum_{n=1}^{\infty} n^{\gamma-1} a_n < \infty$, *we have*

$$\int_0^{\pi} x^{-1} |f^{(\gamma-1)}(x)| \mathrm{d}x = \infty.$$

The proof of these two results is a combination of the following propositions.

PROPOSITION 5. For any nonnegative sequence $\{a_n\}$ satisfying (1) and

$$0 < \sum_{n=1}^{\infty} a_n < \infty,$$

we have

$$\int_0^{\pi} x^{-1} |g_0(x)| dx = \infty$$

for the cosine series $g_0(x) = \sum_{n=1}^{\infty} a_n \cos nx$.

Proof. Note now that the series $\sum_{n=1}^{\infty} a_n \cos nx$ uniformly and absolutely converges to $g_0(x)$. We see that

$$\cos jx = 1 + \cos jx - 1 = 1 - 2\sin^2 \frac{jx}{2}.$$

By the inequality $|\sin x| \leqslant |x|$ and Abel's transformation, for $x \in [\frac{\pi}{n+1}, \frac{\pi}{n})$, we clearly see that

$$\begin{aligned}|g_0(x)| &\geqslant \sum_{j=1}^{n} a_j - 2\sum_{j=1}^{n} a_j \sin^2 \frac{jx}{2} - (n+1)\sum_{j=n+1}^{\infty} |\Delta a_j| - (n+1)a_{n+1} \\ &\geqslant \sum_{j=1}^{n} a_j - x\sum_{j=1}^{n} ja_j - (n+1)\sum_{j=n+1}^{\infty} |\Delta a_j| - (n+1)a_{n+1}.\end{aligned}$$

Hence

$$\begin{aligned}\int_0^{\pi} x^{-1}|g_0(x)|\mathrm{d}x &\geqslant \sum_{n=1}^{\infty} \frac{n}{\pi} \int_{\pi/(n+1)}^{\pi/n} |g_0(x)|\mathrm{d}x \\ &\geqslant \sum_{n=1}^{\infty} \frac{n}{\pi} \frac{\pi}{n(n+1)} \sum_{j=1}^{n} a_j - \sum_{n=1}^{\infty} \frac{n}{\pi} \frac{\pi^2}{2} \frac{2n+1}{n^2(n+1)^2} \sum_{j=1}^{n} ja_j \\ &\quad - \sum_{n=1}^{\infty} \sum_{j=n+1}^{\infty} |\Delta a_j| - \sum_{n=1}^{\infty} a_{n+1} \\ &=: J - (J_1 + J_2 + J_3).\end{aligned}$$

A similar calculation to the proof of Theorem 2 yields that

$$J_1 \leqslant M\sum_{n=1}^{\infty} a_n, \;\; J_2 \leqslant M\sum_{n=1}^{\infty} a_n, \;\; J_3 \leqslant \sum_{n=1}^{\infty} a_n.$$

Since $\sum\limits_{n=1}^{\infty} a_n$ is convergent, say, $\sum\limits_{n=1}^{\infty} a_n = B > 0$, then there is an N_0 such that

$$\sum_{n=1}^{N_0} a_n \geqslant \frac{B}{2}.$$

For arbitrarily large N, one get

$$J \geqslant M\sum_{n=1}^{\infty} n^{-1} \sum_{j=1}^{n} a_j \geqslant M\sum_{n=N_0}^{N} n^{-1} \sum_{j=1}^{N_0} a_j \geqslant MB\log(N/N_0).$$

Combining all the above estimates, we derive that

$$\int_0^{\pi} x^{-1}|g_0(x)|\mathrm{d}x \geqslant MB\log(N/N_0)$$

for any sufficiently large N. Proposition 5 is proved. □

PROPOSITION 6. Suppose that a nonnegative sequence $\{a_n\}$ satisfies condition (1), and consider the trigonometric series

$$S(x) \equiv \sum_{n=1}^{\infty} a_n \sin nx, \tag{10}$$

and its sum function is denoted by $f(x)$. Then, $x^{-1}f(x) \in L_{2\pi}$ and $\{a_n\}$ is the Fourier coefficients of $f(x)$ if and only if

$$\sum_{n=1}^{\infty} a_k < \infty.$$

This was proved in [6, Theorem 2].

Finally, we pose an open problem for coefficients that may not be necessarily nonnegative.

PROBLEM 7. Suppose that a *real sequence* $\{a_n\}$ satisfies condition (1), and consider the trigonometric series (10), and its sum function is denoted by $f(x)$. Then, whether it is true that $x^{-1}f(x) \in L_{2\pi}$ and $\{a_n\}$ is the Fourier coefficients of $f(x)$ if and only if

$$\sum_{n=1}^{\infty} |a_n| < \infty?$$

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