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SOME FAMILIES OF GENERALIZED MATHIEU–TYPE POWER SERIES, ASSOCIATED PROBABILITY DISTRIBUTIONS AND RELATED INEQUALITIES INVOLVING COMPLETE MONOTONICITY AND LOG–CONVEXITY

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Dedicated to Professor Neven Elezović on the occasion of his sixtieth birthday

(*Communicated by M. Praljak*)

Abstract. By making use of the familiar Mathieu series and its generalizations, the authors derive a number of new integral representations and present a systematic study of probability density functions and probability distributions associated with some generalizations of the Mathieu series. In particular, the mathematical expectation, variance and the characteristic functions, related to the probability density functions of the considered probability distributions are derived. As a consequence, some interesting inequalities involving complete monotonicity and log-convexity are derived.

1. Introduction

The following familiar infinite series

$$S(r)=\sum_{n=1}^{\infty}\frac{2n}{(n^2+r^2)^2}\quad \left(r\in\mathbb{R}^+\right) \tag{1}$$

is named after Emile Leonard Mathieu (1835–1890), who investigated it in his 1890 work [13] on elasticity of solid bodies. Let $\mathbb{C}$, $\mathbb{R}^+$, $\mathbb{N}$ and $\mathbb{Z}_0^-$ be the sets of complex numbers, positive real numbers, positive integers, and non-positive integers, respectively. Integral representations of (1) is given by (see [9])

$$S(r)=\frac{1}{r}\int_0^{\infty}\frac{t\sin(rt)}{e^t-1}dt. \tag{2}$$

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Several interesting problems and solutions dealing with integral representations and bounds for the following slight generalization of the Mathieu series with a fractional power

$$S_\mu(r) = \sum_{n=1}^{\infty} \frac{2n}{(n^2+r^2)^\mu} \quad \left(r \in \mathbb{R}^+; \mu > 1\right) \tag{3}$$

can be found in the works by Cerone and Lenard [4], Diananda [7], and Tomovski and Trencevski [21]. Motivated essentially by the works of Cerone and Lenard [4], Srivastava and Tomovski in [18] defined a family of generalized Mathieu series

$$S_\mu^{(\alpha,\beta)}(r;\mathrm{a}) = S_\mu^{(\alpha,\beta)}(r;\{a_k\}_{k=1}^\infty) = \sum_{n=1}^{\infty} \frac{2a_n^\beta}{(a_n^\alpha+r^2)^\mu} \quad \left(r,\alpha,\beta,\mu \in \mathbb{R}^+\right) \tag{4}$$

where it is tacitly assumed that the positive sequence

$$\mathrm{a} = \{a_n\}_{n=1}^\infty = \{a_1,a_2,a_3,\ldots\} \left(\lim_{n\to\infty} a_n = \infty\right)$$

is so chosen that the infinite series in definition (4) converges, that is, that the following auxiliary series

$$\sum_{n=1}^{\infty} \frac{1}{a_n^{\mu\alpha-\beta}}$$

is convergent. Comparing the definitions (1), (3) and (4), we see that $S_2(r) = S(r)$ and $S_\mu(r) = S_\mu^{(2,1)}(r,\{n\}_{n=1}^\infty)$. Furthermore, the special cases $S_2^{(2,1)}(r;\{a_n\}_{n=1}^\infty)$, $S_\mu(r) = S_\mu^{(2,1)}(r;\{n\}_{n=1}^\infty)$, $S_\mu^{(2,1)}(r;\{n^\gamma\}_{n=1}^\infty)$ and $S_\mu^{(\alpha,\alpha/2)}(r;\{n\}_{n=1}^\infty)$ were investigated by Cerone-Lenard [4], Diananda [7]; and Tomovski [22]. For more details the interested reader is referred to the papers [8, 15, 18, 5, 20, 21, 22, 23, 24, 25].

In this paper we consider a power series

$$S_{\mu,\nu}^{(\alpha,\beta)}(r,\mathrm{a};z) = S_{\mu,\nu}^{(\alpha,\beta)}(r,\{a_k\}_{k=1}^\infty;z) = \sum_{n=1}^{\infty} \frac{2a_n^\beta(\nu)_n z^n}{(a_n^\alpha+r^2)^\mu\, n!} \tag{5}$$

$$\left(r,\alpha,\beta,\mu \in \mathbb{R}^+; |z| \leqslant 1\right).$$

We denote

$$S_{\mu,\nu}^{(\alpha,\beta)}(r,\mathrm{a};1) \equiv S_{\mu,\nu}^{(\alpha,\beta)}(r,\mathrm{a}), \quad S_{\mu,1}^{(\alpha,\beta)}(r,\mathrm{a};1) \equiv S_{\mu}^{(\alpha,\beta)}(r,\mathrm{a}) \tag{6}$$

$$S_{\mu,\nu}^{(\alpha,\beta)}(r,\mathrm{a};-1) \equiv \widetilde{S}_{\mu,\nu}^{(\alpha,\beta)}(r,\mathrm{a}), \quad S_{\mu,1}^{(\alpha,\beta)}(r,\mathrm{a};-1) \equiv \widetilde{S}_{\mu}^{(\alpha,\beta)}(r,\mathrm{a}) \tag{7}$$

For $a_n = n$, $\alpha = 2$, $\beta = 1$, $\nu = 1$ and μ with $\mu+1$ the series (5) was introduced and considered by Tomovski and Pogany in [25].

This paper is organized as follows: in Sections 2 and 3, we present new integral and series representations for generalised Mathieu series. In particular, we present a new type integral for the Mathieu series for a special case. In section 4, we introduce, develop and investigate probability distribution functions (PDF) associated with the

Mathieu series and their generalizations. As consequences some inequalities are derived. In Section 5, we show the complete monotonicity and log-convexity properties for generalised Mathieu series. Moreover, as consequences of these results, we presented some functional inequalities as well as lower and upper bounds for generalised Mathieu series.

2. Integral expression of Mathieu series

Our first main results is the next theorem.

THEOREM 1. *Let* $r, \alpha, \beta, \nu, \mu > 0$ *and* $\gamma(\mu\alpha - \beta) > 1$*. Then the Mathieu type power series* $S^{(\alpha,\beta)}_{\mu,\nu}\left(r, \{k^\gamma\}_{k=1}^\infty; z\right)$ *has the integral representation*

$$S^{(\alpha,\beta)}_{\mu,\nu}\left(r, \{k^\gamma\}_{k=1}^\infty; z\right) = \frac{2\nu z}{\Gamma(\mu)} \int_0^\infty \frac{t^{\gamma[\mu\alpha-\beta]-1} e^{-t}}{\left(1-ze^{-t}\right)^{\nu+1}} {}_1\Psi_1\left[(\mu,1); (\gamma(\mu\alpha-\beta), \gamma\alpha); -r^2 t^{\gamma\alpha}\right] dt, \tag{8}$$

where ${}_p\Psi_q$ *denotes the Fox-Wright generalization of the hypergeometric* ${}_pF_q$ *function with* p *numerator and* q *denominator parameters, defined by* [18, p. 50, Eq. 1.5 (21)]

$${}_p\Psi_q\left[(\alpha_1, A_1), \ldots, (\alpha_p, A_p); (\beta_1, B_1), \ldots, (\beta_q, B_q); z\right] = \sum_{n=0}^\infty \frac{\prod_{j=1}^p \Gamma(\alpha_j + nA_j)}{\prod_{j=1}^q \Gamma(\beta_j + nB_j)} \cdot \frac{z^n}{n!}, \tag{9}$$

with

$$\left(A_j \in \mathbb{R}^+,\ j = 1, \ldots, p;\ B_j \in \mathbb{R}^+,\ j = 1, \ldots, q;\ 1 + \sum_{j=1}^q B_j - \sum_{j=1}^p A_j > 0\right).$$

Proof. First of all, we find from the definition (5) that

$$S^{(\alpha,\beta)}_{\mu,\nu}\left(r, \{a_k\}_{k=1}^\infty; z\right) = 2\sum_{m=0}^\infty \binom{\mu+m-1}{m} \left(-r^2\right)^m \sum_{n=1}^\infty \frac{(\nu)_n z^n}{a_n^{(\mu+m)\alpha-\beta} n!} \tag{10}$$

So,

$$\begin{aligned} S^{(\alpha,\beta)}_{\mu,\nu}\left(r, \{k^\gamma\}_{k=1}^\infty; z\right) &= 2z \sum_{m=0}^\infty \binom{\mu+m-1}{m} \left(-r^2\right)^m \sum_{n=0}^\infty \frac{(\nu)_{n+1} z^n}{(n+1)^{[(\mu+m)\alpha-\beta]\gamma} (n+1)!} \\ &= 2\nu z \sum_{m=0}^\infty \binom{\mu+m-1}{m} \left(-r^2\right)^m \sum_{n=0}^\infty \frac{(\nu+1)_n z^n}{(n+1)^{[(\mu+m)\alpha-\beta]\gamma+1} n!} \\ &= 2\nu z \sum_{m=0}^\infty \binom{\mu+m-1}{m} \left(-r^2\right)^m \Phi^*_{\nu+1}\left(z, [(\mu+m)\alpha-\beta]\gamma+1, 1\right). \end{aligned}$$

Here $\Phi^*_{\nu+1}$ denotes a Hurwitz-Lerch Zeta function, defined by Lin and Srivastava [12]

$$\Phi^*_\nu(z, s, a) = \sum_{n=0}^\infty \frac{(\nu)_n}{n!} \frac{z^n}{(n+a)^s}$$

$$\left(\nu \in \mathbb{C};\ a \in \mathbb{C}\setminus\mathbb{Z}_0^-,\ s \in \mathbb{C},\ |z|<1;\ \Re(s-\nu)>1 \text{ when } |z|=1\right).$$

A special case of Φ_ν^* with $\nu=1$ give us the well-known Hurwitz-Lerch Zeta function

$$\Phi(z,s,a)=\sum_{n=0}^{\infty}\frac{z^n}{(n+a)^s}$$

$$\left(a \in \mathbb{C}\setminus\mathbb{Z}_0^-,\ s \in \mathbb{C},\ |z|<1;\ \Re(s)>1 \text{ when } |z|=1\right).$$

Now, by making use of the familiar integral representation (see [12])

$$\Phi_\nu^*(z,s,a)=\frac{1}{\Gamma(s)}\int_0^\infty \frac{t^{s-1}e^{-at}}{(1-ze^{-t})^\nu}dt$$

$$(\Re(a)>0;\ \Re(s)>0 \text{ when } |z|\leqslant 1,\ z\neq 1;\ \Re(s)>1 \text{ when } z=1)$$

we get

$$\begin{aligned}
&S_{\mu,\nu}^{(\alpha,\beta)}(r,\{k^\gamma\}_{k=1}^\infty;z)\\
&=2\nu z\sum_{m=0}^{\infty}\frac{\binom{\mu+m-1}{m}}{\Gamma(\gamma[(\mu+m)\alpha-\beta])}(-r^2)^m\int_0^\infty \frac{t^{\gamma[(\mu+m)\alpha-\beta]-1}e^{-t}}{(1-ze^{-t})^{\nu+1}}dt\\
&=2\nu z\int_0^\infty \frac{t^{\gamma[\mu\alpha-\beta]-1}e^{-t}}{(1-ze^{-t})^{\nu+1}}\left[\sum_{m=0}^{\infty}\frac{\binom{\mu+m-1}{m}}{\Gamma(\gamma[(\mu+m)\alpha-\beta])}\left(-r^2t^{\gamma\alpha}\right)^m\right]dt\\
&=\frac{2\nu z}{\Gamma(\mu)}\int_0^\infty \frac{t^{\gamma[\mu\alpha-\beta]-1}e^{-t}}{(1-ze^{-t})^{\nu+1}}\,{}_1\Psi_1\left[(\mu,1);(\gamma(\mu\alpha-\beta),\gamma\alpha);-r^2t^{\gamma\alpha}\right]dt.
\end{aligned}$$

This ends the proof. □

3. Concluding Remarks

1. As the convergence interval of $S_{\mu,\nu}^{(\alpha,\beta)}(r,\{k^\gamma\}_{k=1}^\infty;z)$ is $[-1,1]$, we easily conclude the following representations

$$S_{\mu,\nu}^{(\alpha,\beta)}(r,\{k^\gamma\})\equiv\frac{2\nu}{\Gamma(\mu)}\int_0^\infty \frac{t^{\gamma[\mu\alpha-\beta]-1}e^{-t}}{(1-e^{-t})^{\nu+1}}\,{}_1\Psi_1\left[(\mu,1);(\gamma(\mu\alpha-\beta),\gamma\alpha);-r^2t^{\gamma\alpha}\right]dt, \tag{11}$$

and

$$\widetilde{S}_{\mu,\nu}^{(\alpha,\beta)}(r,\{k^\gamma\})\equiv\frac{2\nu}{\Gamma(\mu)}\int_0^\infty \frac{t^{\gamma[\mu\alpha-\beta]-1}e^{-t}}{(1+e^{-t})^{\nu+1}}\,{}_1\Psi_1\left[(\mu,1);(\gamma(\mu\alpha-\beta),\gamma\alpha);-r^2t^{\gamma\alpha}\right]dt. \tag{12}$$

2. Specially, for $z=e^{-x}<1$, we find that

$$\begin{aligned}S_{\mu,\nu}^{(\alpha,\beta)}\left(r,\{k^\gamma\}_{k=1}^\infty;e^{-x}\right)&=\frac{2\nu e^{-x}}{\Gamma(\mu)}\int_0^\infty \frac{t^{\gamma(\mu\alpha-\beta)-1}e^{-t}}{\left(1-e^{-(x+t)}\right)^{\nu+1}}\\
&\quad\times{}_1\Psi_1\left[(\mu,1);(\gamma(\mu\alpha-\beta),\gamma\alpha);-r^2t^{\gamma\alpha}\right]dt,\end{aligned} \tag{13}$$

and

$$\widetilde{S}_{\mu,\nu}^{(\alpha,\beta)}\left(r,\{k^{\gamma}\}_{k=1}^{\infty};e^{-x}\right)=\frac{2\nu e^{-x}}{\Gamma(\mu)}\int_0^{\infty}\frac{t^{\gamma(\mu\alpha-\beta)-1}e^{-t}}{\left(1+e^{-(x+t)}\right)^{\nu+1}}$$
$$\times\ {}_1\Psi_1\left[(\mu,1);(\gamma(\mu\alpha-\beta),\gamma\alpha);-r^2t^{\gamma\alpha}\right]dt. \qquad (14)$$

3. In a similar manner, we get

$$\begin{aligned}
&S_{\mu,\nu}^{(\alpha,\beta)}\left(r,\left\{k^{q/\alpha}\right\}_{k=1}^{\infty};z\right)\\
&=2\nu\sum_{m=0}^{\infty}\binom{\mu+m-1}{m}\left(-r^2\right)^m\Phi_{\nu+1}^{*}\left(z,q\left[(\mu+m)-\frac{\beta}{\alpha}\right]\gamma,1\right)\\
&=2\nu\int_0^{\infty}\frac{t^{q\left[\mu-\frac{\beta}{\alpha}\right]-1}e^{-t}}{(1-ze^{-t})^{\nu+1}}\left[\sum_{m=0}^{\infty}\frac{\binom{\mu+m-1}{m}}{\Gamma\left(q\left[(\mu+m)-\frac{\beta}{\alpha}\right]\right)}\left(-r^2t^q\right)^m\right]dt\\
&=\frac{2\nu}{\Gamma\left(q\left[\mu-\frac{\beta}{\alpha}\right]\right)}\int_0^{\infty}\frac{t^{q\left[\mu-\frac{\beta}{\alpha}\right]-1}e^{-t}}{(1-ze^{-t})^{\nu+1}}\,{}_1F_q\left(\mu;\Delta\left(q;q\left[\mu-\frac{\beta}{\alpha}\right]\right);-r^2\left(\frac{t}{q}\right)^q\right)dt.
\end{aligned}$$

$$\left(r,\alpha,\beta,\nu\in\mathbb{R}^+;\ \mu-\frac{\beta}{\alpha}>q^{-1};\ q\in\mathbb{N}\right),$$

where for convenience, $\Delta(q;\lambda)$ abbreviates the array of q parameters

$$\frac{\lambda}{q},\frac{\lambda+1}{q},\ldots,\frac{\lambda+q-1}{q}\quad(q\in\mathbb{N}).$$

4. For $q=2$, this integral representation, can easily be simplified to the form:

$$S_{\mu,\nu}^{(\alpha,\beta)}\left(r,\left\{k^{2/\alpha}\right\}_{k=1}^{\infty};z\right)=\frac{2\nu}{\Gamma\left(2\left[\mu-\frac{\beta}{\alpha}\right]\right)}\int_0^{\infty}\frac{t^{2\left[\mu-\frac{\beta}{\alpha}\right]-1}e^{-t}}{(1-ze^{-t})^{\nu+1}}$$
$$\times\ {}_1F_2\left(\mu;\mu-\frac{\beta}{\alpha},\mu-\frac{\beta}{\alpha}+\frac{1}{2};-\frac{r^2t^2}{4}\right)dt,$$

$$\left(r,\alpha,\beta,\nu\in\mathbb{R}^+;\ \mu-\frac{\beta}{\alpha}>\frac{1}{2}\right).$$

5. In a similar manner, a limit case, when $\beta\to 0$ would formally yield the formula:

$$S_{\mu,\nu}^{(\alpha,0)}\left(r,\left\{k^{2/\alpha}\right\}_{k=1}^{\infty};z\right)=\frac{2\nu}{\Gamma(2\mu)}\int_0^{\infty}\frac{t^{2\mu-1}e^{-t}}{(1-ze^{-t})^{\nu+1}}\,{}_0F_1\left(-;,\mu+\frac{1}{2};-\frac{r^2t^2}{4}\right)dt$$

$$\left(r,\alpha,\nu\in\mathbb{R}^+;\mu>\frac{1}{2}\right).$$

6. On the other hand, we have

$$S_{2,\nu}^{(2,1)}\left(r,\{k\}_{k=1}^{\infty};z\right)=\sum_{n=1}^{\infty}\frac{[(n+ir)+(n-ir)]\,(\nu)_n z^n}{[(n+ir)(n-ir)]^2\, n!}$$
$$=\frac{1}{2ir}\left[\Phi_\nu^*(z,2,ir)-\Phi_\nu^*(z,2,-ir)\right] \tag{15}$$

As a matter of fact, in terms of the Riemann-Liouville fractional derivative operator D_z^ν, defined by

$$D_z^\mu\{f(z)\}=\begin{cases}\frac{1}{\Gamma(-\mu)}\int_0^z (z-t)^{-\mu-1} f(t)\,dt & (\Re(\mu)<0)\\ \frac{d^m}{dz^m}\left\{D_z^{\mu-m} f(z)\right\} & (m-1\leqslant \Re(\mu)<m) \quad (m\in\mathbb{N})\end{cases}$$

it is easily seen from the series definitions in (5) and (15) that

$$\Phi_\nu^*(z,s,a)=\frac{1}{\Gamma(\nu)}D_z^{\nu-1}\left\{z^{\nu-1}\Phi(z,s,a)\right\}$$

we obtain

$$S_{2,\nu}^{(2,1)}\left(r;\{n\}_{n=1}^{\infty};z\right)=\frac{1}{2ir\Gamma(\nu)}\left\{D_z^{\nu-1}\left[z^{\nu-1}\Phi(z,2,ir)\right]-D_z^{\nu-1}\left[z^{\nu-1}\Phi(z,2,-ir)\right]\right\}, \tag{16}$$

where $0\leqslant Re(\nu)<1$.

7. In [10], the authors defined generalized β-Mittag-Leffler functions

$$E_{\beta,\nu,\gamma}(x)=\sum_{k=0}^{\infty}\frac{x^k}{[\Gamma(\nu k+\gamma)]^\beta}. \tag{17}$$

For $\nu=\gamma=1$ and $\beta\in\mathbb{N}$, we get the hyper-Bessel function. We define a new family of β-Mittag-Leffler functions

$$E_{\beta,\nu,\gamma}^{(\tau)}(x)=\sum_{k=0}^{\infty}\frac{(\tau)_k x^k}{k![\Gamma(\nu k+\gamma)]^\beta}. \tag{18}$$

Specially, for $\tau=1$, we obtain $E_{\beta,\nu,\gamma}^{(1)}(x)=E_{\beta,\nu,\gamma}(x)$.

The Mathieu series $S_{\mu,\nu}^{(\alpha,\beta)}\left(r,\{\Gamma(\gamma n+\delta)\}_{n=1}^{\infty},z\right)$ admits the following series representation:

$$S_{\mu,\nu}^{(\alpha,\beta)}\left(r,\{\Gamma(\gamma n+\delta)\}_{n=1}^{\infty},z\right)$$
$$=\sum_{m=0}^{\infty}\binom{\mu+m-1}{m}\left(-r^2\right)^m\sum_{n=1}^{\infty}\frac{(\nu)_n z^n}{[\Gamma(\gamma n+\delta)]^{(\mu+m)\alpha-\beta}\, n!}$$
$$=\sum_{m=0}^{\infty}\binom{\mu+m-1}{m}\left(-r^2\right)^m\left[E_{(\mu+m)\alpha-\beta,\gamma,\delta}^{(\nu)}(z)-\frac{1}{[\Gamma(\delta)]^{(\mu+m)\alpha-\beta}}\right].$$

8. The Mathieu series $S_{\mu,1}^{(\alpha,\beta)}(r,\{\Gamma(\gamma n+\delta)\}_{n=1}^{\infty},z)$ admits the following series representation:

$$\begin{aligned} S_{\mu,1}^{(\alpha,\beta)}(r,\{\Gamma(\gamma n+\delta)\}_{n=1}^{\infty},z) &= \sum_{m=0}^{\infty}\binom{\mu+m-1}{m}\left(-r^2\right)^m \sum_{n=1}^{\infty}\frac{z^n}{[\Gamma(\gamma n+\delta)]^{(\mu+m)\alpha-\beta}} \\ &= z\sum_{m=0}^{\infty}\binom{\mu+m-1}{m}\left(-r^2\right)^m \sum_{n=0}^{\infty}\frac{z^n}{[\Gamma(\gamma n+\gamma+\delta)]^{(\mu+m)\alpha-\beta}} \\ &= z\sum_{m=0}^{\infty}\binom{\mu+m-1}{m}\left(-r^2\right)^m E_{(\mu+m)\alpha-\beta,\gamma,\gamma+\delta}(z). \end{aligned}$$

In the next Theorem we give an integral expression of Mathieu series $S_{3/2,1}^{(\alpha,\beta)}(r;\mathbf{a};1)$, by using the Fourier transform.

THEOREM 2. *Let $r,\alpha,\beta>0$ and $\mathbf{a}=(a_k)_{k\geqslant 1}$ be a sequences such that the function*

$$f_r^{(\alpha,\beta)}(t)=\sum_{n=1}^{\infty}\frac{2(a_n^{\alpha}t^2-r^2)}{a_n^{\frac{\alpha}{2}-\beta}(a_n^{\alpha}t^2+r^2)^2} \tag{19}$$

converges for all $t>0$. Then we have

$$S_{3/2,1}^{(\alpha,\beta)}(r;\mathbf{a},1)=\frac{2}{\pi}\int_1^{\infty}t\sqrt{t^2-1}\,f_r^{(\alpha,\beta)}(t)dt.$$

Proof. In [26, Eq. 6.6], the author give the Fourier transform of the function $\varphi_{c,\mu}(x)=\frac{1}{(c^2+x^2)^{\mu}}$ with $c>0$ and $\mu>1/2$:

$$\mathscr{F}\varphi_{c,\mu}(\xi)=\frac{2^{1-\mu}}{\Gamma(\mu)c^{\mu-1/2}}|\xi|^{\mu-1/2}K_{\mu-1/2}(c|\xi|), \tag{20}$$

where K_{α} is the modified Bessel function of the second kind. On the other hand, using the representation integral (see for example [2, Theorem 4.17]),

$$K_{\alpha}(x)=\frac{1}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\left(\frac{x}{2}\right)^{\alpha}\int_1^{+\infty}e^{-xt}\left(t^2-1\right)^{\alpha-\frac{1}{2}}dt,\quad \alpha>-\frac{1}{2},\quad x>0. \tag{21}$$

and Fubini's Theorem that the even function $x^{\alpha}K_{\alpha}(x)$ belongs to $L^1[0,\infty)$. So, by using the inversion formula and (20) we deduce that

$$\begin{aligned} \varphi_{c,\mu}(x) &= \frac{2^{1-\mu}}{\sqrt{2\pi}\,\Gamma(\mu)\,c^{\mu-1/2}}\mathscr{F}\left(|\xi|^{\mu-1/2}K_{\mu-1/2}(c|\xi|\right)(x), \\ &= \frac{2^{2-\mu}}{\sqrt{2\pi}\,\Gamma(\mu)\,c^{\mu-1/2}}\int_0^{\infty}\cos(x\xi)\xi^{\mu-1/2}K_{\mu-1/2}(c\xi)d\xi \end{aligned} \tag{22}$$

In view of the representation integral (see [26, Def. 5.10])

$$K_{\mu}(z)=\int_0^{\infty}e^{-z\cosh(t)}\cosh(\mu t)dt, \tag{23}$$

and (22) we obtain

$$\varphi_{c,\mu}(x)=\frac{2^{2-\mu}}{\sqrt{2\pi}\Gamma(\mu)c^{\mu-1/2}}\int_0^\infty\cosh((\mu-1/2)t)\left[\int_0^\infty\cos(x\xi)\xi^{\mu-1/2}e^{-c\xi\cosh(t)}d\xi\right]dt. \tag{24}$$

Now, let $\mu=3/2$. Combining the following formula (see [6, Eq. 2. p. 391])

$$\int_0^\infty\cos(x\xi)\xi e^{-c\cosh(t)\xi}d\xi=\frac{c^2\cosh(t)^2-x^2}{(c^2\cosh(t)^2+x^2)^2} \tag{25}$$

and (24) we obtain

$$\begin{aligned}\varphi_{c,3/2}(x)&=\frac{1}{\sqrt{\pi}\,\Gamma(3/2)\,c}\int_0^\infty\cosh(t)\frac{(c^2\cosh(t)^2-x^2)}{(c^2\cosh(t)^2+x^2)^2}dt\\&=\frac{1}{\sqrt{\pi}\,\Gamma(3/2)\,c}\int_1^\infty t\sqrt{t^2-1}\frac{(c^2t^2-x^2)}{(c^2t^2+x^2)^2}dt.\end{aligned} \tag{26}$$

Letting $c=a_n^{\frac{\alpha}{2}}$ and $x=r$ in the (26), we get

$$\frac{2a_n^\beta}{(a_n^\alpha+r^2)^{3/2}}=\frac{2}{\pi}\int_1^\infty t\sqrt{t^2-1}\frac{2a_n^\beta(a_n^\alpha t^2-r^2)}{a_n^{\alpha/2}(a_n^\alpha t^2+r^2)^2}dt.$$

The interchanging between integral and summation gives the desired result. □

4. Mathieu probability distribution

The main objective of this section is to introduce, develop and investigate probability distribution functions (PDF) associated with the Mathieu series and their generalizations. We define a discrete random variable X defined on some fixed standard probability space $(\Omega,\mathcal{F},P)$ possesing a Mathieu distribution with parameter $r>0$,

$$P_{\mu,\nu}^{(\alpha,\alpha)}(n,r)=P(X=n)=\frac{2n^\alpha(\nu)_n}{(n^\alpha+r^2)^\mu\,n!}\frac{1}{S_{\mu,\nu}^{(\alpha,\alpha)}(r)} \tag{27}$$

$$\left(r,\alpha,\mu,\nu\in\mathbb{R}^+\right),\ n\in\mathbb{N},$$

where $S_{\mu,\nu}^{(\alpha,\alpha)}(r)=S_{\mu,\nu}^{(\alpha,\alpha)}(r,\{n\}_{n=1}^\infty)$. $P_{\mu,\nu}^{(\alpha,\alpha)}(n,r)$ is normalized, since

$$\sum_{n=1}^\infty P_{\mu,\nu}^{(\alpha,\alpha)}(n,r)=\sum_{n=1}^\infty\frac{2n^\alpha(\nu)_n}{(n^\alpha+r^2)^\mu\,n!}\frac{1}{S_{\mu,\nu}^{(\alpha,\alpha)}(r)}=\frac{S_{\mu,\nu}^{(\alpha,\alpha)}(r)}{S_{\mu,\nu}^{(\alpha,\alpha)}(r)}=1.$$

THEOREM 3. *The expected value EX of a Mathieu distribution $P_{\mu,\nu}^{(\alpha,\alpha)}(n,r)$ is $\frac{S_{\mu,\nu}^{(\alpha,\alpha+1)}(r)}{S_{\mu,\nu}^{(\alpha,\alpha)}(r)}$ and variance $Var(X)$ is given by*

$$\frac{S_{\mu,\nu}^{(\alpha,\alpha+2)}(r)\,S_{\mu,\nu}^{(\alpha,\alpha)}(r)-\left[S_{\mu,\nu}^{(\alpha,\alpha+1)}(r)\right]^2}{\left(S_{\mu,\nu}^{(\alpha,\alpha)}(r)\right)^2}.$$

Moreover the following Turán type inequality holds true

$$S_{\mu,\nu}^{(\alpha,\alpha+2)}(r)\,S_{\mu,\nu}^{(\alpha,\alpha)}(r) \geqslant \left(S_{\mu,\nu}^{(\alpha,\alpha+1)}(r)\right)^2. \tag{28}$$

Proof. By computation we have

$$EX=\sum_{n=1}^{\infty} nP(X=n)=\sum_{n=1}^{\infty}\frac{2n^{\alpha+1}(\nu)_n}{(n^\alpha+r^2)^\mu\, n!}\frac{1}{S_{\mu,\nu}^{(\alpha,\alpha)}(r)}=\frac{S_{\mu,\nu}^{(\alpha,\alpha+1)}(r)}{S_{\mu,\nu}^{(\alpha,\alpha)}(r)},$$

and

$$EX^2=\sum_{n=1}^{\infty} n^2P(X=n)=\sum_{n=1}^{\infty}\frac{2n^{\alpha+2}(\nu)_n}{(n^\alpha+r^2)^\mu\, n!}\frac{1}{S_{\mu,\nu}^{(\alpha,\alpha)}(r)}=\frac{S_{\mu,\nu}^{(\alpha,\alpha+2)}(r)}{S_{\mu,\nu}^{(\alpha,\alpha)}(r)}.$$

Thus

$$Var(X)=EX^2-(EX)^2=\frac{S_{\mu,\nu}^{(\alpha,\alpha+2)}(r)\,S_{\mu,\nu}^{(\alpha,\alpha)}(r)-\left[S_{\mu,\nu}^{(\alpha,\alpha+1)}\right]^2}{\left(S_{\mu,\nu}^{(\alpha,\alpha)}(r)\right)^2}.$$

Using the fact that $Var(x)$ is nonnegative, we easy get that the Turán type inequality (28) holds true. □

THEOREM 4. *The characteristic funtion of the Mathieu distribution $P_{\mu+1,\nu}^{(2,1)}(n,r)$, $r>0$ is given by*

$$f_x(t)=\frac{\sqrt{\pi}}{(2r)^{\mu-1/2}\Gamma(\mu+1)\,S_{\mu,\nu}(r)}\int_0^\infty \frac{e^{u\nu}u^{\mu+1/2}}{(e^u-e^{it})^\nu}J_{\mu-1/2}(ru)\,du \quad \left(Re(\mu)>-\frac{1}{2}\right), \tag{29}$$

where $J_{\mu-1/2}(.)$ is the Bessel function.

Proof. Using the formula:

$$\frac{2n}{(n^2+r^2)^{\mu+1}}=\frac{\sqrt{\pi}}{(2r)^{\mu-1/2}}\int_0^\infty e^{-nt}t^{\mu+1/2}J_{\mu-1/2}(rt)\,dt$$

$$\left(\Re(\mu)>-\frac{1}{2}\right)$$

we obtain

$$\begin{aligned} f_x(t)&=\sum_{n=1}^{\infty}e^{itn}\frac{2n(\nu)_n}{(n^2+r^2)^{\mu+1}\,n!}\frac{1}{S_{\mu,\nu}(r)}\\ &=\frac{\sqrt{\pi}}{(2r)^{\mu-1/2}\Gamma(\mu+1)\,S_{\mu,\nu}(r)}\int_0^\infty u^{\mu+1/2}J_{\mu-1/2}(ru)\left[\sum_{n=1}^{\infty}\frac{(\nu)_n}{n!}\left(e^{-u}e^{it}\right)^n\right]du\\ &=\frac{\sqrt{\pi}}{(2r)^{\mu-1/2}\Gamma(\mu+1)\,S_{\mu,\nu}(r)}\int_0^\infty \frac{e^{u\nu}u^{\mu+1/2}}{(e^u-e^{it})^\nu}J_{\mu-1/2}(ru)\,du. \end{aligned}$$

So, the proof of Theorem 4 is complete. □

COROLLARY 1. *Let $\nu>0$ and $\mu>1$. Then the following inequality holds*

$$S_{\mu,\nu}^{(2,1)}(r,\{n\})\,S_{\mu,\nu}^{(2,3)}(r,\{n\})\geqslant\left[S_{\mu,\nu}^{(2,2)}(r,\{n\})\right]^2. \tag{30}$$

In particular,

$$S_{2,\nu}^{(2,1)}(r,\{n\})\,S_{2,\nu}^{(2,3)}(r,\{n\})\geqslant\left[S_{2,\nu}^{(2,2)}(r,\{n\})\right]^2.$$

Proof. We consider a special case

$$P_{\mu,\nu}^{(2,1)}(n,r)=P(Y=n)=\frac{2n(\nu)_n}{(n^2+r^2)^{\mu}\,n!}\frac{1}{S_{\mu,\nu}^{(2,1)}(r)} \tag{31}$$

$$\left(r,\mu,\nu\in\mathbb{R}^+\right),\ n\in\mathbb{N}.$$

Then

$$EY=\frac{S_{\mu,\nu}^{(2,2)}(r,\{n\})}{S_{\mu,\nu}^{(2,1)}(r,\{n\})}$$

$$EY^2=\frac{S_{\mu,\nu}^{(2,3)}(r,\{n\})}{S_{\mu,\nu}^{(2,1)}(r,\{n\})}$$

Applying the elementary inequality $EY^2\geqslant(EY)^2$, we obtain

$$S_{\mu,\nu}^{(2,1)}(r,\{n\})\,S_{\mu,\nu}^{(2,3)}(r,\{n\})\geqslant\left[S_{\mu,\nu}^{(2,2)}(r,\{n\})\right]^2. \quad \square$$

THEOREM 5. *Let $\nu\geqslant1$. Then the following inequality*

$$S_{1,\nu}^{(2,1)}(r,\{n\})\,S_{2,\nu}^{(2,1)}(r,\{n\})\geqslant 2r^2S_{3,1}^{(2,1)}(r,\{n\})+\left[S_{2,1}^{(2,2)}(r,\{n\})\right]^2 \tag{32}$$

holds true.

Proof. From Corollary 1, we have

$$\left[S_{1,\nu}^{(2,1)}(r,\{n\})-r^2S_{2,\nu}^{(2,1)}(r,\{n\})\right]S_{2,\nu}^{(2,1)}(r,\{n\})\geqslant\left[S_{2,\nu}^{(2,2)}(r,\{n\})\right]^2. \tag{33}$$

On the other hand, using the fact that the function $\nu\mapsto(\nu)_n$ is increasing on $[1,\infty)$, we deduce that

$$\left[S_{2,\nu}^{(2,1)}(r,\{n\})\right]^2\geqslant\left[S_{2,1}^{(2,1)}(r,\{n\})\right]^2, \tag{34}$$

and

$$\left[S_{2,\nu}^{(2,2)}(r,\{n\})\right]^2\geqslant\left[S_{2,1}^{(2,2)}(r,\{n\})\right]^2, \tag{35}$$

In [28], Wilkins proved the following inequality

$$\left[\sum_{n=1}^{\infty}\frac{n}{(n^2+r^2)^2}\right]^2 \geqslant \sum_{n=1}^{\infty}\frac{n}{(n^2+r^2)^3}. \tag{36}$$

Combining (34) and (36) we get

$$\left[S_{2,\nu}^{(2,1)}(r,\{n\})\right]^2 \geqslant 2S_{3,1}^{(2,1)}(r,\{n\}). \tag{37}$$

In view of (37) and (35) we get the inequality (32). □

5. Functional inequalities for Mathieu series

Before we present our main results in this section, we recall some standard definitions and basic facts. A non-negative function f defined on $(0,\infty)$ is called completely monotonic if it has derivatives of all orders and

$$(-1)^n f^{(n)}(x) \geqslant 0, \;\; n \geqslant 1$$

and $x>0$ [16, 3, 14]. This inequality is known to be strict unless f is a constant. By the celebrated Bernstein theorem, a function is completely monotonic if and only if it is the Laplace transform of a non-negative measure [16, Theorem 1.4].

THEOREM 6. *Let $\alpha,\beta,\nu,\mu>0$ and $0\leqslant z\leqslant 1$. Then the function $r\mapsto S_{\mu,\nu}^{(\alpha,\beta)}(\sqrt{r},\mathbf{a},z)$ is completely monotonic and log-convex on $(0,\infty)$. In particular, for $r_1,r_2,\nu>0$, the following chain of inequalities:*

$$\begin{aligned}\left[S_{\mu,\nu}^{(\alpha,\beta)}\left(\sqrt{\frac{r_1+r_2}{2}},\mathbf{a},z\right)\right]^2 &\leqslant S_{\mu,\nu}^{(\alpha,\beta)}(\sqrt{r_1},\mathbf{a},z)S_{\mu,\nu}^{(\alpha,\beta)}(\sqrt{r_2},\mathbf{a},z)\\ &\leqslant 2\zeta_{\mu,\nu}(\alpha,\beta,z)S_{\mu,\nu}^{(\alpha,\beta)}(\sqrt{r_1+r_2},\mathbf{a},z)\end{aligned} \tag{38}$$

holds true if the following auxiliary series

$$\zeta_{\nu,\mu}(\alpha,\beta,z)=\sum_{n=1}^{\infty}\frac{(\nu)_n z^n}{n!a_n^{\alpha\mu-\beta}}$$

is convergent.

Proof. As the function $r\mapsto \frac{2a_n^{\beta}(\nu)_n z^n}{n!(a_n^{\alpha}+r)^{\mu}}$ is completely monotonic on $(0,\infty)$ for all $\alpha,\beta,\mu,\nu>0$. Using the fact that sums of completely monotonic functions are completely monotonic too, we deduce that the function $r\mapsto S_{\mu,\nu}^{(\alpha,\beta)}(\sqrt{r},\mathbf{a},z)$ is completely monotonic and log-convex on $(0,\infty)$, since every completely monotonic function is log-convex (see [27, p. 167]). Thus for all $r_1,r_2>0$, and $t\in[0,1]$ we get

$$S_{\mu,\nu}^{(\alpha,\beta)}\left(\sqrt{\frac{tr_1+(1-t)r_2}{2}},\mathbf{a},z\right) \leqslant [S_{\mu,\nu}^{(\alpha,\beta)}(\sqrt{r_1},\mathbf{a},z)]^t[S_{\mu,\nu}^{(\alpha,\beta)}(\sqrt{r_2},\mathbf{a},z)]^{1-t}. \tag{39}$$

Choosing $t = 1/2$, the above inequality reduce to the first inequality in (38). For the second inequality in (38), we observe that the function $r \mapsto S_{\mu,\nu}^{(\alpha,\beta)}(\sqrt{r}, \mathbf{a}, z)$ is decreasing on $(0,\infty)$, and thus

$$S_{\mu,\nu}^{(\alpha,\beta)}(\sqrt{r}, \mathbf{a}, z) \leqslant S_{\mu,\nu}^{(\alpha,\beta)}(0, \mathbf{a}, z) = 2\sum_{n=1}^{\infty} \frac{(\nu)_n z^n}{n! a_n^{\alpha\mu-\beta}} = 2\zeta_{\mu,\nu}(\alpha,\beta,z).$$

So, the function $r \mapsto \frac{S_{\mu,\nu}^{(\alpha,\beta)}(\sqrt{r},\mathbf{a},z)}{2\zeta_{\mu,\nu}(\alpha,\beta,z)}$ maps $(0,\infty)$ into $(0,1)$ and its completely monotonic on $(0,\infty)$. On the other hand, according to Kimberling [11], if a function f, defined on $(0,\infty)$, is continuous and completely monotonic on $(0,\infty)$ into $(0,1)$, then the $\log f$ is super-additive, that is for all $x, y > 0$, we have

$$\log f(x+y) \geqslant \log f(x) + \log f(y) \text{ or } f(x+y) \geqslant f(x) f(y).$$

Therefore we conclude the second inequality in (38). □

LEMMA 1. [1] *Let* $f, g : [a,b] \to \mathbb{R}$, *be two continuous functions which are differentiable on* (a,b). *Further, let* $g\prime(x) \neq 0$ *on* (a,b). *If* f'/g' *is increasing (or decreasing) on* (a,b), *then the functions*

$$\frac{f(x)-f(a)}{g(x)-g(a)} \text{ and } \frac{f(x)-f(b)}{g(x)-g(b)},$$

are also increasing (or decreasing) on (a,b).

COROLLARY 2. *Let* $\alpha, \beta, \mu > 0$ *and* $0 \leqslant z \leqslant 1$. *Then the following inequality*

$$2\zeta_{\mu,\nu}(\alpha,\beta,z) e^{-\mu \frac{\zeta_{\mu+1,\nu}(\alpha,\beta,z)}{2\zeta_{\mu,\nu}(\alpha,\beta,z)} r^2} \leqslant S_{\mu,\nu}^{(\alpha,\beta)}(r, \mathbf{a}, z) \tag{40}$$

holds true if the following auxiliary series

$$\zeta_{\mu,\nu}(\alpha,\beta,z) = \sum_{n=1}^{\infty} \frac{(\nu)_n z^n}{n! a_n^{\alpha\mu-\beta}}$$

is convergent.

Proof. Since the function $r \mapsto \frac{S_{\mu,\nu}^{(\alpha,\beta)}(\sqrt{r},\mathbf{a},z)}{2\zeta_{\mu,\nu}(\alpha,\beta,z)}$ is log-convex on $(0,\infty)$ for al $\alpha, \beta, \nu > 0$, we obtain that $r \mapsto \frac{(S_{\mu,\nu}^{(\alpha,\beta)}(\sqrt{r},\mathbf{a},z))'}{S_{\mu,\nu}^{(\alpha,\beta)}(\sqrt{r},\mathbf{a},z)}$ is increasing on $(0,\infty)$. Let

$$F(r) = \log\left(\frac{S_{\mu,\nu}^{(\alpha,\beta)}(\sqrt{r}, \mathbf{a}, z)}{2\zeta_{\mu,\nu}(\alpha,\beta,z)}\right), \text{ and } G(r) = r.$$

Then the function

$$H(r) = \frac{F(r)}{G(r)} = \frac{F(r) - F(0)}{G(r) - G(0)},$$

is increasing on $(0,\infty)$, by means of Lemma 1. So, by using the l'Hospital's rule we get

$$\log\left(\frac{S_{\mu,\nu}^{(\alpha,\beta)}(\sqrt{r},\mathbf{a},z)}{2\zeta_{\mu,\nu}(\alpha,\beta,z)}\right) \geqslant rF'(0) = -r\mu\frac{\zeta_{\mu+1,\nu}(\alpha,\beta,z)}{2\zeta_{\mu,\nu}(\alpha,\beta,z)}. \tag{41}$$

Therefore we conclude the asserted inequality (40). □

REMARKS.

1. Taking $z=\beta=\nu=1$, $\mathbf{a}=(n)_{n\geqslant 1}$ and $\alpha=2$ in (40), we obtain the following inequality

$$2\zeta(2\mu-1)\exp\left\{-\mu\frac{\zeta(2\mu+1)}{2\mu-1}r^2\right\} \leqslant S_\mu(r), \quad r>0, \tag{42}$$

where $\zeta(.)$ denotes the Riemann zeta function defined by

$$\zeta(p) = \sum_{n=1}^{\infty}\frac{1}{n^p}.$$

2. We note that if we choose $\mu=2$ and $\mu=3/2$ in (42) we obtain the following inequalities

$$2\zeta(3)\exp\left\{-2\frac{\zeta(5)}{\zeta(3)}r^2\right\} \leqslant S(r), \tag{43}$$

and

$$\frac{\pi^2}{3}\exp\left\{-\frac{\pi^2 r^2}{10}\right\} \leqslant S_{3/2}(r), \tag{44}$$

holds true for all $r>0$.

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