

ALMOST EVERYWHERE STRONG SUMMABILITY OF CUBIC PARTIAL SUMS OF D-DIMENSIONAL WALSH-FOURIER SERIES

USHANGI GOGINAVA

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Abstract. In this paper we study the a. e. strong summability of the cubic partial sums of the d -dimensional Walsh-Fourier series of the functions belonging to $L(\log^+ L)^{d-1}$.

1. Introduction

We shall denote the set of all non-negative integers by \mathbb{N} , the set of all integers by \mathbb{Z} and the set of dyadic rational numbers in the unit interval $\mathbb{I} := [0, 1)$ by \mathbb{Q} . In particular, each element of \mathbb{Q} has the form $\frac{p}{2^n}$ for some $p, n \in \mathbb{N}$, $0 \leq p \leq 2^n$. Denote $I_N := [0, 2^{-N})$, $I_N(x) := I_N \dot{+} x$, where by $\dot{+}$ we denote dyadic addition (see [37], [14]).

Let $r_0(x)$ be the function defined by

$$r_0(x) = \begin{cases} 1, & \text{if } x \in [0, 1/2) \\ -1, & \text{if } x \in [1/2, 1) \end{cases}, \quad r_0(x+1) = r_0(x).$$

The Rademacher system is defined by

$$r_n(x) = r_0(2^n x), \quad n \geq 1.$$

Let w_0, w_1, \dots represent the Walsh functions, i.e. $w_0(x) = 1$ and if $k = 2^{n_1} + \dots + 2^{n_s}$ is a positive integer with $n_1 > n_2 > \dots > n_s$ then

$$w_k(x) = r_{n_1}(x) \cdots r_{n_s}(x).$$

Given $x \in \mathbb{I}$, the expansion

$$x = \sum_{k=0}^{\infty} x_k 2^{-(k+1)}, \quad (1)$$

where each $x_k = 0$ or 1 , will be called a dyadic expansion of x . If $x \in \mathbb{I} \setminus \mathbb{Q}$, then (1) is uniquely determined. For the dyadic expansion $x \in \mathbb{Q}$ we choose the one for which $\lim_{k \rightarrow \infty} x_k = 0$.

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The dyadic addition of $x, y \in \mathbb{I}$ in terms of the dyadic expansion of x and y is defined by

$$\rho(x, y) := x \dot{+} y = \sum_{k=0}^{\infty} |x_k - y_k| 2^{-(k+1)}.$$

If $f \in L^1(\mathbb{I})$, then

$$\hat{f}(n) = \int_{\mathbb{I}} f(x) w_n(x) dx$$

is the n -th Fourier coefficient of f .

The partial sums of Fourier series with respect to the Walsh system are defined by

$$S_M(x; f) = \sum_{m=0}^{M-1} \hat{f}(m) w_m(x).$$

For $n \in \mathbb{N}$ let us introduce the projections

$$E_n(x; f) := S_{2^n}(x; f) = 2^n \int_{I_n(x)} f(s) ds \quad (f \in L_1(\mathbb{I}), x \in \mathbb{I}),$$

$$E^*(x; f) := \sup_{n \in \mathbb{N}} E_n(x; |f|).$$

Let $\mathbb{I}^d = [0, 1]^d$ denote the cube in the d -dimensional Euclidean space R^d . The elements of R^d will be denoted by $\vec{x} = (x_1, \dots, x_d)$. For any $\vec{x} = (x_1, \dots, x_d)$ and $\vec{y} = (y_1, \dots, y_d)$ the vector $(x_1 \dot{+} y_1, \dots, x_d \dot{+} y_d)$ of the space R^d is denoted by $\vec{x} \dot{+} \vec{y}$.

Let $M = \{1, 2, \dots, d\}$, $B = \{l_1, \dots, l_r\}$, $l_i < l_{i+1}$, $i = 1, \dots, r - 1$, $B \subset M$, $B' = M \setminus B$, $|B| = \text{card}(B)$.

For any $x \in R^d$ and $B \subset M$, the symbol \vec{x}_B will stand for the point of R^d whose coordinates with indices from B coincide with the corresponding coordinates of \vec{x} , and those with indices from B' are zero. Besides, $\vec{x}_M := \vec{x}$.

If $\vec{x} \in \mathbb{I}^d$ and $I_k(x_i)$, $i = 1, 2, \dots, d$, are dyadic intervals containing x_i , then the set

$$I_k(\vec{x}) = I_k(x_1) \times I_k(x_2) \times \dots \times I_k(x_d)$$

is a dyadic d -dimensional cube.

We denote by $L(\log^+ L)^\alpha(\mathbb{I}^d)$ the class of measurable functions f , with

$$\int_{\mathbb{I}^d} |f| (\log^+ |f|)^\alpha < \infty, \quad \alpha > 0,$$

where $\log^+ u := \mathbb{I}_{(1, \infty)} \log u$ and \mathbb{I}_E is character function of the set E .

The rectangular partial sums of d -dimensional Walsh-Fourier series are defined as follows:

$$S_{m_1, \dots, m_d}(x_1, \dots, x_d; f) = \sum_{j_1=0}^{m_1-1} \dots \sum_{j_d=0}^{m_d-1} \hat{f}(j_1, \dots, j_d) \prod_{i=1}^d w_{j_i}(x_i),$$

where the number

$$\widehat{f}(j_1, \dots, j_d) = \int_{\mathbb{I}^d} f(x_1, \dots, x_d) \prod_{i=1}^d w_{j_i}(x_i) dx_1 \cdots dx_d$$

is said to be the (j_1, \dots, j_d) th Walsh-Fourier coefficient of f .

Set

$$E_n^{(i)}(x_1, \dots, x_d; f) := S_{2^n}^{(i)}(x_1, \dots, x_d; f) \\ := 2^n \int_{I_n(x_i)} f(x_1, \dots, x_{i-1}, s, x_{i+1}, \dots, x_d) ds, \quad i = 1, \dots, d,$$

$$E_n(x_1, \dots, x_d; f) := S_{2^n, \dots, 2^n}(x_1, \dots, x_d; f) \\ = 2^{nd} \int_{I_n(x_1) \times \dots \times I_n(x_d)} f(s_1, \dots, s_d) ds_1 \cdots ds_d,$$

$$E_n^{(B)}(x_1, \dots, x_d; f) := S_{2^n, \dots, 2^n}^{(B)}(x_1, \dots, x_d; f) \\ := 2^{n|B|} \int_{I_n(x_{1_r}) \times \dots \times I_n(x_{r_r})} f(\vec{s}_B \dot{+} \vec{x}_{B'}) ds_{1_r} \cdots ds_{r_r}, \quad B \subset M, B \neq \emptyset.$$

The dyadic maximal functions is given by

$$E_* (x_1, \dots, x_d; f) := \sup_{n \in \mathbb{N}} E_n(x_1, \dots, x_d; |f|),$$

$$E_*^{(B)}(x_1, \dots, x_d; f) := \sup_{n \in \mathbb{N}} E_n^{(B)}(x_1, \dots, x_d; |f|).$$

We denote by $L_0(\mathbb{I}^d)$ the Lebesgue space of functions that are measurable and finite almost everywhere on \mathbb{I}^d , $|A|$ is the Lebesgue measure of the set $A \subset \mathbb{I}^d$.

Let $B \subset M, B \neq \emptyset$ and $f \in L(\log^+ L)^{|B|}(\mathbb{I}^d)$. Then it is well-known that (see [43])

$$\int_{\mathbb{I}^d} E_*^{(B)}(f)(x_1, \dots, x_d; f) dx_1 \cdots dx_d \tag{2} \\ \leq \int_{\mathbb{I}^d} |f(x_1, \dots, x_d)| (\log^+ |f(x_1, \dots, x_d)|)^{|B|} dx_1 \cdots dx_d + c_d$$

and for $B \subset M, B \neq \emptyset$ and $f \in L(\log^+ L)^{|B|-1}(\mathbb{I}^d)$,

$$\left| \left\{ (x_1, \dots, x_d) \in \mathbb{I}^d : E_*^{(B)}(f)(x_1, \dots, x_d; f) > c\lambda \right\} \right| \tag{3} \\ \leq \frac{c}{\lambda} \left(\int_{\mathbb{I}^d} |f((x_1, \dots, x_d))| (\log^+ |f((x_1, \dots, x_d))|)^{|B|-1} dx_1 \cdots dx_d + c_d \right)$$

Denote by $S_n^T(x, f)$ the partial sums of the trigonometric Fourier series of f and let

$$\sigma_n^T(x, f) = \frac{1}{n+1} \sum_{k=0}^n S_k^T(x, f)$$

be the $(C, 1)$ means. Fejér [3] proved that $\sigma_n^T(f)$ converges to f uniformly for any 2π -periodic continuous function. Lebesgue in [23] established almost everywhere convergence of $(C, 1)$ means if $f \in L_1(\mathbb{T})$, $\mathbb{T} := [-\pi, \pi)$. The strong summability problem, i.e. the convergence of the strong means

$$\frac{1}{n+1} \sum_{k=0}^n |S_k^T(x, f) - f(x)|^p, \quad x \in \mathbb{T}, \quad p > 0, \quad (4)$$

was first considered by Hardy and Littlewood in [20]. They showed that for any $f \in L_r(\mathbb{T})$ ($1 < r < \infty$) the strong means tend to 0 a.e., if $n \rightarrow \infty$. The Fourier series of $f \in L_1(\mathbb{T})$ is said to be (H, p) -summable at $x \in T$, if the values (4) converge to 0 as $n \rightarrow \infty$. The (H, p) -summability problem in $L_1(\mathbb{T})$ has been investigated by Marcinkiewicz [28] for $p = 2$, and later by Zygmund [48] for the general case $1 \leq p < \infty$. Almost everywhere (H, p) -summability of Walsh-Fourier series with $p > 0$ was proved by F. Schipp in [35].

Almost everywhere Φ -summability of Fourier series with respect to trigonometric and Walsh systems with condition was proved by Oskolkov in [29], Rodin [32], Schipp [33], Karagulyan [21], Gát, Goginava, Karagulyan [8].

For quadratic partial sums of two-dimensional trigonometric Fourier series Marcinkiewicz [28] has proved, that if $f \in L \log^+ L(\mathbb{T}^2)$, $\mathbb{T} := [-\pi, \pi)^2$, then

$$\lim_{n \rightarrow \infty} \frac{1}{n+1} \sum_{k=0}^n (S_{kk}^T(x, y, f) - f(x, y)) = 0$$

for a. e. $(x, y) \in \mathbb{T}^2$. Zhizhiashvili [47] improved this result showing that class $L \log^+ L(\mathbb{T}^2)$ can be replaced by $L_1(\mathbb{T}^2)$ (for d-dimensional trigonometric Fourier series see Dyachenko [2], for two-dimensional Walsh system see Weisz [42], and in case of d-dimensional Walsh-Fourier series see Goginava [9]). Almost Everywhere strong summability of quadratical partial sums of two-dimensional trigonometric Fourier series was proved by Goginava, Gogoladze, Karagulyan in [13] and in case of two-dimensional Walsh system by Gát, Goginava and karagulyan in [7]. In this paper we study the a. e. strong summability of the cubic partial sums of the d-dimensional Walsh-Fourier series.

The results on strong summation and approximation of trigonometric Fourier series have been extended for several other orthogonal systems, see Schipp [34, 35, 36], Fridli and Schipp [4, 5], Leindler [24, 25, 26, 27], Totik [39, 40, 41], Rodin [30, 31, 32], Weisz [45, 44, 46], Gabisonia [6], Goginava, Gogoladze [12, 11], Gogoladze [16, 17], Glukhov [18], Goginava [9, 10].

2. Main results

THEOREM 1. *Let $f \in L(\log^+ L)^{d-1}(\mathbb{I}^d)$, $p > 0$ and $d \geq 1$. Then*

$$\left| \left\{ (x_1, \dots, x_d) \in \mathbb{I}^d : \sup_{n \in \mathbb{N}} \left(\frac{1}{2^n} \sum_{m=0}^{2^n-1} |S_{m, \dots, m}(x_1, \dots, x_d; f)|^p \right)^{1/p} > \lambda \right\} \right| \leq \frac{c(d, p)}{\lambda} \left(1 + \int_{\mathbb{I}^d} |f(x_1, \dots, x_d)| \left((\log^+ |f(x_1, \dots, x_d)|) \right)^{d-1} dx_1 \cdots dx_d \right)$$

By making use the well-known density argument due to Marcinkiewicz and Zygmund we can show that Theorem 2 follows from Theorem 1.

THEOREM 2. *Let $f \in L(\log^+ L)^{d-1}(\mathbb{I}^d)$ and $d \geq 1$. Then for any $p > 0$*

$$\lim_{n \rightarrow \infty} \frac{1}{2^n} \sum_{m=0}^{2^n-1} |S_{m, \dots, m}(x_1, \dots, x_d; f) - f(x_1, \dots, x_d)|^p = 0$$

a.e. on \mathbb{I}^d .

We note that from the theorem of Getsadze [15] and Konyagin [22] it follows that the class $L(\log L)^{d-1}(\mathbb{I}^d)$ in the last theorem is necessary in the context of strong summability question. That is, it is not possible to give a larger convergence space than $L(\log L)^{d-1}(\mathbb{I}^d)$.

We also note that in the case of two-dimensional trigonometric system Sjölin [38] (for d-dimensional case see Antonov [1]) proved that for every $r > 1$ and d- variable function $f \in L_r(\mathbb{T}^d)$ the almost everywhere convergence $S_{n, \dots, n} f \rightarrow f$ ($n \rightarrow \infty$) holds. Since this issue with respect to the Walsh system is still open, then in this point of view Theorem 2 may seem more interesting.

3. Proofs of main results

In [33] Schipp introduced the following operator

$$V_n^{(p)}(x; f) := \left(\frac{1}{2^{n(q-1)}} \int_{\mathbb{I}} \left| \sum_{k=0}^{n-1} 2^k \mathbb{I}_{I_k}(t) S_{2^n}(x + t + e_k; f) \right|^q dt \right)^{1/q},$$

where

$$p > 1, \quad 1/p + 1/q = 1, \quad e_j := 2^{-j-1}.$$

In order to prove Theorem 2 we need the following lemma (for $p = 2$, see [33], for $p > 2$ see [7]).

LEMMA 1. Let $p \geq 2$. Then

$$\sup_{\lambda > 0} \lambda \left| \left\{ x \in \mathbb{I} : V_*^{(p)}(x; |f|) > \lambda \right\} \right| \leq c(p) \|f\|_1.$$

Proof of Theorem 1. It is easy to show that

$$\begin{aligned} & \left(\sum_{m=0}^{2^n-1} |S_{m,\dots,m}(x_1, \dots, x_d; f)|^p \right)^{1/p} \\ &= \left(\sum_{m=0}^{2^n-1} |S_{m,\dots,m}(x_1, \dots, x_d; S_{2^n, \dots, 2^n}(f))|^p \right)^{1/p} \\ &= \sup_{\{\alpha_{mn}(x_1, \dots, x_d)\}} \left| \sum_{m=0}^{2^n-1} \alpha_{mn}(x_1, \dots, x_d) S_{m,\dots,m}(x_1, \dots, x_d; S_{2^n, \dots, 2^n}(f)) \right| \end{aligned} \tag{5}$$

by taking the supremum over all $\{\alpha_{mn}(x_1, \dots, x_d)\}$ for which

$$\left(\sum_{m=0}^{2^n-1} |\alpha_{mn}(x_1, \dots, x_d)|^q \right)^{1/q} \leq 1, \quad 1/p + 1/q = 1. \tag{6}$$

Let

$$\varepsilon_{ji} := \begin{cases} -1, & \text{if } j = 0, 1, \dots, i-1 \\ 1, & \text{if } j = i \end{cases}.$$

In ([33]) Schipp proved that

$$D_m(t) = \sum_{k=0}^{n-1} \mathbb{I}_{I_k \setminus J_{k+1}}(t) \sum_{j=0}^k \varepsilon_{kj} 2^{j-1} w_m(t + e_j) - \frac{1}{2} w_m(t) + \left(m + \frac{1}{2}\right) \mathbb{I}_{I_n}(t), \quad m < 2^n. \tag{7}$$

Then we can write

$$\begin{aligned} & J_n^{(p)}(x_1, \dots, x_d) \\ &:= \left| \sum_{m=0}^{2^n-1} \alpha_{mn}(x_1, \dots, x_d) S_{m,\dots,m}(x_1, \dots, x_d; S_{2^n, \dots, 2^n}(f)) \right| \\ &= \left| \int_{\mathbb{I}^d} S_{2^n, \dots, 2^n}(x_1 \dot{+} t_1, \dots, x_d \dot{+} t_d; f) \sum_{m=0}^{2^n-1} \alpha_{mn}(x_1, \dots, x_d) \prod_{j=1}^d D_m(t_j) dt_1 \cdots dt_d \right| \\ &\leq \sum_{B \subset M, B \neq \emptyset} \left| \int_{\mathbb{I}^d} S_{2^n, \dots, 2^n}(x_1 \dot{+} t_1, \dots, x_d \dot{+} t_d; f) \sum_{m=0}^{2^n-1} \alpha_{mn}(x_1, \dots, x_d) \right. \\ &\quad \times \prod_{i \in B} \sum_{k=0}^{n-1} \mathbb{I}_{I_k \setminus J_{k+1}}(t_i) \sum_{j=0}^k \varepsilon_{kj} 2^j w_m(t + e_j) \prod_{s \in B'} \frac{(-1)^{|B'|}}{2} w_m(t_s) dt_1 \cdots dt_d \left. \right| \\ &\quad + \sum_{B \subset M, B \neq M} \left| \int_{\mathbb{I}^d} S_{2^n, \dots, 2^n}(x_1 \dot{+} t_1, \dots, x_d \dot{+} t_d; f) \sum_{m=0}^{2^n-1} \alpha_{mn}(x_1, \dots, x_d) \right. \end{aligned} \tag{8}$$

$$\begin{aligned}
 & \times \left| \prod_{i \in B} \sum_{k=0}^{n-1} \mathbb{I}_{I_k \setminus I_{k+1}}(t_i) \sum_{j=0}^k \varepsilon_{kj} 2^j w_m(t + e_j) \prod_{s \in B'} \left(m + \frac{1}{2}\right) \mathbb{I}_{I_n}(t_s) dt_1 \cdots dt_d \right| \\
 & + \sum_{B \subset M, B \neq \emptyset} \left| \int_{\mathbb{I}^d} S_{2^n, \dots, 2^n}(x_1 + t_1, \dots, x_d + t_d; f) \sum_{m=0}^{2^n-1} \alpha_{mn}(x_1, \dots, x_d) \right. \\
 & \times \left. \prod_{i \in B} \frac{(-1)^{|B|}}{2} w_m(t_i) \prod_{s \in B'} \left(m + \frac{1}{2}\right) \mathbb{I}_{I_n}(t_s) dt_1 \cdots dt_d \right| \\
 = & \sum_{B \subset M, B \neq \emptyset} J_{nB1}^{(p)}(x_1, \dots, x_d) + \sum_{B \subset M, B \neq M} J_{nB2}^{(p)}(x_1, \dots, x_d) + \sum_{B \subset M, B \neq \emptyset} J_{nB3}^{(p)}(x_1, \dots, x_d).
 \end{aligned}$$

Let $B = M$. Since

$$\sum_{k=j}^{n-1} \mathbb{I}_{I_k \setminus I_{k+1}}(t) \leq \mathbb{I}_{I_j}(t)$$

we can write

$$\begin{aligned}
 & J_{nM1}^{(p)}(x_1, \dots, x_d) \tag{9} \\
 = & \left| \int_{\mathbb{I}^d} S_{2^n, \dots, 2^n}(x_1 + t_1, \dots, x_d + t_d; f) \sum_{m=0}^{2^n-1} \alpha_{mn}(x_1, \dots, x_d) \right. \\
 & \times \left. \prod_{i=1}^d \sum_{k=0}^{n-1} \mathbb{I}_{I_k \setminus I_{k+1}}(t_i) \sum_{j=0}^k \varepsilon_{kj} 2^j w_m(t_i + e_j) dt_1 \cdots dt_d \right| \\
 \leq & \sum_{k_1=0}^{n-1} \cdots \sum_{k_d=0}^{n-1} \sum_{j_1=0}^{k_1} \cdots \sum_{j_d=0}^{k_d} 2^{j_1 + \cdots + j_d} \int_{I_{k_1} \setminus I_{k_1+1} \times \cdots \times I_{k_d} \setminus I_{k_d+1}} S_{2^n, \dots, 2^n}(x_1 + t_1, \dots, x_d + t_d; |f|) \\
 & \times \left| \sum_{m=0}^{2^n-1} \alpha_{mn}(x_1, \dots, x_d) w_m(t_1 + \cdots + t_d + e_{j_1} + \cdots + e_{j_d}) \right| dt_1 \cdots dt_d \\
 \leq & \sum_{j_1=0}^{n-1} \cdots \sum_{j_d=0}^{n-1} 2^{j_1 + \cdots + j_d} \int_{I_{j_1} \times \cdots \times I_{j_d}} S_{2^n, \dots, 2^n}(x_1 + t_1, \dots, x_d + t_d; |f|) \\
 & \times \left| \sum_{m=0}^{2^n-1} \alpha_{mn}(x_1, \dots, x_d) w_m(t_1 + \cdots + t_d + e_{j_1} + \cdots + e_{j_d}) \right| dt_1 \cdots dt_d \\
 = & \sum_{\sigma} \sum_{0 \leq j_{\sigma(1)} \leq \cdots \leq j_{\sigma(d)} < n} 2^{j_1 + \cdots + j_d} \int_{I_{j_1} \times \cdots \times I_{j_d}} S_{2^n, \dots, 2^n}(x_1 + t_1, \dots, x_d + t_d; |f|) \\
 & \times \left| \sum_{m=0}^{2^n-1} \alpha_{mn}(x_1, \dots, x_d) w_m(t_1 + \cdots + t_d + e_{j_1} + \cdots + e_{j_d}) \right| dt_1 \cdots dt_d \\
 := & \sum_{\sigma} J_{nM1\sigma}^{(p)}(x_1, \dots, x_d),
 \end{aligned}$$

where the sum is taken over all rearrangements $\sigma := \{\sigma(k)\}_{k=1}^d$ of the set $\{1, 2, \dots, d\}$.

Let $\sigma(k) = d + 1 - k$ for some $k = 1, \dots, d$. Then we have

$$\begin{aligned}
 & J_{nM1\sigma}^{(p)}(x_1, \dots, x_d) \tag{10} \\
 &= \sum_{0 \leq j_d \leq \dots \leq j_1 < n} 2^{j_1 + \dots + j_d} \int_{I_{j_1} \times \dots \times I_{j_d}} S_{2^n, \dots, 2^n}(x_1 \dot{+} t_1, \dots, x_d \dot{+} t_d; |f|) \\
 &\quad \times \left| \sum_{m=0}^{2^n-1} \alpha_{mn}(x_1, \dots, x_d) w_m(t_1 \dot{+} \dots \dot{+} t_d \dot{+} e_{j_1} \dot{+} \dots \dot{+} e_{j_d}) \right| dt_1 \dots dt_d.
 \end{aligned}$$

Denote

$$\begin{aligned}
 u_{d-1} &:= t_{d-1} \dot{+} t_d \dot{+} e_{j_d}; \\
 u_{d-2} &:= t_{d-2} \dot{+} u_{d-1}; \\
 u_{d-3} &:= t_{d-3} \dot{+} u_{d-2}; \\
 &\quad \vdots \\
 u_1 &:= t_1 \dot{+} u_2.
 \end{aligned}$$

Then it is easy to see that

$$u_k \in I_{j_d}, \quad k = 1, 2, \dots, d - 1.$$

Hence, we can write

$$\begin{aligned}
 & J_{nM1\sigma}^{(p)}(x_1, \dots, x_d) \tag{11} \\
 &= \sum_{0 \leq j_d \leq \dots \leq j_1 < n} 2^{j_1 + \dots + j_d} \\
 &\quad \times \int_{I_{j_1} \times \dots \times I_{j_d}} S_{2^n, \dots, 2^n}(x_1 \dot{+} t_1, \dots, x_{d-1} \dot{+} t_{d-1}, x_d \dot{+} u_{d-1} \dot{+} t_{d-1} \dot{+} e_{j_d}; |f|) \\
 &\quad \times \left| \sum_{m=0}^{2^n-1} \alpha_{mn}(x_1, \dots, x_d) \right. \\
 &\quad \left. \times w_m(t_1 \dot{+} \dots \dot{+} t_{d-2} \dot{+} u_{d-1} \dot{+} e_{j_1} \dot{+} \dots \dot{+} e_{j_{d-1}}) \right| dt_1 \dots dt_{d-1} du_{d-1} \\
 &= \sum_{0 \leq j_d \leq \dots \leq j_1 < n} 2^{j_1 + \dots + j_d} \\
 &\quad \times \int_{I_{j_1} \times \dots \times I_{j_d}} S_{2^n, \dots, 2^n}(x_1 \dot{+} t_1, \dots, x_{d-1} \dot{+} t_{d-1}, x_d \dot{+} u_{d-2} \dot{+} t_{d-1} \dot{+} t_{d-2} \dot{+} e_{j_d}; |f|) \\
 &\quad \times \left| \sum_{m=0}^{2^n-1} \alpha_{mn}(x_1, \dots, x_d) \right. \\
 &\quad \left. \times w_m(t_1 \dot{+} \dots \dot{+} t_{d-3} \dot{+} u_{d-2} \dot{+} e_{j_1} \dot{+} \dots \dot{+} e_{j_{d-1}}) \right| dt_1 \dots dt_{d-1} du_{d-2}
 \end{aligned}$$

$$\begin{aligned}
 &= \dots = \\
 &= \sum_{0 \leq j_d \leq \dots \leq j_1 < n} 2^{j_1 + \dots + j_d} \\
 &\quad \times \int_{I_{j_1} \times \dots \times I_{j_d}} S_{2^n, \dots, 2^n} (x_1 \dot{+} t_1, \dots, x_{d-1} \dot{+} t_{d-1}, x_d \dot{+} t_1 \dot{+} \dots \dot{+} t_{d-1} \dot{+} u_1 \dot{+} e_{j_d}; |f|) \\
 &\quad \times \left| \sum_{m=0}^{2^n-1} \alpha_{mn} (x_1, \dots, x_d) w_m (u_1 \dot{+} e_{j_1} \dot{+} \dots \dot{+} e_{j_{d-1}}) \right| dt_1 \dots dt_{d-1} du_1.
 \end{aligned}$$

Set

$$F(x_1, \dots, x_d) := f(x_1, \dots, x_{d-1}, x_1 \dot{+} \dots \dot{+} x_d)$$

and

$$\begin{aligned}
 &A_{j_1, \dots, j_{d-1}}(x_1, \dots, x_d; f) \\
 &:= 2^{j_1 + \dots + j_{d-1}} \int_{I_{j_1} \times \dots \times I_{j_{d-1}}} |f(x_1 \dot{+} t_1, \dots, x_{d-1} \dot{+} t_{d-1}, x_d \dot{+} t_1 \dot{+} \dots \dot{+} t_{d-1})| dt_1 \dots dt_{d-1}.
 \end{aligned}$$

It is evident that

$$\begin{aligned}
 &A_{j_1, \dots, j_{d-1}}(x_1, \dots, x_{d-1}, x_1 \dot{+} \dots \dot{+} x_d; f) \tag{12} \\
 &:= 2^{j_1 + \dots + j_{d-1}} \\
 &\quad \times \int_{I_{j_1} \times \dots \times I_{j_{d-1}}} |f(x_1 \dot{+} t_1, \dots, x_{d-1} \dot{+} t_{d-1}, x_1 \dot{+} \dots \dot{+} x_d \dot{+} t_1 \dot{+} \dots \dot{+} t_{d-1})| dt_1 \dots dt_{d-1} \\
 &= 2^{j_1 + \dots + j_{d-1}} \int_{I_{j_1} \times \dots \times I_{j_{d-1}}} F(x_1 \dot{+} t_1, \dots, x_{d-1} \dot{+} t_{d-1}, x_d) dt_1 \dots dt_{d-1} \\
 &\leq E_*^{(1)} \left(E_*^{(2)} \left(\dots \left(E_*^{(d-1)}(x_1, \dots, x_d; F) \right) \dots \right) \right).
 \end{aligned}$$

Denote

$$A(x_1, \dots, x_d; f) := \sup_{j_1, \dots, j_{d-1} \in \mathbb{N}} A_{j_1, \dots, j_{d-1}}(x_1, \dots, x_d; f).$$

Then from (2) and (12) for $f \in L(\log^+ L)^{d-1}(\mathbb{I}^d)$ we obtain

$$\begin{aligned}
 &\int_{\mathbb{I}^d} A(x_1, \dots, x_d; f) dx_1 \dots dx_d \tag{13} \\
 &= \int_{\mathbb{I}^d} A(x_1, \dots, x_{d-1}, x_1 \dot{+} \dots \dot{+} x_d; f) dx_1 \dots dx_d \\
 &\leq \int_{\mathbb{I}^d} E_*^{(1)} \left(E_*^{(2)} \left(\dots \left(E_*^{(d-1)}(x_1, \dots, x_d; F) \right) \dots \right) \right) dx_1 \dots dx_d
 \end{aligned}$$

$$\begin{aligned} &\leq c_1(d) + c_2(d) \int_{\mathbb{I}^d} |F(x_1, \dots, x_d)| (\log^+ |F(x_1, \dots, x_d)|)^{d-1} dx_1 \cdots dx_d \\ &= c_1(d) + c_2(d) \int_{\mathbb{I}^d} |f(x_1, \dots, x_d)| (\log^+ |f(x_1, \dots, x_d)|)^{d-1} dx_1 \cdots dx_d. \end{aligned}$$

From (11) we can write

$$\begin{aligned} &J_{nM1\sigma}^{(p)}(x_1, \dots, x_d) \tag{14} \\ &= \sum_{0 \leq j_d \leq \dots \leq j_1 < n} 2^{j_d} \int_{I_{j_d}} 2^{j_1 + \dots + j_{d-1}} \left(\int_{I_{j_1} \times \dots \times I_{j_{d-1}}} S_{2^{n-d}, \dots, 2^n}(x_1 + t_1, \dots, x_{d-1} + t_{d-1}, \right. \\ &\quad \left. x_d + t_1 + \dots + t_{d-1} + u + e_{j_d}; |f|) dt_1 \cdots dt_{d-1} \right) \\ &\quad \times \left| \sum_{m=0}^{2^n-1} \alpha_{mn}(x_1, \dots, x_d) w_m(u + e_{j_1} + \dots + e_{j_{d-1}}) \right| du \\ &= \sum_{0 \leq j_d \leq \dots \leq j_1 < n} 2^{j_d} \left(\int_{I_{j_d}} 2^{j_1 + \dots + j_{d-1}} \left(\int_{I_{j_1} \times \dots \times I_{j_{d-1}}} 2^{nd} \int_{I_n \times \dots \times I_n} |f(x_1 + t_1 + s_1, \dots, \right. \right. \\ &\quad \left. \left. x_{d-1} + t_{d-1} + s_{d-1}, x_d + t_1 + \dots + t_{d-1} + u + e_{j_d} + s_d) ds_1 \cdots ds_d \right) dt_1 \cdots dt_{d-1} \right) \\ &\quad \times \left| \sum_{m=0}^{2^n-1} \alpha_{mn}(x_1, \dots, x_d) w_m(u + e_{j_1} + \dots + e_{j_{d-1}}) \right| du \\ &\leq \sum_{0 \leq j_d \leq \dots \leq j_1 < n} 2^{j_d} \int_{I_{j_d}} 2^{nd} \left(\int_{I_n \times \dots \times I_n} (2^{j_1 + \dots + j_{d-1}} \int_{I_{j_1} \times \dots \times I_{j_{d-1}}} |f(x_1 + t_1 + s_1, \dots, \right. \\ &\quad \left. x_{d-1} + t_{d-1} + s_{d-1}, x_d + t_1 + \dots + t_{d-1} + u + e_{j_d} + s_d) dt_1 \cdots dt_{d-1}) ds_1 \cdots ds_d \right) \\ &\quad \times \left| \sum_{m=0}^{2^n-1} \alpha_{mn}(x_1, \dots, x_d) w_m(u + e_{j_1} + \dots + e_{j_{d-1}}) \right| du \\ &= \sum_{0 \leq j_d \leq \dots \leq j_1 < n} 2^{j_d} \int_{I_{j_d}} 2^n \left(\int_{I_n} (2^{j_1 + \dots + j_{d-1}} \int_{I_{j_1} \times \dots \times I_{j_{d-1}}} |f(x_1 + t_1, \dots, x_{d-1} + t_{d-1}, \right. \\ &\quad \left. x_d + t_1 + \dots + t_{d-1} + u + e_{j_d} + s_d) dt_1 \cdots dt_{d-1}) ds_d \right) \\ &\quad \times \left| \sum_{m=0}^{2^n-1} \alpha_{mn}(x_1, \dots, x_d) w_m(u + e_{j_1} + \dots + e_{j_{d-1}}) \right| du \end{aligned}$$

$$\begin{aligned}
 &\leq \sum_{0 \leq j_d \leq \dots \leq j_1 < n} 2^{j_d} \int_{I_{j_d}} 2^n \left(\int_{I_n} A(x_1, \dots, x_{d-1}, x_d + u + e_{j_d} + s_d) ds_d \right) \\
 &\quad \times \left| \sum_{m=0}^{2^n-1} \alpha_{mn}(x_1, \dots, x_d) w_m(u + e_{j_1} + \dots + e_{j_{d-1}}) \right| du \\
 &= \sum_{0 \leq j_d \leq \dots \leq j_1 < n} 2^{j_d} \int_{I_{j_d}} S_{2^n}^{(d)}(x_1, \dots, x_{d-1}, x_d + u + e_{j_d}; A) \\
 &\quad \times \left| \sum_{m=0}^{2^n-1} \alpha_{mn}(x_1, \dots, x_d) w_m(u + e_{j_1} + \dots + e_{j_{d-1}}) \right| du \\
 &\leq \sum_{j_{d-1} \leq \dots \leq j_1 < n} \int_{\mathbb{I}} \sum_{j_d \leq j_{d-1}} 2^{j_d} \mathbb{I}_{I_{j_d}}(u) S_{2^n}^{(d)}(x_1, \dots, x_{d-1}, x_d + u + e_{j_d}; A) \\
 &\quad \times \left| \sum_{m=0}^{2^n-1} \alpha_{mn}(x_1, \dots, x_d) w_m(u + e_{j_1} + \dots + e_{j_{d-1}}) \right| du \\
 &\leq \sum_{j_{d-1} \leq \dots \leq j_1 < n} \left(\int_{\mathbb{I}} \left(\sum_{j_d \leq j_{d-1}} 2^{j_d} \mathbb{I}_{I_{j_d}}(u) S_{2^n}^{(d)}(x_1, \dots, x_{d-1}, x_d + u + e_{j_d}; A) \right)^q du \right)^{1/q} \\
 &\quad \times \left(\int_{\mathbb{I}} \left| \sum_{m=0}^{2^n-1} \alpha_{mn}(x_1, \dots, x_d) w_m(u + e_{j_1} + \dots + e_{j_{d-1}}) \right|^p du \right)^{1/p}.
 \end{aligned}$$

First use Hölder’s inequality and Hausdorff-Young inequality from (6) we have ($p \geq 2, 1/p + 1/q = 1$)

$$\begin{aligned}
 &\left(\int_{\mathbb{I}} \left| \sum_{m=0}^{2^n-1} \alpha_{mn}(x_1, \dots, x_d) w_m(u + e_{j_1} + \dots + e_{j_{d-1}}) \right|^p du \right)^{1/p} \\
 &= \sup_{\|h\|_q \leq 1} \left| \int_{\mathbb{I}} \sum_{m=0}^{2^n-1} \alpha_{mn}(x_1, \dots, x_d) w_m(u + e_{j_1} + \dots + e_{j_{d-1}}) h(u) du \right| \\
 &= \sup_{\|h\|_q \leq 1} \left| \sum_{m=0}^{2^n-1} \alpha_{mn}(x_1, \dots, x_d) w_m(e_{j_1}) \dots w_m(e_{j_{d-1}}) \widehat{h}(m) \right| \\
 &\leq \sup_{\|h\|_q \leq 1} \left(\sum_{m=0}^{2^n-1} |\alpha_{mn}(x_1, \dots, x_d)|^q \right)^{1/q} \left(\sum_{m=0}^{2^n-1} |\widehat{h}(m)|^p \right)^{1/p} \leq c \sup_{\|h\|_q \leq 1} \|h\|_q < \infty.
 \end{aligned}$$

Hence, by (14) we obtain

$$\begin{aligned}
 J_{nM1\sigma}^{(p)}(x_1, \dots, x_d) &\leq c(d) \sum_{j_1=0}^{n-1} \sum_{j_2=0}^{j_1} \dots \sum_{j_{d-1}=0}^{j_{d-2}} 2^{j_{d-1}/p} V_d^{(p)}(x_1, \dots, x_d; A) \quad (15) \\
 &\leq c 2^{n/p} p^{d-1} V_d^{(p)}(x_1, \dots, x_d; A).
 \end{aligned}$$

Consequently,

$$\sup_{n \in \mathbb{N}} \frac{J_{nM1\sigma}^{(p)}(x_1, \dots, x_d)}{2^{n/p}} \leq c(d, p) V_d^{(p)}(x_1, \dots, x_d; A). \tag{16}$$

Set

$$\Omega := \left\{ (x_1, \dots, x_d) \in \mathbb{I}^d : V_d^{(p)}(x_1, \dots, x_d; A) > c_1 \lambda \right\}.$$

Then from Lemma 1 we can write

$$\begin{aligned} |\Omega| &= \int_{\mathbb{I}^d} \mathbb{I}_\Omega(x_1, \dots, x_d) dx_1 \cdots dx_d = \int_{\mathbb{I}^{d-1}} \left(\int_{\mathbb{I}} \mathbb{I}_\Omega(x_1, \dots, x_d) dx_d \right) dx_2 \cdots dx_{d-1} \\ &\leq \int_{\mathbb{I}^{d-1}} \left(\frac{c}{\lambda} \int_{\mathbb{I}} |A(x_1, \dots, x_d)| dx_d \right) dx_2 \cdots dx_{d-1} \\ &= \left(\frac{c}{\lambda} \int_{\mathbb{I}^d} |A(x_1, \dots, x_d)| dx_1 \cdots dx_d \right)^{1/2}. \end{aligned}$$

Hence, from (13) we obtain $(f \in L(\log^+ L)^{d-1}(\mathbb{I}^d))$

$$\begin{aligned} &\left| \left\{ (x_1, \dots, x_d) \in \mathbb{I}^d : V_d^{(p)}(x_1, \dots, x_d; A) > \lambda \right\} \right| \\ &\leq \frac{c(d, p)}{\lambda} \left(1 + \int_{\mathbb{I}^d} |f(x_1, \dots, x_d)| (\log^+ |f(x_1, \dots, x_d)|)^{d-1} dx_1 \cdots dx_d \right). \end{aligned}$$

From (16) we conclude that

$$\begin{aligned} &\left| \left\{ (x_1, \dots, x_d) \in \mathbb{I}^d : \sup_{n \in \mathbb{N}} \frac{J_{nM1\sigma}^{(p)}(x_1, \dots, x_d)}{2^{n/p}} > c\lambda \right\} \right| \tag{17} \\ &\leq \frac{c(d, p)}{\lambda} \left(1 + \int_{\mathbb{I}^d} |f(x_1, \dots, x_d)| (\log^+ |f(x_1, \dots, x_d)|)^{d-1} dx_1 \cdots dx_d \right) \end{aligned}$$

for $\sigma(k) = d + 1 - k$.

Similarly we can prove that for all other summands in (9) the estimation (17) holds. So, we have

$$\begin{aligned} &\left| \left\{ (x_1, \dots, x_d) \in \mathbb{I}^d : \sup_{n \in \mathbb{N}} \frac{J_{nM1\sigma}^{(p)}(x_1, \dots, x_d)}{2^{n/p}} > c\lambda \right\} \right| \tag{18} \\ &\leq \frac{c(d, p)}{\lambda} \left(1 + \int_{\mathbb{I}^d} |f(x_1, \dots, x_d)| (\log^+ |f(x_1, \dots, x_d)|)^{d-1} dx_1 \cdots dx_d \right). \end{aligned}$$

Let $B \subset M, B \neq M$. Then analogously, we can prove that

$$\left| \left\{ (x_1, \dots, x_d) \in \mathbb{I}^d : \sup_{n \in \mathbb{N}} \frac{J_{nB1}^{(p)}(x_1, \dots, x_d)}{2^{n/p}} > c\lambda \right\} \right| \tag{19}$$

$$\leq \frac{c(d, p)}{\lambda} \left(1 + \int_{\mathbb{I}^d} |f(x_1, \dots, x_d)| (\log^+ |f(x_1, \dots, x_d)|)^{d-1} dx_1 \cdots dx_d \right).$$

Now, we estimate $J_{nB2}^{(p)}(x_1, \dots, x_d)$. Let $B = \emptyset$. Then we have

$$J_{nB2}^{(p)}(x_1, \dots, x_d)$$

$$= \left| \int_{\mathbb{I}^d} S_{2^n, \dots, 2^n}(x_1 \dot{+} t_1, \dots, x_d \dot{+} t_d; f) \sum_{m=0}^{2^n-1} \alpha_{mn}(x_1, \dots, x_d) \prod_{s \in M} \left(m + \frac{1}{2}\right) \mathbb{I}_{I_n}(t_s) dt_1 \cdots dt_d \right|$$

$$\leq \int_{\mathbb{I}^d} S_{2^n, \dots, 2^n}(x_1 \dot{+} t_1, \dots, x_d \dot{+} t_d; |f|) \left| \sum_{m=0}^{2^n-1} \alpha_{mn}(x_1, \dots, x_d) \left(m + \frac{1}{2}\right)^d \prod_{s \in M} \mathbb{I}_{I_n}(t_s) dt_1 \cdots dt_d \right|$$

$$\leq c(d) 2^{nd+n/p} \int_{I_n \times \cdots \times I_n} |f(x_1 \dot{+} t_1, \dots, x_d \dot{+} t_d)| dt_1 \cdots dt_d \left(\sum_{m=0}^{2^n-1} |\alpha_{mn}(x_1, \dots, x_d)|^q \right)^{1/q}$$

$$\leq c(d) 2^{n/p} E_*(x_1, \dots, x_d; f).$$

Hence, from (2), we have

$$\left| \left\{ (x_1, \dots, x_d) \in \mathbb{I}^d : \sup_{n \in \mathbb{N}} \frac{J_{nB2}^{(p)}(x_1, \dots, x_d)}{2^{n/p}} > c_1\lambda \right\} \right| \tag{20}$$

$$\leq \frac{c(d, p)}{\lambda} \left(1 + \int_{\mathbb{I}^d} |f(x_1, \dots, x_d)| (\log^+ |f(x_1, \dots, x_d)|)^{d-1} dx_1 \cdots dx_d \right).$$

Let $B := \{l_1, \dots, l_r\} \subset M, l_i < l_{i+1}, i = 1, \dots, r-1, B \neq \emptyset, B' = M \setminus B := \{k_1, \dots, k_s\}, r + s = d$. Since

$$\int_{[0, 2^{-n}]^{|B'|}} S_{2^n, \dots, 2^n}(x_1 \dot{+} t_1, \dots, x_d \dot{+} t_d; |f|) dt_{k_1} \cdots dt_{k_s} = 2^{-n|B'|} S_{2^n, \dots, 2^n}(\vec{x}_M \dot{+} \vec{t}_B; |f|). \tag{21}$$

Repeating the arguments in the same way as in estimation $J_{nB1}^{(p)}(x_1, \dots, x_d)$ we get

$$J_{nB2}^{(p)}(x_1, \dots, x_d) \tag{22}$$

$$\leq \sum_{j_1=0}^{n-1} \cdots \sum_{j_r=0}^{n-1} 2^{j_1 + \cdots + j_r} \int_{I_{j_1} \times \cdots \times I_{j_r}} \int_{[0, 2^{-n}]^{|B'|}} S_{2^n, \dots, 2^n}(x_1 \dot{+} t_1, \dots, x_d \dot{+} t_d; |f|)$$

$$\begin{aligned} & \times \left| \sum_{m=0}^{2^n-1} \alpha_{mn}(x_1, \dots, x_d) w_m(t_{l_1} \dot{+} \dots \dot{+} t_{l_r} \dot{+} e_{j_{l_1}} \dot{+} \dots \dot{+} e_{j_{l_r}}) \left(m + \frac{1}{2}\right)^{|B'|} \right| dt_1 \dots dt_d \\ & \leq \sum_{j_{l_1}=0}^{n-1} \dots \sum_{j_{l_r}=0}^{n-1} 2^{j_1+\dots+j_r} 2^{-n|B'|} \int_{I_{j_{l_1}} \times \dots \times I_{j_{l_r}}} S_{2^n, \dots, 2^n}^{(B)} \left(\vec{x}_M \dot{+} \vec{t}_B; E_*^{(B')}(|f|)\right) \\ & \times \left| \sum_{m=0}^{2^n-1} \alpha_{mn}(x_1, \dots, x_d) w_m(t_{l_1} \dot{+} \dots \dot{+} t_{l_r} \dot{+} e_{j_{l_1}} \dot{+} \dots \dot{+} e_{j_{l_r}}) \left(m + \frac{1}{2}\right)^{|B'|} \right| dt_{l_1} \dots dt_{l_r}. \end{aligned}$$

Hence, from (2) and (3) we have

$$\begin{aligned} & \left| \left\{ (x_1, \dots, x_d) \in \mathbb{I}^d : \sup_{n \in \mathbb{N}} \frac{J_{nB2}^{(p)}(x_1, \dots, x_d)}{2^{n/p}} > c\lambda \right\} \right| \tag{23} \\ & \leq \frac{c(d, p)}{\lambda} \left(1 + \int_{\mathbb{I}^d} |E_*^{B'}(x_1, \dots, x_d; f)| \left(\log^+ |E_*^{B'}(x_1, \dots, x_d; f)|\right)^{|B'|-1} dx_1 \dots dx_d \right) \\ & \leq \frac{c(d, p)}{\lambda} \left(1 + \int_{\mathbb{I}^d} |f(x_1, \dots, x_d)| \left(\log^+ |f(x_1, \dots, x_d)|\right)^{d-1} dx_1 \dots dx_d \right). \end{aligned}$$

Analogously, we can write

$$\begin{aligned} J_{nB3}^{(p)}(x_1, \dots, x_d) & \leq \int_{\mathbb{I}^{|B|} [0, 2^{-n}]^{|B'|}} \int S_{2^n, \dots, 2^n}(x_1 \dot{+} t_1, \dots, x_d \dot{+} t_d; |f|) \tag{24} \\ & \times \left| \sum_{m=0}^{2^n-1} \alpha_{mn}(x_1, \dots, x_d) w_m(t_{l_1} \dot{+} \dots \dot{+} t_{l_r}) \left(m + \frac{1}{2}\right)^{|B'|} \right| dt_1 \dots dt_d \\ & \leq \frac{1}{2^{|B'|}} \int_{\mathbb{I}^{|B|}} S_{2^n, \dots, 2^n}^{(B)}(x_M \dot{+} t_B; E_*^{(B')}(|f|)) \\ & \times \left| \sum_{m=0}^{2^n-1} \alpha_{mn}(x_1, \dots, x_d) w_m(t_{l_1} \dot{+} \dots \dot{+} t_{l_r}) \left(m + \frac{1}{2}\right)^{|B'|} \right| dt_{l_1} \dots dt_{l_r}, \end{aligned}$$

$$\begin{aligned} & \left| \left\{ (x_1, \dots, x_d) \in \mathbb{I}^d : \sup_{n \in \mathbb{N}} \frac{J_{nB3}^{(p)}(x_1, \dots, x_d)}{2^{n/p}} > c_1\lambda \right\} \right| \\ & \leq \frac{c(d, p)}{\lambda} \left(1 + \int_{\mathbb{I}^d} |f(x_1, \dots, x_d)| \left(\log^+ |f(x_1, \dots, x_d)|\right)^{d-1} dx_1 \dots dx_d \right). \end{aligned}$$

Combining (8), (17)–(24) we complete the proof of Theorem 1. \square

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Ushangi Goginava

Department of Mathematics, Faculty of Exact and Natural Sciences
Ivane Javakhishvili Tbilisi State University
Chavchavadze str. 1, Tbilisi 0128, Georgia
e-mail: zazagoginava@gmail.com