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BOUNDS OF GENERALIZED RELATIVE OPERATOR ENTROPIES

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Abstract. In this paper, we identify upper and lower bounds of the generalized relative operator entropy based on the notion of perspectives. Moreover, we find upper and lower bounds of the Tsallis relative operator entropy to specify the bounds of the relative operator entropy.

1. Introduction and preliminaries

In [1], the quantum Tsallis relative entropy was defined by

$$D_q(\rho|\sigma) := \frac{1-\mathrm{Trace}[\rho^q\sigma^{1-q}]}{1-q}$$

for two density matrices ρ and σ and $0<q<1$. One can see that it is one parameter extension of the quantum relative entropy defined by Umegaki [22]

$$U(\rho|\sigma) := \mathrm{Trace}[\rho(\log\rho-\log\sigma)].$$

The quantum relative entropy is a very important quantity in quantum information theory [15]. It satisfies many significant relations such as monotonicity property under quantum channels [14]. In information theory, more than 30 measures of entropies generalizing Shannon's entropy, as parametric, trigonometric and weighted entropies have been introduced. Shannon entropy quantifies the expected value of information contained in a stochastic variable, measuring the uncertainty associated with such a variable. Hence, it provides an estimation of the average amount of information loss if the value of the stochastic variable is not known.

In the statistical physics, the Tsallis entropy was defined in [20] by $S_q(X) = -\sum_x p(x)^q \ln_q p(x)$ with one parameter q as an extension of Shannon entropy, where q-logarithm is defined by $\ln_q(x) = \frac{x^{1-q}-1}{1-q}$ for any nonnegative real number q and x, and $p(x) = p(X=x)$ is the probability distribution of the given random variable X. As $q\to 1$, the Tsallis entropy $S_q(X)$ converges to the Shannon entropy $-\sum_x p(x)\log p(x)$. This notion has an important role in non-extensive statistics, which is often called Tsallis statistics. However, the notion of entropy is essential not only in thermodynamical physics and statistical physics but also in information theory and analytical mathematics

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such as operator theory and probability theory. Mainly, the relative entropy is fundamental in the sense that it produces the entropy and the mutual information as special cases.

A relative operator entropy of strictly positive operators A and B on a Hilbert space was introduced in the noncommutative information theory by Fujii and Kamei [10] by

$$S(A|B) := A^{\frac{1}{2}}(\ln A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}}.$$

The generalized relative operator entropy for strictly positive operators A,B and $q \in \mathbb{R}$ defined in [7] by setting

$$S_q(A|B) := A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^q(\ln A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}}.$$

In particular, when $q=0$, it leads to the relative operator entropy $S(A|B)$. Furthermore, it is an easy exercise to realize that $S_1(A|B) = -S(B|A)$.

Furuta obtained the parametric extension of operator Shannon inequality and its reverse one [7]. Some refinements and precise estimations of these parametric extensions of Shannon inequality and its reverse one and an extension of operator Shannon type inequality proved in [18]. In [17], Nikoufar determined upper and lower bounds of the relative operator (α,β)-entropy and Tsallis relative operator (α,β)-entropy according to operator (α,β)-geometric mean introduced in [16]. Drogomir in [3] provided some bounds for the following difference

$$S(A|B) - \frac{\ln m}{M-m}(MA-B) - \frac{\ln M}{M-m}(B-mA), \tag{1}$$

where A,B are two strictly positive operators such that $mA \leqslant B \leqslant MA$ for some $m,M > 0$ with $m < M$. Motivated by the fact that in general $S(A|B)$ is not equal to $S(B|A)$, he established in [2] some bounds for

$$\frac{m\ln m}{M-m}(MA-B) + \frac{M\ln M}{M-m}(B-mA) + S(B|A) \tag{2}$$

under the same assumptions for the operators A and B in [3].

In this paper, we identify upper and lower bounds of the generalized relative operator entropy $S_q(A|B)$ for $0 < q \leqslant 1$ based on the notion of perspective of some functions. In particular, our bounds confirm the bounds established by Dragomir for $S(B|A)$ in (2). Moreover, we find upper and lower bounds of the Tsallis relative operator $T_\lambda(A|B)$ to specify bounds of the relative operator entropy $S(A|B)$ in (1). Our results confirm and generalize the presented results in [2] and improve the upper bound of $S(A|B)$ proved in [3]. This upper bound for $S(A|B)$ is sharper than Dragomir's upper bound.

We organize the paper in the following way. In section 2, we prove that the function $t^q \ln t$ is convex on an interval $\mathbb{J}_q$ for $0 < q \leqslant 1$. Then we show that

$$\frac{m^q \ln m}{M-m}(MA-B) + \frac{M^q \ln M}{M-m}(B-mA) - S_q(A|B) \geqslant 0, \tag{3}$$

where A and B are two strictly positive operators such that $mA \leqslant B \leqslant MA$ for some $m,M \in \mathbb{J}_q$ with $m < M$ and $0 < q \leqslant 1$. Furthermore, we provide some upper and positive lower bounds for the difference appeared in (3) under the same assumptions.

In particular, when we put $q \to 1$ our results recover Dragomir's results announced in [2] for $m, M > 0$ with $m < M$. On the other hand, if $q \to 0$, we reach the difference (1) in a negative sign. Unfortunately, we will have $\mathbb{J}_0 = \emptyset$ so in this case we can not claim that our results generalize the presented results in [3]. For this reason, in section 3, we consider the Tsallis relative operator entropy introduced by Yanagi et al. [21] and defined by

$$T_\lambda(A|B) := \frac{A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\lambda A^{\frac{1}{2}} - 1}{\lambda},$$

which is a generalization of the relative operator entropy $S(A|B)$ in the sense that

$$\lim_{\lambda \to 0} T_\lambda(A|B) = S(A|B).$$

Hence, we determine some upper and positive lower bounds for the following difference

$$T_\lambda(A|B) - \frac{m^\lambda - 1}{\lambda(M-m)}(MA - B) - \frac{M^\lambda - 1}{\lambda(M-m)}(B - mA), \tag{4}$$

where A and B are two strictly positive operators such that $mA \leqslant B \leqslant MA$ for some $m, M > 0$ with $m < M$ and $0 < \lambda \leqslant 1$. Then by letting $\lambda \to 0$ we determine some precise bounds for $S(A|B)$.

Let throughout this paper $B(\mathscr{H})$ denote the C^*-algebra of all bounded linear operators on a Hilbert space $\mathscr{H}$ with inner product $\langle \cdot, \cdot \rangle$. A self–adjoint operator A in $B(\mathscr{H})$ is said to be positive, written $A \geqslant 0$, if $\langle Ah, h \rangle \geqslant 0$ for $h \in \mathscr{H}$. If moreover A is invertible, then A is said to be strictly positive, written $A > 0$. For self–adjoint operators A and B in $B(\mathscr{H})$, we write $A \geqslant B$ (resp. $A > B$) if $A - B$ is positive (resp. strictly positive).

2. Bounds of the generalized relative operator entropy

The notion of operator perspective function introduced in [6] by Effros consists of commuting operators and proved the perspective of an operator convex function is operator convex as a function. A fully non-commutative perspective of the one variable function f defined in [5] by setting

$$P_f(A, B) = A^{1/2} f(A^{-1/2}BA^{-1/2})A^{1/2}$$

and the generalized perspective of two variables (associated with f and h) defined by

$$P_{f\Delta h}(A, B) = h(A)^{1/2} f(h(A)^{-1/2}Bh(A)^{-1/2})h(A)^{1/2},$$

where A is a strictly positive operator and B is a self-adjoint operator on a Hilbert space $\mathscr{H}$ with spectra in the closed interval $\mathbb{I}$ containing 0. So, the main results of [6] are generalized in [5] for the non-commutative case where the necessary and sufficient conditions for joint convexity (concavity) of the perspective and generalized perspective functions are established. As an application of these results, Nikoufar et al. [16] gave the simplest proof of Lieb concavity theorem and Ando convexity theorem (see also [19]). The axiomatic theory for connections have been disscused by Kubo and Ando

[13]. They proved the existence of an affine order isomorphism between the class of connections and the class of positive operator monotone functions.

The following theorem proved in [17, Theorem 2.1] for the real valued functions r, s, k, and h defined on the closed interval $\mathbb{I}$.

THEOREM 1. *Let r, s, k, and h be real valued functions on the closed interval $\mathbb{I}$ such that $h > 0$. If $r(t) \leqslant s(t) \leqslant k(t)$ for $t \in \mathbb{I}$, then*

$$P_{r\Delta h}(A,B) \leqslant P_{s\Delta h}(A,B) \leqslant P_{k\Delta h}(A,B)$$

for every strictly positive operator A and every self-adjoint operator B.

In particular we obtain the following corollary whenever $h(t) = t$:

COROLLARY 1. *Let r, s, and k be real valued functions on the closed interval $\mathbb{I}$. If $r(t) \leqslant s(t) \leqslant k(t)$ for $t \in \mathbb{I}$, then*

$$P_r(A,B) \leqslant P_s(A,B) \leqslant P_k(A,B)$$

for every strictly positive operator A and every self-adjoint operator B.

REMARK 1. Dragomir in [4] proved that if $\phi : D \to \mathbb{R}$ is a convex function defined on a convex subset $D \subset \mathbb{R}$, then

$$2r\Big[\frac{\phi(x)+\phi(y)}{2} - \phi\Big(\frac{x+y}{2}\Big)\Big] \leqslant (1-c)\phi(x) + c\phi(y) - \phi((1-c)x+cy)$$
$$\leqslant 2R\Big[\frac{\phi(x)+\phi(y)}{2} - \phi\Big(\frac{x+y}{2}\Big)\Big]$$

for any $x, y \in D$ and $c \in [0,1]$, where $r = \min\{c, 1-c\}$ and $R = \max\{c, 1-c\}$.

For the sake of simplified writing throughout this paper, we define

$$r(u) := \min\left\{\frac{u-m}{M-m}, \frac{M-u}{M-m}\right\} = \frac{1}{2} - \left|\frac{u-\frac{M+m}{2}}{M-m}\right|,$$
$$R(u) := \max\left\{\frac{u-m}{M-m}, \frac{M-u}{M-m}\right\} = \frac{1}{2} + \left|\frac{u-\frac{M+m}{2}}{M-m}\right|,$$
$$K_q(m,M) := \frac{m^q \ln m + M^q \ln M}{2} - \left(\frac{M+m}{2}\right)^q \ln\left(\frac{M+m}{2}\right),$$
$$W_\lambda(m,M) := \frac{(m+M)^\lambda - 2^\lambda}{2^\lambda \lambda} - \frac{m^\lambda + M^\lambda - 2}{2\lambda},$$
$$W_0(m,M) := \ln \frac{m+M}{2\sqrt{mM}},$$

where $0 < m < M$, $0 < q \leqslant 1$, and $0 < \lambda \leqslant 1$.

Define $\omega(t) := t^q \ln t$ for $0 \leqslant q \leqslant 1$, where $\ln t$ is the natural logarithm function and consider

$$\mathbb{J}_q := \{t \geqslant 0 : \omega''(t) \geqslant 0\}.$$

Note that the function ω is convex on $\mathbb{J}_q$. By a simple calculation, we realize that $\mathbb{J}_q := [0, e^{\frac{2q-1}{q(1-q)}}]$, where $0 \leqslant q \leqslant 1$. Consequently, $\mathbb{J}_1 = [0, \infty)$ and $\mathbb{J}_0 = \emptyset$.

LEMMA 1. *The function* $\omega(t) = t^q \ln t$ *is convex on* $\mathbb{J}_q := [0, e^{\frac{2q-1}{q(1-q)}}]$ *for* $0 \leqslant q \leqslant 1$.

The concavity of the function $\ln t$ means geometrically that the points of the graph of the restriction of $\ln t$ on $[m,M]$ are on the chord joining the end points $(m, \ln m)$ and $(M, \ln M)$. Then

$$\ln m + \frac{\ln M - \ln m}{M - m}(x - m) \leqslant \ln x \tag{5}$$

for all $x \in [m,M]$. By rewriting the left hand side of (5) we obtain

$$\frac{\ln M}{M-m}(x-m) + \frac{\ln m}{M-m}(M-x) \leqslant \ln x$$

for all $x \in [m,M]$ and taking the perspective in the sense of Corollary 1 we get the difference (1) is positive. Indeed, the term $\frac{\ln m}{M-m}(MA-B) + \frac{\ln M}{M-m}(B-mA)$ appeared in the difference (1) is the perspective of the line joining the points $(m, \ln m)$ and $(M, \ln M)$. The points of the graph of the convex function $t^q \ln t$ on $[m,M] \subseteq \mathbb{J}_q$ are under the chord joining the end points $(m, m^q \ln m)$ and $(M, M^q \ln M)$. So, by taking the perspective we achieve (3). Applying the same approach for the concave function $\frac{t^\lambda - 1}{\lambda}, 0 < \lambda \leqslant 1$, we identify the difference appeared in (4) is positive.

Note that convexity of the function $\omega(t) = t^q \ln t$ on $\mathbb{J}_q$ shows that

$$K_q(m,M) \geqslant 0 \tag{6}$$

for $m, M \in \mathbb{J}_q$ with $0 < m < M$. In the following theorem, if we put $q \to 1$, then we obtain [2, Theorem 3]. However, this theorem is a generalization of Dragomir's result.

THEOREM 2. *Let* A *and* B *be two strictly positive operators such that* $mA \leqslant B \leqslant MA$ *for some* $m, M \in \mathbb{J}_q$ *with* $0 < m < M$. *Then*

$$\begin{aligned} 0 &\leqslant \frac{m^q \ln m}{M-m}(MA-B) + \frac{M^q \ln M}{M-m}(B-mA) - S_q(A|B) \\ &\leqslant \frac{M^{q-1}(1+q\ln M) - m^{q-1}(1+q\ln m)}{M-m} P_f(A,B) \\ &\leqslant \frac{1}{4}(M-m)\left(M^{q-1}(1+q\ln M) - m^{q-1}(1+q\ln m)\right)A, \end{aligned}$$

where $f(t) = (t-m)(M-t)$.

Proof. We apply [2, Lemma 1] for the function $g(t) = t^q \ln t$, $t \in \mathbb{J}_q$. Then

$$\begin{aligned} 0 &\leqslant (1-c)g(x) + cg(y) - g((1-c)x + cy) \\ &\leqslant c(1-c)(y-x)(g'_-(y) - g'_+(x)), \end{aligned} \tag{7}$$

where $c \in [0,1]$ and $x, y \in [m,M]$. Replacing $x = m$, $y = M$, and $c = \frac{u-m}{M-m}$ in (7), we get

$$\begin{aligned} 0 &\leqslant \frac{m^q \ln m}{M-m}(M-u) + \frac{M^q \ln M}{M-m}(u-m) - u^q \ln u \\ &\leqslant \frac{M^{q-1}(1+q\ln M) - m^{q-1}(1+q\ln m)}{M-m} f(u). \end{aligned} \tag{8}$$

The function $f(u)$ attains its maximum value at $u = \frac{M+m}{2}$ and the maximum value is $\frac{1}{4}(M-m)^2$. So,

$$\begin{aligned}&\frac{M^{q-1}(1+q\ln M)-m^{q-1}(1+q\ln m)}{M-m}f(u)\\&\leqslant \frac{1}{4}(M-m)\left(M^{q-1}(1+q\ln M)-m^{q-1}(1+q\ln m)\right).\end{aligned} \tag{9}$$

Combining inequalities (8), (9) and regarding Corollary 1 and taking the perspective, we conclude the result. □

We define the operator q-entropy by $H_q(A) := A^q \ln A$ for a positive operator A and $0 < q \leqslant 1$. In particular, $H_1(A)$ is the well known von Neumann entropy. Note that $S_q(I,A) = H_q(A)$. For commutative strictly positive operators A and B, we denote by $E_q(A,B)$ the generalized relative operator entropy and so

$$E_q(A,B) := A^{1-q}B^q(\ln B - \ln A).$$

The following corollary is a direct consequence of (8) and (9).

COROLLARY 2. *If A is a strictly positive operator such that $mI \leqslant A \leqslant MI$ for some $m,M \in \mathbb{J}_q$ with $0 < m < M$, then*

$$\begin{aligned}0 &\leqslant \frac{m^q \ln m}{M-m}(MI-A) + \frac{M^q \ln M}{M-m}(A-mI) - H_q(A)\\&\leqslant \frac{M^{q-1}(1+q\ln M)-m^{q-1}(1+q\ln m)}{M-m}(A-mI)(MI-A)\\&\leqslant \frac{1}{4}(M-m)\left(M^{q-1}(1+q\ln M)-m^{q-1}(1+q\ln m)\right)I.\end{aligned}$$

The following theorem is a generalization of the result announced by Dragomir. Indeed, if we put $q \to 1$, then we reach [2, Theorem 2]. As we remarked in (6), $K_q(m,M) \geqslant 0$.

THEOREM 3. *Let A and B be two strictly positive operators such that $mA \leqslant B \leqslant MA$ for some $m,M \in \mathbb{J}_q$ with $0 < m < M$. Then we have*

$$\begin{aligned}2K_q(m,M)P_r(A,B) &\leqslant \frac{m^q \ln m}{M-m}(MA-B) + \frac{M^q \ln M}{M-m}(B-mA) - S_q(A|B)\\&\leqslant 2K_q(m,M)P_R(A,B).\end{aligned}$$

Proof. If we take in Remark 1 the convex function $\phi(t) = t^q \ln t$, $t \in \mathbb{J}_q$, then we have

$$\begin{aligned}&2r\Big[\frac{x^q\ln x + y^q \ln y}{2} - \left(\frac{x+y}{2}\right)^q \ln\left(\frac{x+y}{2}\right)\Big]\\&\leqslant (1-c)x^q\ln x + cy^q \ln y - ((1-c)x+cy)^q \ln((1-c)x+cy)\\&\leqslant 2R\Big[\frac{x^q\ln x + y^q \ln y}{2} - \left(\frac{x+y}{2}\right)^q \ln\left(\frac{x+y}{2}\right)\Big]\end{aligned} \tag{10}$$

for any $x,y \in \mathbb{J}_q$ and $c \in [0,1]$, where $r = \min\{c, 1-c\}$ and $R = \max\{c, 1-c\}$. Replacing $x = m$, $y = M$, and $c = \frac{u-m}{M-m}$ with $u \in [m,M]$ in (10), we deduce

$$2K_q(m,M)r(u) \leqslant m^q \ln m \frac{M-u}{M-m} + M^q \ln M \frac{u-m}{M-m} - u^q \ln u \leqslant 2K_q(m,M)R(u). \tag{11}$$

Making use of Corollary 1 and taking the perspective, we get the desired inequalities. □

When the strictly positive operators A and B are commutative, we deduce $P_r(A,B) = Ar(BA^{-1})$, $P_R(A,B) = AR(BA^{-1})$, and $S_q(A|B) = E_q(A,B)$.

COROLLARY 3. *Let A and B be two strictly positive operators such that $mA \leqslant B \leqslant MA$ for some $m,M \in \mathbb{J}_q$ with $0 < m < M$. If A and B commute, then*

$$2K_q(m,M)Ar(BA^{-1}) \leqslant \frac{m^q \ln m}{M-m}(MA-B) + \frac{M^q \ln M}{M-m}(B-mA) - E_q(A,B) \leqslant 2K_q(m,M)AR(BA^{-1}).$$

The following corollary is a direct consequence of (11).

COROLLARY 4. *If A is a positive operator such that $mI \leqslant A \leqslant MI$ for some $m,M \in \mathbb{J}_q$ with $0 < m < M$, then*

$$2K_q(m,M)r(A) \leqslant \frac{m^q \ln m}{M-m}(MI-A) + \frac{M^q \ln M}{M-m}(A-mI) - H_q(A) \leqslant 2K_q(m,M)R(A).$$

According to the following theorem, if we let $q \to 1$, then we identify [2, Theorem 4].

THEOREM 4. *Let A and B be two strictly positive operators such that $mA \leqslant B \leqslant MA$ for some $m,M \in \mathbb{J}_q$ with $0 < m < M$. Then*

$$\begin{aligned} 0 &\leqslant \frac{1}{2}M^{q-2}(2q-1+q(q-1)\ln M)P_f(A,B) \\ &\leqslant \frac{m^q \ln m}{M-m}(MA-B) + \frac{M^q \ln M}{M-m}(B-mA) - S_q(A|B) \\ &\leqslant \frac{1}{2}m^{q-2}(2q-1+q(q-1)\ln m)P_f(A,B), \end{aligned}$$

where $f(t) = (t-m)(M-t)$.

Proof. Using [2, Lemma 2] for the function $g(t) = t^q \ln t$, $t \in \mathbb{J}_q$, we get

$$\frac{1}{2}c(1-c)d(y-x)^2 \leqslant (1-c)g(x) + cg(y) - g((1-c)x+cy) \leqslant \frac{1}{2}c(1-c)D(y-x)^2, \tag{12}$$

where $c \in [0,1]$, $x,y \in [m,M]$, $d \leqslant g''(t) \leqslant D$ for some constants d,D, and any $t \in [m,M]$. Substitute $x = m$, $y = M$, $c = \frac{u-m}{M-m}$, $d = M^{q-2}(2q-1+q(q-1)\ln M) > 0$ and $D = m^{q-2}(2q-1+q(q-1)\ln m) > 0$ in (12), to get

$$0 \leqslant \frac{1}{2}(u-m)(M-u)d \leqslant \frac{M-u}{M-m}m^q \ln m + \frac{u-m}{M-m}M^q \ln M - u^q \ln u$$
$$\leqslant \frac{1}{2}(u-m)(M-u)D. \tag{13}$$

Due to Corollary 1 and replacing d,D, we reach the desired inequalities. □

The following corollary is a direct consequence of (13).

COROLLARY 5. *If A is a positive operator such that $mI \leqslant A \leqslant MI$ for some $m,M \in \mathbb{J}_q$ with $0 < m < M$, then*

$$0 \leqslant \frac{1}{2}M^{q-2}(2q-1+q(q-1)\ln M)(A-mI)(MI-A)$$
$$\leqslant \frac{m^q \ln m}{M-m}(MI-A) + \frac{M^q \ln M}{M-m}(A-mI) - H_q(A)$$
$$\leqslant \frac{1}{2}m^{q-2}(2q-1+q(q-1)\ln m)(A-mI)(MI-A).$$

REMARK 2. Note that as we remarked above in Theorems 2, 3, and 4 if $q \to 1$, then the results in Theorems 3, 2, and 4 of [2] are satisfied, respectively. On the other hand, when $q \to 0$ the set $\mathbb{J}_0$ is an empty set. However, in Theorems 2, 3, and 4, if $q \to 0$, we can not identify the upper and lower bounds of $S(A|B)$ on an empty set. So, in the next section, we declare a method to determine the upper and lower bounds of $S(A|B)$.

3. Bounds of the Tsallis relative operator entropy

Furuichi et al. [12] obtained the following inequalities (see also [11]):

$$T_{-\lambda}(A|B) \leqslant S(A|B) \leqslant T_\lambda(A|B), \tag{14}$$

$$A - AB^{-1}A \leqslant T_\lambda(A|B) \leqslant B - A \tag{15}$$

and Zou [23] refined (14) and (15) as follows:

$$A - AB^{-1}A \leqslant T_{-\lambda}(A|B) \leqslant S(A|B) \leqslant T_\lambda(A|B) \leqslant B - A,$$

where A and B are two strictly positive operators and $0 < \lambda \leqslant 1$. For more inequalities on the Tsallis relative operator entropy the reader is referred to [8, 9, 17].

Dragomir established a main and natural question that how far the terms in the difference (1) and so provided some bounds for this difference as follows.

THEOREM 5. [3, Theorem 2] *Let A and B be two strictly positive operators such that $mA \leqslant B \leqslant MA$ for some m,M with $0<m<M$. Then*

$$K\Big(\frac{M}{m}\Big)P_r(A,B) \leqslant S(A|B) - \frac{\ln m}{M-m}(MA-B) - \frac{\ln M}{M-m}(B-mA) \leqslant K\Big(\frac{M}{m}\Big)P_R(A,B), \tag{16}$$

where $K(h)=\frac{(h+1)^2}{4h}$, $h>0$ is the Kantrovich constant.

REMARK 3. As mentioned in Remark 1, if $\phi : D \to \mathbb{R}$ is a concave function defined on a convex set $D \subset \mathbb{R}$, then

$$2r\Big[\phi\Big(\frac{x+y}{2}\Big) - \frac{\phi(x)+\phi(y)}{2}\Big] \leqslant \phi((1-c)x+cy) - ((1-c)\phi(x)+c\phi(y)) \leqslant 2R\Big[\phi\Big(\frac{x+y}{2}\Big) - \frac{\phi(x)+\phi(y)}{2}\Big]$$

for any $x,y \in D$ and $c \in [0,1]$, where $r=\min\{c,1-c\}$ and $R=\max\{c,1-c\}$.

We determine some upper and positive lower bounds for the difference (4) and apply them to obtain some bounds for the difference (1).

THEOREM 6. *Let A and B be two strictly positive operators such that $mA \leqslant B \leqslant MA$ for some m,M with $0<m<M$. Then*

$$0 \leqslant T_\lambda(A|B) - \frac{m^\lambda - 1}{\lambda(M-m)}(MA-B) - \frac{M^\lambda-1}{\lambda(M-m)}(B-mA) \leqslant \frac{m^{\lambda-1}-M^{\lambda-1}}{M-m}P_f(A,B) \leqslant \frac{1}{4}(M-m)(m^{\lambda-1}-M^{\lambda-1})A,$$

where $0<\lambda\leqslant 1$ and $f(t)=(t-m)(M-t)$.

Proof. We apply [2, Lemma 1] for the concave function $\phi : \mathbb{I} \subset \mathbb{R} \to \mathbb{R}$. Then we get

$$0 \leqslant \phi((1-c)x+cy) - (1-c)\phi(x) - c\phi(y) \leqslant c(1-c)(y-x)(\phi'_+(x) - \phi'_-(y)) \tag{17}$$

for x,y in the interior of $\mathbb{I}$ and $c \in [0,1]$. Taking $\phi(t)=\frac{t^\lambda-1}{\lambda}$, $t>0$ in (17), we yield

$$0 \leqslant \frac{((1-c)x+cy)^\lambda - 1}{\lambda} - (1-c)\frac{x^\lambda-1}{\lambda} - c\frac{y^\lambda-1}{\lambda} \leqslant c(1-c)(y-x)(x^{\lambda-1}-y^{\lambda-1}). \tag{18}$$

Replacing $x=m$, $y=M$, and $c=\frac{u-m}{M-m}$ with $u\in[m,M]$ in (18), we deduce

$$0 \leqslant \frac{u^\lambda-1}{\lambda} - \frac{m^\lambda-1}{\lambda(M-m)}(M-u) - \frac{M^\lambda-1}{\lambda(M-m)}(u-m) \leqslant \frac{m^{\lambda-1}-M^{\lambda-1}}{M-m}f(u).$$

Taking into account that the maximum value of $f(u)$ is $\frac{1}{4}(M-m)^2$ and using Corollary 1, we get the desired inequalities. $\square$

We notice that concavity of the functions $\frac{t^\lambda-1}{\lambda}$ for $0<\lambda\leqslant 1$ and $\ln t$ ensure

$$W_\lambda(m,M)\geqslant 0 \text{ and } W_0(m,M)\geqslant 0$$

for $m,M>0$ with $m<M$, respectively. The following theorem is a generalization of Theorem 5:

THEOREM 7. *Let A and B be two strictly positive operators such that $mA\leqslant B\leqslant MA$ for some $m,M>0$ with $m<M$. Then*

$$\begin{aligned} 2W_\lambda(m,M)P_r(A,B) &\leqslant T_\lambda(A|B)-\frac{m^\lambda-1}{\lambda(M-m)}(MA-B)-\frac{M^\lambda-1}{\lambda(M-m)}(B-mA)\\ &\leqslant 2W_\lambda(m,M)P_R(A,B), \end{aligned}$$

where $0<\lambda\leqslant 1$.

Proof. If we take in Remark 3, $\phi(t)=\frac{t^\lambda-1}{\lambda}$ for $t>0$ and $0<\lambda\leqslant 1$, then $\phi(t)$ is concave and we have

$$\begin{aligned} 2r\Big[\frac{(x+y)^\lambda-2^\lambda}{2^\lambda\lambda}-\frac{x^\lambda+y^\lambda-2}{2\lambda}\Big] &\leqslant \frac{((1-c)x+cy)^\lambda-1}{\lambda}-(1-c)\frac{x^\lambda-1}{\lambda}-c\frac{y^\lambda-1}{\lambda}\\ &\leqslant 2R\Big[\frac{(x+y)^\lambda-2^\lambda}{2^\lambda\lambda}-\frac{x^\lambda+y^\lambda-2}{2\lambda}\Big] \end{aligned} \tag{19}$$

for any $x,y>0$ and $c\in[0,1]$, where $r=\min\{c,1-c\}$ and $R=\max\{c,1-c\}$. Replacing $x=m$, $y=M$, and $c=\frac{u-m}{M-m}$ with $u\in[m,M]$ in (19), we deduce

$$\begin{aligned} 2W_\lambda(m,M)r(u) &\leqslant \frac{u^\lambda-1}{\lambda}-\frac{m^\lambda-1}{\lambda(M-m)}(M-u)-\frac{M^\lambda-1}{\lambda(M-m)}(u-m)\\ &\leqslant 2W_\lambda(m,M)R(u). \end{aligned}$$

Applying Corollary 1, we get the desired inequalities. $\square$

Note that if λ tends to zero in our Theorems 6 and 7, we obtain corollaries 6 and 7, respectively. In Remark 4, we show that Corollary 6 is the same as [3, Theorem 3] and this fact declares that [3, Theorem 3] is a consequence of our Theorem 6. Moreover, Corollary 7 identifies an upper bound for the relative operator entropy which is sharper than the upper bound determined in Theorem 5.

COROLLARY 6. *Let A and B be two strictly positive operators such that $mA\leqslant B\leqslant MA$ for some $m,M>0$ with $m<M$. Then*

$$\begin{aligned} 0 &\leqslant S(A|B)-\frac{\ln m}{M-m}(MA-B)-\frac{\ln M}{M-m}(B-mA)\\ &\leqslant \frac{1}{Mm}P_f(A,B)\leqslant \frac{(M-m)^2}{4mM}A, \end{aligned}$$

where $f(t)=(t-m)(M-t)$.

COROLLARY 7. *Let A and B be two strictly positive operators such that $mA \leqslant B \leqslant MA$ for some $m, M > 0$ with $m < M$. Then*

$$2W_0(m,M)P_r(A,B) \leqslant S(A|B) - \frac{\ln m}{M-m}(MA-B) - \frac{\ln M}{M-m}(B-mA) \leqslant 2W_0(m,M)P_R(A,B). \tag{20}$$

REMARK 4. We remark that Corollary 6 confirms [3, Theorem 3], since

$$\frac{(M-m)^2}{4mM} = K\left(\frac{M}{m}\right) - 1.$$

Moreover, since $2\ln x \leqslant x^2$, $x > 0$, so for $x = \frac{M+m}{2\sqrt{mM}}$ we conclude that

$$2W_0(m,M) \leqslant K\left(\frac{M}{m}\right).$$

This shows that our determined upper bound $2W_0(m,M)$ in (20) is sharper than the upper bound $K(\frac{M}{m})$ established by Dragomir in (16). Consequently, we refine (16) as follows:

$$K\left(\frac{M}{m}\right)P_r(A,B) \leqslant S(A|B) - \frac{\ln m}{M-m}(MA-B) - \frac{\ln M}{M-m}(B-mA) \leqslant 2W_0(m,M)P_R(A,B).$$

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