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ORLICZ–BRUNN–MINKOWSKI INEQUALITY FOR POLAR BODIES AND DUAL STAR BODIES

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Abstract. In this paper, we establish the Orlicz-Brunn-Minkowski inequality for polar bodies and dual star bodies. These results can be considered as 'polar' counterparts of the existing Orlicz-Brunn-Minkowski inequality for convex bodies and star bodies.

1. Introduction

The classical Brunn-Minkowski inequality states that if K and L are convex bodies in $\mathbb{R}^n$, then

$$V(K+L)^{1/n} \geqslant V(K)^{1/n} + V(L)^{1/n}, \tag{1}$$

with equality if and only if K and L are homothetic, i.e., they coincide up to translation and dilatation. Here $K+L = \{x+y : x \in K, y \in L\}$, and V denotes the volume. As the cornerstone of the Brunn-Minkowski theory, the Brunn-Minkowski inequality is a far-reaching generalization of the isoperimetric inequality.

In the early 1960's, Firey [2] introduced the concept of L_p-addition $+_p$. It is defined for $p \geqslant 1$ by

$$h(K+_p L, x)^p = h(K,x)^p + h(L,x)^p, \tag{2}$$

for all $x \in \mathbb{R}^n$ and K, L convex bodies in $\mathbb{R}^n$ containing the origin in their interior, where $h(M,\cdot)$ denotes the support function of the set M. In the same paper, the L_p-Brunn-Minkowski inequality was established: if $p \geqslant 1$, and K, L are convex bodies in $\mathbb{R}^n$ containing the origin in their interior, then

$$V(K+_p L)^{p/n} \geqslant V(K)^{p/n} + V(L)^{p/n}, \tag{3}$$

with equality if and only if K and L are dilatates. When $p=1$, (3) reduces to (1). In the mid 1990's, it was shown in [8, 9] that when L_p-addition is combined with volume the result is an embryonic L_p-Brunn-Minkowski theory. This theory has expanded rapidly and is still extensively studied (see e.g. [5, 6]).

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The dual Brunn-Minkowksi theory for star bodies was initiated by Lutwak [7] in the 1970's. The corresponding L_p-radial addition $\widetilde{+}_p$ are defined for $p \in \mathbb{R}\backslash\{0\}$ by

$$\rho^p_{K\widetilde{+}_pL}(x) = \rho^p_K(x) + \rho^p_L(x), \tag{4}$$

for $x \in \mathbb{R}^n\backslash\{o\}$ and $K, L \subset \mathbb{R}^n$ star bodies with respect to the origin, where $\rho(M,\cdot)$ is the radial function of the set M. The dual L_p-Brunn-Minkowski inequality states that: if K, L are star bodies with respect to the origin, and $0 < p \leqslant n$, then

$$V(K\widetilde{+}_pL)^{p/n} \leqslant V(K)^{p/n} + V(L)^{p/n}. \tag{5}$$

The reverse inequality holds when either $p > n$ or $p < 0$. Equality holds when $p \neq n$ if and only if K, L are dilatates.

Let Φ_2 be the set of all convex functions $\varphi : [0,\infty)^2 \to [0,\infty)$ that are strictly increasing in each component and such that $\varphi(o) = 0$. Let $\widetilde{\Phi}_2$ be the set of all continuous functions $\varphi : [0,\infty)^2 \to [0,\infty)$ that are strictly increasing in each component and such that $\varphi(o) = 0$ and $\lim_{t\to\infty}\varphi(tx) = \infty$, for each $x \in [0,\infty)^2\backslash\{o\}$. Let $\widetilde{\Psi}_2$ be the set of all continuous functions $\varphi : [0,\infty)^2 \to [0,\infty)$ that are strictly decreasing in each component and such that $\lim_{t\to 0}\varphi(tx) = \infty$ and $\lim_{t\to\infty}\varphi(tx) = 0$, for each $x \in [0,\infty)^2\backslash\{o\}$.

The Orlicz-Brunn-Minkowski theory was launched by Lutwak, Yang and Zhang in a series of papers [10, 11]. The study of the Orlicz-Brunn-Minkowski theory has been considerably developed in the recent years (see e.g. [3, 4]). In 2014, Gardner, Hug, and Weil [3] introduced the concept of Orlicz addition $+_\varphi$. This is defined for $\varphi \in \Phi_2$ by

$$\varphi\Big(\frac{h_K(x)}{h_{K+_\varphi L}(x)}, \frac{h_L(x)}{h_{K+_\varphi L}(x)}\Big) = 1, \tag{6}$$

for $x \in \mathbb{R}^n$ and K, L convex bodies in $\mathbb{R}^n$ containing the origin in their interior. As shown in [3, Lemma 4.2], this addition is well defined, i.e., $K +_\varphi L$ is a convex body.

Very recently, Gardner, Hug, Weil and Ye [4] introduced the concept of radial Orlicz addition $\widetilde{+}_\varphi$. This is defined for $\varphi \in \widetilde{\Phi}_2 \cup \widetilde{\Psi}_2$ by

$$\varphi\Big(\frac{\rho_K(x)}{\rho_{K\widetilde{+}_\varphi L}(x)}, \frac{\rho_L(x)}{\rho_{K\widetilde{+}_\varphi L}(x)}\Big) = 1, \tag{7}$$

for $x \in \mathbb{R}^n\backslash\{o\}$ and $K, L \subset \mathbb{R}^n$ star bodies with respect to the origin.

In [3], Gardner, Hug and Weil also established the following Orlicz-Brunn-Minkowski inequality for convex bodies (see also Xi, Jin, Leng [15]).

THEOREM 1. *Let $\varphi \in \Phi_2$. If K, L are compact sets in $\mathbb{R}^n$ with $V(K)V(L) > 0$, then*

$$\varphi\Big(\Big(\frac{V(K)}{V(K+_\varphi L)}\Big)^{1/n}, \Big(\frac{V(L)}{V(K+_\varphi L)}\Big)^{1/n}\Big) \leqslant 1. \tag{8}$$

When φ is strictly convex, equality holds if and only if K, L are convex bodies containing the origin in their interior and are dilatates of each other.

When $\varphi(x_1,x_2)=x_1^p+x_2^p$ for $p\geqslant 1$, Orlicz addition (6) reduce to L_p-addition (2) and hence (8) yields (3).

The Orlicz-Brunn-Minkowski inequality for star bodies was established by Gardner, Hug, Weil and Ye [4].

THEOREM 2. *Let $\varphi\in\widetilde{\Phi}_2\cup\widetilde{\Psi}_2$ and let K,L be star bodies with respect to the origin. If $\varphi_0(x_1,x_2)=\varphi(x_1^{1/n},x_2^{1/n})$ is concave then*

$$\varphi\Big(\Big(\frac{V(K)}{V(K\widetilde{+}_\varphi L)}\Big)^{1/n},\Big(\frac{V(L)}{V(K\widetilde{+}_\varphi L)}\Big)^{1/n}\Big)\geqslant 1. \tag{9}$$

If φ_0 is convex, then the reverse inequality holds.

When φ_0 is strictly concave (or convex, as appropriate), equality holds if and only if K,L are dilatates.

When $\varphi(x_1,x_2)=x_1^p+x_2^p$ for $p\in\mathbb{R}\backslash\{0\}$, radial Orlicz addition (7) reduce to L_p-radial addition (4) and hence (9) yields (5).

The purpose of this article is to establish the following Orlicz-Brunn-Minkowski inequality for polar bodies and dual star bodies.

THEOREM 3. *Let $\varphi\in\Phi_2$. If K,L are convex bodies in $\mathbb{R}^n$ containing the origin in their interior, then*

$$\varphi\Big(\Big(\frac{V(K^*)}{V([K+_\varphi L]^*)}\Big)^{-1/n},\Big(\frac{V(L^*)}{V([K+_\varphi L]^*)}\Big)^{-1/n}\Big)\leqslant 1. \tag{10}$$

When φ is strictly convex, equality holds if and only if K,L are dilatates.

Here K^* denotes the polar set of the convex body K. Taking $\varphi(x_1,x_2)=x_1^p+x_2^p$ for $p\geqslant 1$, (10) yields the following L_p-Brunn-Minkowski inequality for polar bodies due to Hernández Cifre and Yepes Nicolás [6]: if $p\geqslant 1$, and K,L are convex bodies in $\mathbb{R}^n$ containing the origin in their interior, then

$$V([K+_p L]^*)^{-p/n}\geqslant V(K^*)^{-p/n}+V(L^*)^{-p/n}, \tag{11}$$

with equality if and only if K and L are dilatates. This inequality for $p=1$ was obtained by Firey [1] in 1961. Moreover, Saroglou [14] recently established this inequality for $p\geqslant 0$.

THEOREM 4. *Let $\varphi\in\widetilde{\Phi}_2\cup\widetilde{\Psi}_2$ and let K,L be star bodies with respect to the origin. If $\psi_0(x_1,x_2)=\varphi(x_1^{-1/n},x_2^{-1/n})$ is concave then*

$$\varphi\Big(\Big(\frac{V(K^o)}{V([K\widetilde{+}_\varphi L]^o)}\Big)^{-1/n},\Big(\frac{V(L^o)}{V([K\widetilde{+}_\varphi L]^o)}\Big)^{-1/n}\Big)\geqslant 1.$$

If ψ_0 is convex, then the reverse inequality holds.

When ψ_0 is strictly concave (or convex, as appropriate), equality holds if and only if K,L are dilatates.

Here K^o denotes the dual star body of the body K. Taking $\varphi(x_1,x_2) = x_1^p + x_2^p$ for $p \in \mathbb{R}\backslash\{0\}$, we get the L_p-Brunn-Minkowski inequality for dual star bodies:

COROLLARY 1. *If K, L are star bodies with respect to the origin, then, for $-n \leqslant p < 0$,*

$$V([K\widetilde{+}_p L]^o)^{-p/n} \leqslant V(K^o)^{-p/n} + V(L^o)^{-p/n}.$$

The reverse inequality holds when either $p < -n$ or $p > 0$. Equality holds when $p \neq -n$ if and only if K, L are dilatates.

2. Proof of the main results

A convex body is a compact convex set of $\mathbb{R}^n$ with nonempty interior. For a convex body K, the support function $h_K(\cdot) : \mathbb{R}^n \to \mathbb{R}$ is defined by $h_K(x) = \sup\{x \cdot y : y \in K\}$, where $x \cdot y$ denotes the standard inner product of x and y in $\mathbb{R}^n$.

A compact set $K \subset \mathbb{R}^n$ is a star-shaped set (with respect to the origin) if the intersection of every straight line through the origin with K is a line segment. Given a compact star-shaped set $K \subset \mathbb{R}^n$ (with respect to the origin), the radial function $\rho_K(\cdot) : \mathbb{R}^n\backslash\{o\} \to \mathbb{R}$ is defined by $\rho_K(x) = \max\{\lambda \geqslant 0 : \lambda x \in K\}$. If ρ_K is strictly positive and continuous, then we call K a star body (with respect to the origin).

The polar set K^* of a convex body K containing the origin in its interior is the convex body defined by

$$K^* = \{x \in \mathbb{R}^n : x \cdot y \leqslant 1 \text{ for all } y \in K\}.$$

In this case, for every $x \in \mathbb{R}^n\backslash\{o\}$,

$$h_{K^*}(x) = \frac{1}{\rho_K(x)}. \tag{12}$$

The possible way to define the 'polar' body of a star body K was provided by Moszyńska [12] (see also [13]). Let $i : \mathbb{R}^n\backslash\{o\} \to \mathbb{R}^n\backslash\{o\}$ be defined by

$$i(x) := \frac{x}{|x|^2}.$$

Moszyńska [12] introduced the dual star body K^o of a star body K as

$$K^o = \mathrm{cl}(\mathbb{R}^n\backslash i(K)),$$

where cl denotes the closure of the given set. It is easy to verify that for every $u \in S^{n-1}$ (see [12]),

$$\rho_{K^o}(u) = \frac{1}{\rho_K(u)}. \tag{13}$$

In particular, if K is a convex body in $\mathbb{R}^n$ that contains the origin in its interior, then

$$K^* \subset K^o$$

and $K^* = K^o$ if and only if K is a centered ball (see [12]).

After these preparations, we now prove our main results by using Theorem 2.

Proof of Theorem 3. Let $\psi(x_1,x_2) = \varphi(x_1^{-1},x_2^{-1})$. It follows from $\varphi \in \Phi_2$ that ψ is convex and strictly decreasing in each component, and furthermore $\psi \in \widetilde{\Phi}_2 \cup \widetilde{\Psi}_2$. Consequently, $\psi_0(x_1,x_2) = \psi(x_1^{1/n},x_2^{1/n})$ is convex. On the other hand, by (6) and (12), we have

$$
\begin{aligned}
1 &= \varphi\Big(\frac{h_K(x)}{h_{K+_\varphi L}(x)},\frac{h_L(x)}{h_{K+_\varphi L}(x)}\Big)\\
&= \psi\Big(\frac{h_{K+_\varphi L}(x)}{h_K(x)},\frac{h_{K+_\varphi L}(x)}{h_L(x)}\Big) = \psi\Big(\frac{\rho_{K^*}(x)}{\rho_{[K+_\varphi L]^*}(x)},\frac{\rho_{L^*}(x)}{\rho_{[K+_\varphi L]^*}(x)}\Big),
\end{aligned}
$$

for $x \in \mathbb{R}^n \backslash \{o\}$. Then, it follows from the definition of the radial Orlicz addition (7) that

$$
[K+_\varphi L]^* = K^* \widetilde{+}_\psi L^*. \tag{14}
$$

Using Theorem 2 with ψ, K^*, L^* in the place of φ, K, L, respectively, we immediately get

$$
\begin{aligned}
1 &\geqslant \psi\Big(\Big(\frac{V(K^*)}{V(K^*\widetilde{+}_\psi L^*)}\Big)^{1/n},\Big(\frac{V(L^*)}{V(K^*\widetilde{+}_\psi L^*)}\Big)^{1/n}\Big)\\
&= \varphi\Big(\Big(\frac{V(K^*)}{V([K+_\varphi L]^*)}\Big)^{-1/n},\Big(\frac{V(L^*)}{V([K+_\varphi L]^*)}\Big)^{-1/n}\Big).
\end{aligned}
$$

The equality case follows from the equality case of Theorem 2. □

For the L_p-case, relation (14) can be interpreted as $[K+_p L]^* = K^* \widetilde{+}_{-p} L^*$ for $p \geqslant 1$, and hence inequality (11) can be deduced from (5).

We shall mention that another proof of Theorem 3 can be obtained with the approach followed in Section 7 of [3] together with (11) for $p = 1$.

Proof of Theorem 4. Without loss of generality, we may consider the case in which $\varphi \in \widetilde{\Phi}_2$ and $\psi_0(x_1,x_2) = \varphi(x_1^{-1/n},x_2^{-1/n})$ is concave. Then $\psi(x_1,x_2) = \varphi(x_1^{-1},x_2^{-1}) \in \widetilde{\Psi}_2$. On the other hand, by (7), (13) and the fact that the radial functions are homogeneous of degree -1, we have

$$
\begin{aligned}
1 &= \varphi\Big(\frac{\rho_K(x)}{\rho_{K\widetilde{+}_\varphi L}(x)},\frac{\rho_L(x)}{\rho_{K\widetilde{+}_\varphi L}(x)}\Big) = \varphi\Big(\frac{\rho_K(u)}{\rho_{K\widetilde{+}_\varphi L}(u)},\frac{\rho_L(u)}{\rho_{K\widetilde{+}_\varphi L}(u)}\Big)\\
&= \psi\Big(\frac{\rho_{K\widetilde{+}_\varphi L}(u)}{\rho_K(u)},\frac{\rho_{K\widetilde{+}_\varphi L}(u)}{\rho_L(u)}\Big)\\
&= \psi\Big(\frac{\rho_{K^o}(u)}{\rho_{[K\widetilde{+}_\varphi L]^o}(u)},\frac{\rho_{L^o}(u)}{\rho_{[K\widetilde{+}_\varphi L]^o}(u)}\Big) = \psi\Big(\frac{\rho_{K^o}(x)}{\rho_{[K\widetilde{+}_\varphi L]^o}(x)},\frac{\rho_{L^o}(x)}{\rho_{[K\widetilde{+}_\varphi L]^o}(x)}\Big),
\end{aligned}
$$

for $x = ru$ in polar coordinates. Then, it follows from the definition of the radial Orlicz addition (7) that

$$[K\widetilde{+}_\varphi L]^o = K^o\widetilde{+}_\psi L^o.$$

Using Theorem 2 with ψ, K^o, L^o in the place of φ, K, L, respectively, we immediately get

$$\begin{aligned} 1 &\leqslant \psi\Big(\Big(\frac{V(K^o)}{V(K^o\widetilde{+}_\psi L^o)}\Big)^{1/n}, \Big(\frac{V(L^o)}{V(K^o\widetilde{+}_\psi L^o)}\Big)^{1/n}\Big) \\ &= \varphi\Big(\Big(\frac{V(K^o)}{V([K\widetilde{+}_\varphi L]^o)}\Big)^{-1/n}, \Big(\frac{V(L^o)}{V([K\widetilde{+}_\varphi L]^o)}\Big)^{-1/n}\Big). \end{aligned}$$

The equality case follows from the equality case of Theorem 2. □

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