Mathematical
Inequalities
& Applications
Volume 20, Number 4 (2017), 1159–1182 doi:10.7153/mia-2017-20-75

PROOFS OF CERTAIN CONJECTURES OF VUKŠIĆ CONCERNING THE INEQUALITIES FOR MEANS

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(*Communicated by S. Varošanec*)

Abstract. By using the asymptotic expansion method, Vukšić conjectured inequalities between Seiffert means and convex combinations of other means. In this paper, we prove certain conjectures given by Vukšić.

1. Introduction

For $x,y>0$ with $x\neq y$, the first and second Seiffert means $P(x,y)$ and $T(x,y)$ are defined in [16] and [17], respectively by

$$P(x,y)=\frac{x-y}{2\arcsin\frac{x-y}{x+y}}\quad\text{and}\quad T(x,y)=\frac{x-y}{2\arctan\frac{x-y}{x+y}}.$$

In what follows we will assume that the numbers x and y are positive and unequal. Let

$$H=\frac{2xy}{x+y},\ \ G=\sqrt{xy},\ \ L=\frac{x-y}{\ln x-\ln y},\ \ A=\frac{x+y}{2},\ \ Q=\sqrt{\frac{x^2+y^2}{2}},\ \ N=\frac{x^2+y^2}{x+y}$$

be the harmonic, geometric, logarithmic, arithmetic, root-square, and contraharmonic means of x and y, respectively. It is known (see [18]) that

$$H<G<L<P<A<T<Q<N.$$

There is a large number of papers studying inequalities between Seiffert means and convex combinations of other means [5, 6, 7, 14, 15, 18, 19]. For example, Chu et al. [5] established that the double inequality

$$\mu A+(1-\mu)H<P<\nu A+(1-\nu)H$$

holds if and only if $\mu\leqslant 2/\pi$ and $\nu\geqslant 5/6$. Liu and Meng [15] proved that the double inequality

$$(1-\mu)G+\mu N<P<(1-\nu)G+\nu N$$

Mathematics subject classification (2010): 26E60, 26D05.
Keywords and phrases: Seiffert means, inequality.

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Paper MIA-20-75

holds if and only if $\mu \leqslant 2/9$ and $\nu \geqslant 1/\pi$. Chu et al. [6] proved that the double inequality

$$\mu Q+(1-\mu)A<T<\nu Q+(1-\nu)A \tag{1.1}$$

holds if and only if $\mu \leqslant (4-\pi)/\big(\pi(\sqrt{2}-1)\big)$ and $\nu \geqslant 2/3$. The inequality (1.1) was also proved by Witkowski [19].

Recently, Vukšić [18], by using the asymptotic expansion method, gave a systematic study of inequalities of the form

$$(1-\mu)M_1+\mu M_3<M_2<(1-\nu)M_1+\nu M_3,$$

where M_j are chosen from the class of elementary means given above. For example, Vukšić [18, Theorem 3.5, (3.15)] proved the double inequality

$$(1-\mu)H+\mu N<T<(1-\nu)H+\nu N$$

holds if and only if $\mu \leqslant 2/\pi$ and $\nu \geqslant 1/3$. See [4, 9, 10, 11, 12, 13] for more details about comparison of means using asymptotic methods. Also Vukšić [18] has conjectured certain inequalities related to the first and second Seiffert means $P(x,y)$ and $T(x,y)$.

CONJECTURE 1.1. ([18, Conjecture 3.4]) The following double inequalities hold true with the best possible parameters:

$$\frac{\pi-2}{\pi}G+\frac{2}{\pi}A<P<\frac{1}{3}G+\frac{2}{3}A, \tag{1.2}$$

$$\frac{2}{3}G+\frac{1}{3}Q<P<\frac{\pi-\sqrt{2}}{\pi}G+\frac{\sqrt{2}}{\pi}Q, \tag{1.3}$$

$$\frac{3}{4}P+\frac{1}{4}Q<A<\frac{(\sqrt{2}-1)\pi}{\sqrt{2}\pi-2}P+\frac{\pi-2}{\sqrt{2}\pi-2}Q, \tag{1.4}$$

$$\frac{4}{5}L+\frac{1}{5}Q<P<\frac{\pi-\sqrt{2}}{\pi}L+\frac{\sqrt{2}}{\pi}Q, \tag{1.5}$$

$$\frac{7}{8}L+\frac{1}{8}N<P<\frac{\pi-1}{\pi}L+\frac{1}{\pi}N. \tag{1.6}$$

CONJECTURE 1.2. ([18, Conjecture 3.6]) The following double inequalities hold true with the best possible parameters:

$$\frac{1}{4}H+\frac{3}{4}T<A<\frac{4-\pi}{4}H+\frac{\pi}{4}T, \tag{1.7}$$

$$\frac{1}{9}H+\frac{8}{9}Q<T<\frac{\pi-2\sqrt{2}}{\pi}H+\frac{2\sqrt{2}}{\pi}Q, \tag{1.8}$$

$$\frac{\pi-2}{\pi}H+\frac{2}{\pi}N<T<\frac{1}{3}H+\frac{2}{3}N, \tag{1.9}$$

$$\frac{1}{6}G+\frac{5}{6}Q<T<\frac{\pi-2\sqrt{2}}{\pi}G+\frac{2\sqrt{2}}{\pi}Q, \tag{1.10}$$

$$\frac{1}{2}L+\frac{1}{2}T<A<\frac{4-\pi}{4}L+\frac{\pi}{4}T, \tag{1.11}$$

$$\frac{1}{5}L+\frac{4}{5}Q<T<\frac{\pi-2\sqrt{2}}{\pi}L+\frac{2\sqrt{2}}{\pi}Q, \tag{1.12}$$

$$\frac{2\pi-4}{\pi}A+\frac{4-\pi}{\pi}N<T<\frac{2}{3}A+\frac{1}{3}N, \tag{1.13}$$

$$\frac{(2-\sqrt{2})\pi}{2\pi-4}T+\frac{\sqrt{2}\pi-4}{2\pi-4}N<Q<\frac{3}{4}T+\frac{1}{4}N. \tag{1.14}$$

Note that the formulae (1.12) and (1.13) in the original paper [18] contain a typo,which has been corrected here.

The aim of this paper is to offer a proof of these inequalities.

REMARK 1.1. Let $(x-y)/(x+y)=z$, and suppose $x>y$. Then $z\in(0,1)$, and the following identities hold true:

$$\frac{P(x,y)}{A(x,y)}=\frac{z}{\arcsin z},\quad \frac{T(x,y)}{A(x,y)}=\frac{z}{\arctan z},\quad \frac{H(x,y)}{A(x,y)}=1-z^2,\quad \frac{G(x,y)}{A(x,y)}=\sqrt{1-z^2},$$
$$\frac{L(x,y)}{A(x,y)}=\frac{2z}{\ln\frac{1+z}{1-z}},\quad \frac{Q(x,y)}{A(x,y)}=\sqrt{1+z^2},\quad \frac{N(x,y)}{A(x,y)}=1+z^2.$$

The following elementary power series expansions are useful in our investigation.

$$\sin x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!},\qquad |x|<\infty, \tag{1.15}$$

$$\cos x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!},\qquad |x|<\infty, \tag{1.16}$$

$$\tan x=\sum_{n=1}^{\infty}\frac{2^{2n}(2^{2n}-1)|B_{2n}|}{(2n)!}x^{2n-1},\qquad |x|<\frac{\pi}{2}, \tag{1.17}$$

$$\cot x=\frac{1}{x}-\sum_{n=1}^{\infty}\frac{2^{2n}|B_{2n}|}{(2n)!}x^{2n-1},\qquad 0<|x|<\pi, \tag{1.18}$$

$$\csc x=\frac{1}{x}+\sum_{n=1}^{\infty}\frac{2(2^{2n-1}-1)|B_{2n}|}{(2n)!}x^{2n-1},\qquad 0<|x|<\pi, \tag{1.19}$$

where B_n $(n = 0, 1, 2, \ldots)$ are Bernoulli numbers, defined by

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}.$$

The following lemma is also needed in the sequel.

LEMMA 1.1. ([2, 3]) *Let* $-\infty < a < b < \infty$, *and let* $f,\ g : [a,b] \to \mathbb{R}$ *be continuous on* $[a,b]$, *differentiable on* (a,b). *Let* $g'(x) \neq 0$ *on* (a,b). *If* $f'(x)/g'(x)$ *is increasing (decreasing) on* (a,b), *then so are*

$$\frac{f(x) - f(a)}{g(x) - g(a)} \quad \textit{and} \quad \frac{f(x) - f(b)}{g(x) - g(b)}.$$

If $f'(x)/g'(x)$ *is strictly monotone, then the monotonicity in the conclusion is also strict.*

The numerical values given in this paper have been calculated via the computer program MAPLE 13.

2. Proof of Conjecture 1.1

The inequalities (1.2) have been proved in [19]. We here provide an alternative proof.

THEOREM 2.1. *The following double inequality hold:*

$$\frac{\pi - 2}{\pi} G + \frac{2}{\pi} A < P < \frac{1}{3} G + \frac{2}{3} A. \tag{2.1}$$

Proof. By Remark 1.1, (2.1) may be rewritten as

$$\frac{2}{\pi} < \frac{\frac{z}{\arcsin z} - \sqrt{1 - z^2}}{1 - \sqrt{1 - z^2}} < \frac{2}{3}, \qquad 0 < z < 1. \tag{2.2}$$

By an elementary change of variable $z = \sin x$ $(0 < x < \pi/2)$, (2.2) becomes

$$\frac{2}{\pi} < \frac{\frac{\sin x}{x} - \cos x}{1 - \cos x} < \frac{2}{3}, \qquad 0 < x < \frac{\pi}{2}. \tag{2.3}$$

For $0 \leqslant x \leqslant \pi/2$, let

$$f_1(x) = \begin{cases} \dfrac{\sin x}{x} - \cos x, & x \neq 0 \\ 0, & x = 0, \end{cases} \qquad f_2(x) = 1 - \cos x,$$

and let

$$f(x) = \frac{f_1(x)}{f_2(x)} = \frac{\frac{\sin x}{x} - \cos x}{1 - \cos x}, \qquad 0 < x < \frac{\pi}{2}. \tag{2.4}$$

Then,

$$\frac{f_1'(x)}{f_2'(x)} = \frac{\frac{\cos x}{x} - \frac{\sin x}{x^2} + \sin x}{\sin x} = \frac{x\cot x - 1 + x^2}{x^2} =: f_3(x).$$

Using (1.18), we find

$$f_3(x) = \frac{2}{3} - \sum_{n=2}^{\infty} \frac{2^{2n}|B_{2n}|}{(2n)!} x^{2n-2}.$$

Differentiation yields

$$f_3'(x) = -\sum_{n=2}^{\infty} \frac{(2n-2)2^{2n}|B_{2n}|}{(2n)!} x^{2n-3} < 0.$$

Therefore, the functions $f_3(x)$ and $f_1'(x)/f_2'(x)$ are strictly decreasing on $(0,\pi/2)$. By Lemma 1.1, the function

$$f(x) = \frac{f_1(x)}{f_2(x)} = \frac{f_1(x) - f_1(0)}{f_2(x) - f_2(0)}$$

is strictly decreasing on $(0,\pi/2)$, and we have

$$\frac{2}{\pi} = f\left(\frac{\pi}{2}\right) < f(x) = \frac{\frac{\sin x}{x} - \cos x}{1 - \cos x} < \lim_{t\to 0^+} f(t) = \frac{2}{3}$$

for $0 < x < \pi/2$. The proof is complete. □

REMARK 2.1. Let $f(x)$ be given in (2.4). By the monotonicity property of $f(x)$, we here provide a proof of (1.1).

By Remark 1.1, (1.1) may be written as

$$\mu < \frac{\frac{z}{\arctan z} - 1}{\sqrt{1+z^2} - 1} < \nu, \qquad 0 < z < 1.$$

By an elementary change of variable $z = \tan x$ $(0 < x < \pi/4)$, we find

$$\mu < \frac{\frac{\tan x}{x} - 1}{\sec x - 1} = \frac{\frac{\sin x}{x} - \cos x}{1 - \cos x} = f(x) < \nu, \qquad 0 < x < \frac{\pi}{4}.$$

Since $f(x)$ is strictly decreasing on $(0,\pi/4)$, we obtain, for $0 < x < \pi/4$,

$$\frac{4-\pi}{(\sqrt{2}-1)\pi} = f\left(\frac{\pi}{4}\right) < f(x) = \frac{\frac{\tan x}{x} - 1}{\sec x - 1} < \lim_{t\to 0^+} f(t) = \frac{2}{3}.$$

Hence, (1.1) holds if and only if $\mu \leqslant (4-\pi)/\big(\pi(\sqrt{2}-1)\big)$ and $\nu \geqslant 2/3$.

THEOREM 2.2. *The following double inequalities hold true:*

$$\frac{2}{3}G+\frac{1}{3}Q<P<\frac{\pi-\sqrt{2}}{\pi}G+\frac{\sqrt{2}}{\pi}Q \tag{2.5}$$

and

$$\frac{3}{4}P+\frac{1}{4}Q<A<\frac{(\sqrt{2}-1)\pi}{\sqrt{2}\pi-2}P+\frac{\pi-2}{\sqrt{2}\pi-2}Q. \tag{2.6}$$

Proof. By Remark 1.1, (2.5) and (2.6) may be written for $0<z<1$ as

$$\frac{1}{3}<\frac{\frac{z}{\arcsin z}-\sqrt{1-z^2}}{\sqrt{1+z^2}-\sqrt{1-z^2}}<\frac{\sqrt{2}}{\pi} \quad \text{and} \quad \frac{1}{4}<\frac{1-\frac{z}{\arcsin z}}{\sqrt{1+z^2}-\frac{z}{\arcsin z}}<\frac{\pi-2}{\sqrt{2}\pi-2},$$

respectively. By an elementary change of variable $z=\sin x$ $(0<x<\pi/2)$, these two inequalities become

$$\frac{1}{3}<F(x)<\frac{\sqrt{2}}{\pi} \quad \text{and} \quad \frac{1}{4}<H(x)<\frac{\pi-2}{\sqrt{2}\pi-2} \quad \text{for} \quad 0<x<\frac{\pi}{2},$$

where

$$F(x)=\frac{\frac{\sin x}{x}-\cos x}{\sqrt{1+\sin^2 x}-\cos x} \quad \text{and} \quad H(x)=\frac{1-\frac{\sin x}{x}}{\sqrt{1+\sin^2 x}-\frac{\sin x}{x}}.$$

Elementary calculations reveal that

$$\lim_{x\to 0^+}F(x)=\frac{1}{3}, \quad F\left(\frac{\pi}{2}\right)=\frac{\sqrt{2}}{\pi}, \quad \lim_{x\to 0^+}H(x)=\frac{1}{4}, \quad H\left(\frac{\pi}{2}\right)=\frac{\pi-2}{\sqrt{2}\pi-2}.$$

In order prove (2.5) and (2.6), it suffices to show that $F(x)$ and $H(x)$ are both strictly increasing for $0<x<\pi/2$.

Differentiation yields

$$\begin{aligned}
&2x^2\cos x\sqrt{1+\sin^2 x}\left(\sqrt{1+\tan^2 x}-\sqrt{1+\sin^2 x}\right)F'(x)\\
&=x\cos x+\sin x\cos^2 x+(2x^2-2)\sin x-(x-\sin x\cos x)\sqrt{1+\sin^2 x}\\
&>x\cos x+\sin x\cos^2 x+(2x^2-2)\sin x-(x-\sin x\cos x)\left(1+\frac{1}{2}\sin^2 x\right)\\
&=(2x^2-2)\sin x+\sin x\cos^2 x-\frac{1}{2}\sin x\cos^3 x+\frac{3}{4}\sin(2x)+x\cos x+\frac{1}{2}x\cos^2 x-\frac{3}{2}x\\
&=\left(2x^2-\frac{7}{4}\right)\sin x+\frac{5}{8}\sin(2x)+\frac{1}{4}\sin(3x)-\frac{1}{16}\sin(4x)+x\cos x+\frac{1}{4}x\cos(2x)-\frac{5}{4}x\\
&=\frac{13}{180}x^7-\frac{223}{7560}x^9+\frac{1621}{302400}x^{11}-\frac{5189}{8553600}x^{13}+\sum_{n=7}^{\infty}(-1)^{n-1}u_n(x),
\end{aligned}$$

where

$$u_n(x) = \frac{16^n - 3\cdot 9^n - (2n+6)4^n + 32n^2 + 8n + 3}{4\cdot(2n+1)!}x^{2n+1}.$$

Noting that $\frac{1}{2}x^2 < \frac{1}{2}(\frac{\pi}{2})^2 < 2$ holds for $0 < x < \pi/2$, we find that for $0 < x < \pi/2$ and $n \geqslant 7$,

$$\begin{aligned}\frac{u_{n+1}(x)}{u_n(x)} &= \frac{\frac{1}{2}x^2\Big(16\cdot 16^n - 27\cdot 9^n - (8n+32)4^n + 32n^2 + 72n + 43\Big)}{(n+1)(2n+3)\Big(16^n - 3\cdot 9^n - (2n+6)4^n + 32n^2 + 8n + 3\Big)}\\ &< \frac{2\Big(16\cdot 16^n + 32n^2 + 72n + 43\Big)}{(n+1)(2n+3)\Big(16^n - 3\cdot 9^n - (2n+6)4^n\Big)}\\ &= \frac{2(16+R_n)}{(n+1)(2n+3)(1-S_n)},\end{aligned}$$

where

$$R_n = \frac{32n^2+72n+43}{16^n} \quad \text{and} \quad S_n = 3\left(\frac{9}{16}\right)^n + (2n+6)\left(\frac{4}{16}\right)^n.$$

Noting that the sequence $\{R_n\}$ and $\{S_n\}$ are both strictly decreasing for $n \geqslant 7$, we have, for $n \geqslant 7$,

$$0 < R_n \leqslant R_7 = \frac{2115}{268435456} \quad \text{and} \quad 0 < S_n \leqslant S_7 = \frac{14676587}{268435456}.$$

We then obtain that for $0 < x < \pi/2$ and $n \geqslant 7$,

$$\frac{u_{n+1}(x)}{u_n(x)} < \frac{2\left(16 + \frac{2115}{268435456}\right)}{(n+1)(2n+3)\left(1 - \frac{14676587}{268435456}\right)} < 1.$$

Therefore, for fixed $x \in (0, \pi/2)$, the sequence $n \longmapsto u_n(x)$ is strictly decreasing for $n \geqslant 7$. We then obtain that for $0 < x < \pi/2$,

$$\begin{aligned}&2x^2\cos x\sqrt{1+\sin^2 x}\left(\sqrt{1+\tan^2 x} - \sqrt{1+\sin^2 x}\right)F'(x)\\ &\qquad > x^7\left(\frac{13}{180} - \frac{223}{7560}x^2 + \frac{1621}{302400}x^4 - \frac{5189}{8553600}x^6\right) > 0.\end{aligned}$$

Hence, $F(x)$ is strictly increasing for $0 < x < \pi/2$.

Differentiation yields

$$\begin{aligned}\frac{\sqrt{1+\sin^2 x}\left(x\sqrt{1+\sin^2 x} - \sin x\right)^2}{\sin x - x\cos x}H'(x) &= 1 + \frac{\sin x(\sin^2 x - x^2\cos x)}{\sin x - x\cos x} - \sqrt{1+\sin^2 x}\\ &> 1 + \frac{\sin x(\sin^2 x - x^2\cos x)}{\sin x - x\cos x} - \left(1 + \frac{1}{2}\sin^2 x\right) = \frac{\tan x H_1(x)}{2(\tan x - x)},\end{aligned}$$

with

$$H_1(x) = \sin^2 x + x\sin x\cos x - 2x^2\cos x = \frac{17}{180}x^6 - \frac{11}{840}x^8 + \sum_{n=5}^{\infty}(-1)^{n-1}P_n(x),$$

where

$$P_n(x) = \frac{(n+1)4^n - 16n^2 + 8n}{2\cdot(2n)!}x^{2n}.$$

Noting that $2x^2 < 2(\pi/2)^2 < 5$ holds for $0 < x < \pi/2$, we find that for $0 < x < \pi/2$ and $n \geqslant 5$,

$$\begin{aligned}\frac{P_{n+1}(x)}{P_n(x)} &= \frac{2x^2\Big((n+2)4^n - 2(n+1)(2n+1)\Big)}{(2n+1)(n+1)\Big((n+1)4^n - 8n(2n-1)\Big)} \\ &< \frac{5(n+2)4^n}{(2n+1)(n+1)\Big((n+1)4^n - 8n(2n-1)\Big)} \\ &= \frac{5(n+2)}{(2n+1)(n+1)\Big((n+1) - Q_n\Big)},\end{aligned}$$

where

$$Q_n = \frac{8n(2n-1)}{4^n}.$$

Noting that the sequence $\{Q_n\}$ is strictly decreasing for $n \geqslant 5$, we have

$$0 < Q_n \leqslant Q_5 = \frac{45}{128}, \qquad n \geqslant 5.$$

We then obtain that for $0 < x < \pi/2$ and $n \geqslant 5$,

$$\frac{P_{n+1}(x)}{P_n(x)} < \frac{5(n+2)}{(2n+1)(n+1)\Big((n+1) - \frac{45}{128}\Big)} < 1.$$

Therefore, for fixed $x \in (0, \pi/2)$, the sequence $n \longmapsto P_n(x)$ is strictly decreasing for $n \geqslant 5$. We then obtain that, for $0 < x < \pi/2$,

$$H_1(x) > x^6\left(\frac{17}{180} - \frac{11}{840}x^2\right) > 0 \quad \text{and} \quad H'(x) > 0.$$

So, $H(x)$ is strictly increasing for $0 < x < \pi/2$. The proof is complete. □

THEOREM 2.3. *The inequalities*

$$(1-\mu_1)L + \mu_1 Q < P < (1-\nu_1)L + \nu_1 Q \tag{2.7}$$

and

$$(1-\mu_2)L+\mu_2 N < P < (1-\nu_2)L+\nu_2 N \tag{2.8}$$

hold if and only if

$$\mu_1 \leqslant \frac{1}{5}, \quad \nu_1 \geqslant \frac{\sqrt{2}}{\pi}, \quad \mu_2 \leqslant \frac{1}{8}, \quad \nu_2 \geqslant \frac{1}{\pi}. \tag{2.9}$$

Proof. We first prove (2.7) and (2.8) with $\mu_1 = \frac{1}{5}, \nu_1 = \frac{\sqrt{2}}{\pi}, \mu_2 = \frac{1}{8}, \nu_2 = \frac{1}{\pi}$, namely,

$$\frac{4}{5}L+\frac{1}{5}Q < P < \left(1-\frac{\sqrt{2}}{\pi}\right)L+\frac{\sqrt{2}}{\pi}Q \tag{2.10}$$

and

$$\frac{7}{8}L+\frac{1}{8}N < P < \left(1-\frac{1}{\pi}\right)L+\frac{1}{\pi}N. \tag{2.11}$$

We claim that

$$\left(1-\frac{\sqrt{2}}{\pi}\right)G+\frac{\sqrt{2}}{\pi}Q < \left(1-\frac{\sqrt{2}}{\pi}\right)L+\frac{\sqrt{2}}{\pi}Q < \left(1-\frac{1}{\pi}\right)L+\frac{1}{\pi}N. \tag{2.12}$$

This claim shows that, among the second inequalities in (2.5), (2.10) and (2.11), the upper bound

$$\left(1-\frac{\sqrt{2}}{\pi}\right)G+\frac{\sqrt{2}}{\pi}Q$$

is the best, in the sense that it is the smallest one among the three upper bounds in (2.5), (2.10) and (2.11).

Obvious, the left-hand side of (2.12) holds. We now prove the right-hand side of (2.12). Noting that $G < L$ holds, we have

$$\begin{aligned}&\left(1-\frac{1}{\pi}\right)L+\frac{1}{\pi}N-\left\{\left(1-\frac{\sqrt{2}}{\pi}\right)L+\frac{\sqrt{2}}{\pi}Q\right\}\\&\quad=\frac{1}{\pi}\left\{(\sqrt{2}-1)L+N-\sqrt{2}Q\right\} > \frac{1}{\pi}\left\{(\sqrt{2}-1)G+N-\sqrt{2}Q\right\}.\end{aligned}$$

In order prove the right-hand side of (2.12), it suffices to show that

$$(\sqrt{2}-1)G+N > \sqrt{2}Q,$$

which can be written, by Remark 1.1, as

$$(\sqrt{2}-1)\sqrt{1-z^2}+(1+z^2) > \sqrt{2}\sqrt{1+z^2}, \qquad 0<z<1,$$

i.e.,

$$(\sqrt{2}-1)\sqrt{1-t}+(1+t)>\sqrt{2}\sqrt{1+t}, \qquad 0<t<1. \tag{2.13}$$

We find

$$\begin{aligned}&\left((\sqrt{2}-1)\sqrt{1-t}+(1+t)\right)^2-\left(\sqrt{2}\sqrt{1+t}\right)^2\\&=2(\sqrt{2}-1)(1+t)\sqrt{1-t}-(2\sqrt{2}-2+t)(1-t)\end{aligned}$$

and

$$\begin{aligned}&\left(2(\sqrt{2}-1)(1+t)\sqrt{1-t}\right)^2-\left((2\sqrt{2}-2+t)(1-t)\right)^2\\&=t(1-t)\left\{t^2+(7-4\sqrt{2})t+40-28\sqrt{2}\right\}>0 \quad \text{for} \quad 0<t<1.\end{aligned}$$

Hence, (2.13) holds. The claim (2.12) is proved.

By Remark 1.1, the first inequalities in (2.10) and (2.11) can be written for $0<z<1$ as

$$\frac{4}{5}\frac{2z}{\ln\frac{1+z}{1-z}}+\frac{1}{5}\sqrt{1+z^2}<\frac{z}{\arcsin z} \tag{2.14}$$

and

$$\frac{7}{8}\frac{2z}{\ln\frac{1+z}{1-z}}+\frac{1}{8}(1+z^2)<\frac{z}{\arcsin z}, \tag{2.15}$$

respectively.

We first prove (2.14) for $0<z<0.7$. From the well known continued fraction for $\ln\frac{1+x}{1-x}$ (see [8, p. 196, Eq. (11.2.4)]), we find that for $0<x<1$,

$$\frac{2x(15-4x^2)}{3(5-3x^2)}=\cfrac{2x}{1+\cfrac{-\frac{1}{3}x^2}{1+\cfrac{-\frac{4}{15}x^2}{1}}}<\ln\frac{1+x}{1-x}. \tag{2.16}$$

Using (2.16), we have

$$\begin{aligned}\frac{z}{\arcsin z}-\left(\frac{4}{5}\frac{2z}{\ln\frac{1+z}{1-z}}+\frac{1}{5}\sqrt{1+z^2}\right)&>\frac{z}{\arcsin z}-\left\{\frac{4}{5}\frac{3(5-3z^2)}{15-4z^2}+\frac{1}{5}\left(1+\frac{1}{2}z^2\right)\right\}\\&=\frac{z}{\arcsin z}-\frac{150-65z^2-4z^4}{10(15-4z^2)}.\end{aligned}$$

In order to prove (2.14) for $0<z<0.7$, it suffices to show that

$$\theta(z)>0 \quad \text{for} \quad 0<z<0.7,$$

where

$$\theta(z) = \frac{10z(15-4z^2)}{150-65z^2-4z^4} - \arcsin z.$$

Differentiation yields

$$\theta'(z) = \frac{10(2250-825z^2+440z^4-16z^6)}{(150-65z^2-4z^4)^2} - \frac{1}{\sqrt{1-z^2}}.$$

Elementary calculations reveal that, for $0 < z < 0.7$,

$$\begin{aligned}
&\left(\frac{10(2250-825z^2+440z^4-16z^6)}{(150-65z^2-4z^4)^2}\right)^2 - \frac{1}{1-z^2} \\
&\quad = \frac{1}{(1-z^2)\big(150-65z^2-4z^4\big)^4}\Big[120937500-251287500z^2+112209375z^4 \\
&\qquad -25930000z^6+z^8(1066400-42240z^2-256z^4)\Big] > 0.
\end{aligned}$$

We then obtain $\theta'(z) > 0$ for $0 < z < 0.7$. Hence, $\theta(z)$ is strictly increasing for $0 < z < 0.7$, and we have

$$\theta(z) = \frac{10z(15-4z^2)}{150-65z^2-4z^4} - \arcsin z > \theta(0) = 0 \quad \text{for} \quad 0 < z < 0.7.$$

Therefore, (2.14) holds for $0 < z < 0.7$.

Second, we prove (2.14) for $0.7 \leqslant z < 1$. Let

$$\omega(z) = \omega_1(z) + \omega_2(z),$$

where

$$\omega_1(z) = -\left(\frac{4}{5}\frac{2z}{\ln\frac{1+z}{1-z}} + \frac{1}{5}\sqrt{1+z^2}\right) \quad \text{and} \quad \omega_2(z) = \frac{z}{\arcsin z}.$$

Let $0.7 \leqslant r \leqslant z \leqslant s < 1$. Since $\omega_1(z)$ is increasing and $\omega_2(z)$ is decreasing, we obtain

$$\omega(z) \geqslant \omega_1(r) + \omega_2(s) =: \sigma(r,s).$$

We divide the interval $[0.7, 1]$ into 30 subintervals:

$$[0.7, 1] = \bigcup_{k=0}^{29}\left[0.7+\frac{k}{100}, 0.7+\frac{k+1}{100}\right] \quad \text{for} \quad k = 0, 1, 2, \ldots, 29.$$

By direct computation we get

$$\sigma\left(0.7+\frac{k}{100}, 0.7+\frac{k+1}{100}\right) > 0 \quad \text{for} \quad k = 0, 1, 2, \ldots, 29.$$

Hence,

$$\omega(z) > 0 \quad \text{for} \quad z \in \left[0.7 + \frac{k}{100}, 0.7 + \frac{k+1}{100}\right] \quad \text{and} \quad k = 0,1,2,\ldots,29.$$

This implies that $\omega(z)$ is positive on $[0.7, 1)$. This proves (2.14) for $0.7 \leqslant z < 1$. Hence, (2.14) holds for all $0 < z < 1$.

We now prove (2.15). We first prove (2.15) for $0 < z < 0.7$. Using (2.16), we have

$$\frac{z}{\arcsin z} - \left(\frac{7}{8}\frac{2z}{\ln\frac{1+z}{1-z}} + \frac{1}{8}(1+z^2)\right) > \frac{z}{\arcsin z} - \left\{\frac{7}{8}\frac{3(5-3z^2)}{15-4z^2} + \frac{1}{8}(1+z^2)\right\}$$
$$= \frac{z}{\arcsin z} - \frac{30-13z^2-z^4}{2(15-4z^2)}.$$

In order to prove (2.15) for $0 < z < 0.7$, it suffices to show that

$$\Theta(z) > 0 \quad \text{for} \quad 0 < z < 0.7,$$

where

$$\Theta(z) = \frac{2z(15-4z^2)}{30-13z^2-z^4} - \arcsin z.$$

Differentiation yields

$$\Theta'(z) = \frac{2(450-165z^2+97z^4-4z^6)}{(30-13z^2-z^4)^2} - \frac{1}{\sqrt{1-z^2}}.$$

Elementary calculations reveal that, for $0 < z < 0.7$,

$$\left(\frac{2(450-165z^2+97z^4-4z^6)}{(30-13z^2-z^4)^2}\right)^2 - \frac{1}{1-z^2}$$
$$= \frac{(247500-477300z^2)+z^4(212235-50128z^2)+z^8(2274-116z^2-z^4)}{\left(30-13z^2-z^4\right)^4(1-z^2)} > 0.$$

We then obtain $\Theta'(z) > 0$ for $0 < z < 0.7$. Hence, $\Theta(z)$ is strictly increasing for $0 < z < 0.7$, and we have

$$\Theta(z) = \frac{2z(15-4z^2)}{30-13z^2-z^4} - \arcsin z > \Theta(0) = 0 \quad \text{for} \quad 0 < z < 0.7.$$

Therefore, (2.15) holds for $0 < z < 0.7$.

Second, we prove (2.15) for $0.7 \leqslant z < 1$. Let

$$y(z) = y_1(z) + y_2(z),$$

where

$$y_1(z) = -\left(\frac{7}{8}\frac{2z}{\ln\frac{1+z}{1-z}} + \frac{1}{8}(1+z^2)\right) \quad \text{and} \quad y_2(z) = \frac{z}{\arcsin z}.$$

Let $0.7 \leqslant r \leqslant z \leqslant s < 1$. Since $y_1(z)$ is increasing and $y_2(z)$ is decreasing, we obtain

$$y(z) \geqslant y_1(r) + y_2(s) =: \rho(r,s).$$

We divide the interval $[0.7,1]$ into 30 subintervals:

$$[0.7,1] = \bigcup_{k=0}^{29}\left[0.7+\frac{k}{100}, 0.7+\frac{k+1}{100}\right] \quad \text{for} \quad k=0,1,2,\ldots,29.$$

By direct computation we get

$$\rho\left(0.7+\frac{k}{100}, 0.7+\frac{k+1}{100}\right) > 0 \quad \text{for} \quad k=0,1,2,\ldots,29.$$

Hence,

$$y(z) > 0 \quad \text{for} \quad z \in \left[0.7+\frac{k}{100}, 0.7+\frac{k+1}{100}\right] \quad \text{and} \quad k=0,1,2,\ldots,29.$$

This implies that $y(z)$ is positive on $[0.7,1)$. This proves (2.15) for $0.7 \leqslant z < 1$. Hence, (2.15) holds for all $0 < z < 1$.

We then obtain (2.7) and (2.8) with $\mu_1 = \frac{1}{5}$, $\nu_1 = \frac{\sqrt{2}}{\pi}, \mu_2 = \frac{1}{8}, \nu_2 = \frac{1}{\pi}$.

Conversely, if (2.7) and (2.8) are valid, then we get

$$\mu_1 < \frac{P-L}{Q-L} = \frac{\frac{z}{\arcsin z} - \frac{2z}{\ln\frac{1+z}{1-z}}}{\sqrt{1+z^2} - \frac{2z}{\ln\frac{1+z}{1-z}}} < \nu_1 \quad \text{and} \quad \mu_2 < \frac{P-L}{N-L} = \frac{\frac{z}{\arcsin z} - \frac{2z}{\ln\frac{1+z}{1-z}}}{1+z^2 - \frac{2z}{\ln\frac{1+z}{1-z}}} < \nu_2.$$

The limit relations

$$\lim_{z\to 0^+}\frac{\frac{z}{\arcsin z} - \frac{2z}{\ln\frac{1+z}{1-z}}}{\sqrt{1+z^2} - \frac{2z}{\ln\frac{1+z}{1-z}}} = \frac{1}{5}, \quad \lim_{z\to 1^-}\frac{\frac{z}{\arcsin z} - \frac{2z}{\ln\frac{1+z}{1-z}}}{\sqrt{1+z^2} - \frac{2z}{\ln\frac{1+z}{1-z}}} = \frac{\sqrt{2}}{\pi},$$

$$\lim_{z\to 0^+}\frac{\frac{z}{\arcsin z} - \frac{2z}{\ln\frac{1+z}{1-z}}}{1+z^2 - \frac{2z}{\ln\frac{1+z}{1-z}}} = \frac{1}{8}, \quad \lim_{z\to 1^-}\frac{\frac{z}{\arcsin z} - \frac{2z}{\ln\frac{1+z}{1-z}}}{1+z^2 - \frac{2z}{\ln\frac{1+z}{1-z}}} = \frac{1}{\pi}$$

yield

$$\mu_1 \leqslant \frac{1}{5}, \quad \nu_1 \geqslant \frac{\sqrt{2}}{\pi}, \quad \mu_2 \leqslant \frac{1}{8}, \quad \nu_2 \geqslant \frac{1}{\pi}.$$

The proof is complete. □

3. Proof of Conjecture 1.2

THEOREM 3.1. *The following double inequality holds true:*

$$\frac{2\pi-4}{\pi}A+\frac{4-\pi}{\pi}N<T<\frac{2}{3}A+\frac{1}{3}N. \tag{3.1}$$

Proof. By Remark 1.1, (3.1) may be rewritten as

$$\frac{4-\pi}{\pi}<\frac{\frac{z}{\arctan z}-1}{z^2}<\frac{1}{3}\quad \text{for}\quad 0<z<1. \tag{3.2}$$

By an elementary change of variable $z=\tan x\ (0<x<\pi/4)$, (3.2) becomes

$$\frac{4-\pi}{\pi}<U(x)<\frac{1}{3}\quad \text{for}\quad 0<x<\frac{\pi}{4}, \tag{3.3}$$

where

$$U(x)=\frac{\frac{\tan x}{x}-1}{\tan^2 x}.$$

Differentiation yields

$$U'(x)=-\frac{U_1(x)}{x^2\sin^2 x\tan x},$$

where

$$\begin{aligned} U_1(x)&=x\tan x-2x^2+\sin^2 x=x\tan x-\frac{1}{2}\cos(2x)-2x^2+\frac{1}{2}\\ &=\sum_{n=3}^{\infty}\frac{2^{2n-1}\Big(2(2^{2n}-1)|B_{2n}|-(-1)^n\Big)}{(2n)!}x^{2n}. \end{aligned} \tag{3.4}$$

It is well known [1, p. 805] that

$$\frac{2(2n)!}{(2\pi)^{2n}}<|B_{2n}|<\frac{2(2n)!}{(2\pi)^{2n}(1-2^{1-2n})},\qquad n\geqslant 1. \tag{3.5}$$

By the first inequality in (3.5), we find

$$2(2^{2n}-1)|B_{2n}|>2(2^{2n}-1)\frac{2(2n)!}{(2\pi)^{2n}}>1,\qquad n\geqslant 3.$$

We see from (3.4) that

$$U_1(x)>0,\qquad 0<x<\frac{\pi}{4}. \tag{3.6}$$

We then obtain $U'(x)<0$ for $0<x<\pi/4$. Hence, $U(x)$ are strictly decreasing on $(0,\pi/4)$, and we have

$$\frac{4-\pi}{\pi}=U\left(\frac{\pi}{4}\right)<U(x)=\frac{\frac{\tan x}{x}-1}{\tan^2 x}<\lim_{t\to 0^+}U(t)=\frac{1}{3}$$

for $0<x<\pi/4$. The proof is complete. □

REMARK 3.1. Noting that $H+N=2A$ holds, (3.1) can be written as (1.9).

THEOREM 3.2. *The following double inequalities hold true:*

$$\frac{1}{4}H+\frac{3}{4}T<A<\frac{4-\pi}{4}H+\frac{\pi}{4}T, \tag{3.7}$$

$$\frac{1}{9}H+\frac{8}{9}Q<T<\frac{\pi-2\sqrt{2}}{\pi}H+\frac{2\sqrt{2}}{\pi}Q, \tag{3.8}$$

$$\frac{1}{6}G+\frac{5}{6}Q<T<\frac{\pi-2\sqrt{2}}{\pi}G+\frac{2\sqrt{2}}{\pi}Q, \tag{3.9}$$

$$\frac{(2-\sqrt{2})\pi}{2\pi-4}T+\frac{\sqrt{2}\pi-4}{2\pi-4}N<Q<\frac{3}{4}T+\frac{1}{4}N. \tag{3.10}$$

Proof. By Remark 1.1, (3.7), (3.8), (3.9) and (3.10) may be rewritten for $0<z<1$ as

$$\frac{3}{4}<\frac{z^2}{\frac{z}{\arctan z}-(1-z^2)}<\frac{\pi}{4}, \qquad \frac{8}{9}<\frac{\frac{z}{\arctan z}-(1-z^2)}{\sqrt{1+z^2}-(1-z^2)}<\frac{2\sqrt{2}}{\pi},$$
$$\frac{5}{6}<\frac{\frac{z}{\arctan z}-\sqrt{1-z^2}}{\sqrt{1+z^2}-\sqrt{1-z^2}}<\frac{2\sqrt{2}}{\pi}, \qquad \frac{\sqrt{2}\pi-4}{2\pi-4}<\frac{\sqrt{1+z^2}-\frac{z}{\arctan z}}{1+z^2-\frac{z}{\arctan z}}<\frac{1}{4},$$

respectively. By an elementary change of variable $z=\tan x$ $(0<x<\pi/4)$, these four inequalities become

$$\frac{3}{4}<J_1(x)<\frac{\pi}{4}, \quad \frac{8}{9}<J_2(x)<\frac{2\sqrt{2}}{\pi}, \quad \frac{5}{6}<J_3(x)<\frac{2\sqrt{2}}{\pi}, \quad \frac{\sqrt{2}\pi-4}{2\pi-4}<J_4(x)<\frac{1}{4}$$

for $0<x<\pi/4$, where

$$J_1(x)=\frac{\tan^2 x}{\frac{\tan x}{x}-(1-\tan^2 x)}, \quad J_2(x)=\frac{\frac{\tan x}{x}-(1-\tan^2 x)}{\sec x-(1-\tan^2 x)},$$
$$J_3(x)=\frac{\frac{\tan x}{x}-\sqrt{1-\tan^2 x}}{\sec x-\sqrt{1-\tan^2 x}}=\frac{\frac{\sin x}{x}-\sqrt{\cos(2x)}}{1-\sqrt{\cos(2x)}}, \quad J_4(x)=\frac{\sec x-\frac{\tan x}{x}}{\sec^2 x-\frac{\tan x}{x}}.$$

Elementary calculations reveal that

$$\lim_{x\to 0^+}J_1(x)=\frac{3}{4}, \quad J_1\left(\frac{\pi}{4}\right)=\frac{\pi}{4}, \qquad \lim_{x\to 0^+}J_2(x)=\frac{8}{9}, \quad J_2\left(\frac{\pi}{4}\right)=\frac{2\sqrt{2}}{\pi},$$
$$\lim_{x\to 0^+}J_3(x)=\frac{5}{6}, \quad J_3\left(\frac{\pi}{4}\right)=\frac{2\sqrt{2}}{\pi}, \qquad \lim_{x\to 0^+}J_4(x)=\frac{1}{4}, \quad J_4\left(\frac{\pi}{4}\right)=\frac{\sqrt{2}\pi-4}{2\pi-4}.$$

In order prove (3.7), (3.8), (3.9) and (3.10), it suffices to show that $J_1(x)$, $J_2(x)$ and $J_3(x)$ are strictly increasing and $J_4(x)$ is strictly decreasing for $0<x<\pi/4$.

Differentiation yields

$$J_1'(x)=\frac{\sin x\cos xU_1(x)}{U_2(x)},\qquad 0<x<\frac{\pi}{4},$$

where

$$U_1(x)=x\tan x+\sin^2 x-2x^2>0\quad \text{(see (3.6))}$$

and

$$U_2(x)=2x\sin x\cos x-(4x^2-1)\sin^2 x\cos^2 x-4x\cos^3 x\sin x+x^2.$$

We find

$$\begin{aligned}U_2(x)&=-\frac{1}{2}\left(x^2-\frac{1}{4}\right)\big(1-\cos(4x)\big)-\frac{1}{2}x\sin(4x)+x^2\\&=\sum_{n=3}^{\infty}(-1)^{n-1}v_n(x)=\frac{16}{9}x^6-\frac{64}{45}x^8+\sum_{n=5}^{\infty}(-1)^{n-1}v_n(x),\end{aligned}\tag{3.11}$$

where

$$v_n(x)=\frac{2^{4n-5}(n-2)}{n\cdot(2n-2)!}x^{2n}.$$

Elementary calculations reveal that, for $0<x<\pi/4$ and $n\geqslant 5$,

$$\begin{aligned}\frac{v_{n+1}(x)}{v_n(x)}&=\frac{8(n-1)x^2}{(n+1)(2n-1)(n-2)}<\frac{8(n-1)(\pi/4)^2}{(n+1)(2n-1)(n-2)}\\&<\frac{8(n-1)}{(n+1)(2n-1)(n-2)}<1.\end{aligned}$$

Hence, for all $0<x<\pi/4$ and $n\geqslant 5$,

$$\frac{v_{n+1}(x)}{v_n(x)}<1.$$

Therefore, for fixed $x\in(0,\pi/4)$, the sequence $n\longmapsto v_n(x)$ is strictly decreasing for $n\geqslant 5$. We then obtain from (3.11) that

$$U_2(x)>x^6\left(\frac{16}{9}-\frac{64}{45}x^2\right)>0,\quad 0<x<\frac{\pi}{4}.$$

Thus, we have

$$J_1'(x)>0,\quad 0<x<\frac{\pi}{4}.$$

Hence, $J_1(x)$ is strictly increasing for $0 < x < \pi/4$.

Differentiation yields

$$\begin{aligned}
&x^2(1-\cos x)^2(1+2\cos x)^2 J_2'(x)\\
&= 2\sin x\cos^3 x + 2x^2 \sin x\cos^2 x - \sin x\cos x + x^2\sin x - \sin x\cos^2 x - x + x\cos^3 x\\
&= \frac{1}{4}\sin(4x) + \left(\frac{x^2}{2} - \frac{1}{4}\right)\sin(3x) + \frac{1}{4}x\cos(3x) + \left(\frac{3x^2}{2} - \frac{1}{4}\right)\sin x + \frac{3}{4}x\cos x - x\\
&= \frac{1}{15}x^7 - \frac{1}{105}x^9 - \frac{53}{25200}x^{11} + \sum_{n=6}^{\infty}(-1)^n V_n(x), \qquad (3.12)
\end{aligned}$$

where

$$V_n(x) = \frac{6\cdot 16^n - (4n^2 - n + 3)9^n - 36n^2 - 9n + 3}{6(2n+1)!}x^{2n+1}.$$

Noting that $\frac{3}{2}x^2 < \frac{3}{2}(\frac{\pi}{4})^2 < 1$ holds for $0 < x < \pi/4$, we find that for $0 < x < \pi/4$ and $n \geqslant 6$,

$$\begin{aligned}
\frac{V_{n+1}(x)}{V_n(x)} &= \frac{\frac{3}{2}x^2\Big(32\cdot 16^n - (12n^2+21n+18)9^n - (12n^2+27n+14)\Big)}{(n+1)(2n+3)\Big(6\cdot 16^n - (4n^2-n+3)9^n - (36n^2+9n-3)\Big)}\\
&< \frac{32\cdot 16^n}{(n+1)(2n+3)\Big(6\cdot 16^n - (4n^2-n+3)9^n - (36n^2+9n-3)\Big)}\\
&= \frac{32}{(n+1)(2n+3)(6-x_n)},
\end{aligned}$$

where

$$x_n = (4n^2 - n + 3)\left(\frac{9}{16}\right)^n + \frac{36n^2 + 9n - 3}{16^n}.$$

Noting that the sequence $\{x_n\}$ is strictly decreasing for $n \geqslant 6$, we have

$$0 < x_n \leqslant x_6 = \frac{37465917}{8388608}, \qquad n \geqslant 6.$$

We then obtain that, for $0 < x < \pi/4$ and $n \geqslant 6$,

$$\frac{V_{n+1}(x)}{V_n(x)} < \frac{32}{(n+1)(2n+3)\left(6 - \frac{37465917}{8388608}\right)} < 1.$$

Therefore, for fixed $x \in (0, \pi/4)$, the sequence $n \longmapsto V_n(x)$ is strictly decreasing for $n \geqslant 6$. We then obtain from (3.12) that, for $0 < x < \pi/4$,

$$x^2(1-\cos x)^2(1+2\cos x)^2 J_2'(x) > x^7\left(\frac{1}{15} - \frac{1}{105}x^2 - \frac{53}{25200}x^4\right) > 0.$$

Hence, $J_2(x)$ is strictly increasing for $0 < x < \pi/4$.

Differentiation yields

$$x^2\sqrt{\cos(2x)}\left(1-\sqrt{\cos(2x)}\right)^2 J_3'(x) = D_2(x) - D_1(x),$$

where

$$D_2(x) = (\sin x - x\cos x)\cos(2x) + x(x-\sin x)\sin(2x) > 0$$

and

$$D_1(x) = (\sin x - x\cos x)\sqrt{\cos(2x)} > 0$$

for $0 < x < \pi/4$.

We now prove $J_3'(x) > 0$ for $0 < x < \pi/4$, it suffices to show that $D_2(x) > D_1(x)$.

Elementary calculations reveal that

$$\begin{aligned}
\frac{D_2^2(x)-D_1^2(x)}{2\sin x} &= -2x^3\cos^2 x + \sin x + 2\sin x\cos^4 x + 4x^2\sin x\cos^3 x \\
&\quad + (2x^4+x^2-3)\sin x\cos^2 x - x^2\sin(2x) \\
&= -x^3 - x^3\cos(2x) + \left(\frac{1}{2}x^4+\frac{1}{4}x^2+\frac{1}{2}\right)\sin x \\
&\quad + \left(\frac{1}{2}x^4+\frac{1}{4}x^2-\frac{3}{8}\right)\sin(3x) + \frac{1}{2}x^2\sin(4x) + \frac{1}{8}\sin(5x) \\
&= \frac{13}{540}x^9 + \frac{1}{9450}x^{11} - \frac{37}{20160}x^{13} + \frac{108961}{349272000}x^{15} \\
&\quad - \frac{1864237}{108972864000}x^{17} - \frac{493}{583783200}x^{19} + \frac{2419136561}{11204153985024000}x^{21} \\
&\quad - \frac{25139133427}{1300926768261120000}x^{23} + \sum_{n=12}^{\infty}(-1)^n X_n(x), \qquad (3.13)
\end{aligned}$$

where

$$\begin{aligned}
X_n(x) = \Big(&135\cdot 25^n - 54n(2n+1)16^n + (64n^4-64n^3-88n^2-20n-243)9^n \\
&+ 108n(2n-1)(2n+1)4^n + 108(2n-1)(8n^3-4n^2-5n-1)\Big)\frac{x^{2n+1}}{216\cdot(2n+1)!}.
\end{aligned}$$

We find that for $0 < x < \pi/4$ and $n \geqslant 12$,

$$\frac{X_{n+1}(x)}{X_n(x)} = \left(\frac{9x^2}{2}\right)\frac{Y_n}{Z_n} < \frac{9}{2}\left(\frac{\pi}{4}\right)^2\frac{Y_n}{Z_n} < \frac{3Y_n}{Z_n},$$

where

$$Y_n = 375\cdot 25^n - \mathscr{E}_1(n) + \mathscr{E}_2(n) + \mathscr{E}_3(n) + \mathscr{E}_4(n)$$

and

$$Z_n = (n+1)(2n+3)\Big(135\cdot 25^n - \mathscr{E}_5(n) + (64n^4-64n^3-88n^2-20n-243)9^n + 108n(2n-1)(2n+1)4^n + 108(2n-1)(8n^3-4n^2-5n-1)\Big),$$

with

$$\mathscr{E}_1(n) = 96(2n+3)(n+1)16^n, \quad \mathscr{E}_2(n) = (64n^4+192n^3+104n^2-132n-351)9^n,$$
$$\mathscr{E}_3(n) = 48(2n+3)(2n+1)(n+1)4^n, \quad \mathscr{E}_4(n) = 12(2n+1)(8n^3+20n^2+11n-2),$$
$$\mathscr{E}_5(n) = 54n(2n+1)16^n.$$

It is easy to see that, for $n \geqslant 12$,

$$\frac{3Y_n}{Z_n} < \frac{3\Big(375\cdot 25^n + \mathscr{E}_2(n)+\mathscr{E}_3(n)+\mathscr{E}_4(n)\Big)}{(n+1)(2n+3)\Big(135\cdot 25^n - \mathscr{E}_5(n)\Big)} = \frac{3\Big(375+\frac{\mathscr{E}_2(n)}{25^n}+\frac{\mathscr{E}_3(n)}{25^n}+\frac{\mathscr{E}_4(n)}{25^n}\Big)}{(n+1)(2n+3)\Big(135-\frac{\mathscr{E}_5(n)}{25^n}\Big)}.$$

Noting that the sequences $\left\{\frac{\mathscr{E}_j(n)}{25^n}\right\}$ $(j=2,3,4,5)$ are strictly decreasing for $n \geqslant 12$, we have, for $n \geqslant 12$,

$$\begin{aligned} 0 &< \frac{\mathscr{E}_2(n)}{25^n}+\frac{\mathscr{E}_3(n)}{25^n}+\frac{\mathscr{E}_4(n)}{25^n} \leqslant \frac{\mathscr{E}_2(12)}{25^{12}}+\frac{\mathscr{E}_3(12)}{25^{12}}+\frac{\mathscr{E}_4(12)}{25^{12}} \\ &= \frac{472199873062850001}{59604644775390625}+\frac{282662535168}{2384185791015625}+\frac{202008}{2384185791015625} \\ &= \frac{472206939631279401}{59604644775390625} \end{aligned}$$

and

$$0 < \frac{\mathscr{E}_5(n)}{25^n} \leqslant \frac{\mathscr{E}_5(12)}{25^{12}} = \frac{1823957849085050888}{2384185791015625}.$$

We then obtain that for $0 < x < \pi/4$ and $n \geqslant 12$,

$$\frac{X_{n+1}(x)}{X_n(x)} < \frac{3Y_n}{Z_n} < \frac{3\Big(375+\frac{472206939631279401}{59604644775390625}\Big)}{(n+1)(2n+3)\Big(135-\frac{1823957849085050888}{2384185791015625}\Big)} < 1.$$

Therefore, for fixed $x \in (0,\pi/4)$, the sequence $n \longmapsto X_n(x)$ is strictly decreasing for $n \geqslant 12$. We obtain from (3.13) that, for $0 < x < \pi/4$,

$$\begin{aligned} \frac{D_2^2(x)-D_1^2(x)}{2\sin x} &= x^9\left(\frac{13}{540}+\frac{1}{9450}x^2-\frac{37}{20160}x^4\right) \\ &\quad + x^{15}\left(\frac{108961}{349272000}-\frac{1864237}{108972864000}x^2-\frac{493}{583783200}x^4\right) \\ &\quad + x^{21}\left(\frac{2419136561}{11204153985024000}-\frac{25139133427}{1300926768261120000}x^2\right) > 0. \end{aligned}$$

We then obtain that for $0 < x < \pi/4$,

$$D_2(x) > D_1(x) \quad \text{and} \quad J_3'(x) > 0.$$

Hence, $J_3(x)$ is strictly increasing for $0 < x < \pi/4$.

Differentiation yields

$$J_4'(x) = -\frac{I_1(x)}{I_2(x)},$$

where

$$I_1(x) = x^2 \sin x - \sin x \cos x + \sin x \cos^2 x + 2x\cos^2 x - x\cos^3 x - x$$

and

$$I_2(x) = x^2 - x\sin(2x) + \frac{1}{4}\sin^2(2x).$$

We now prove $J_4'(x) < 0$ for $0 < x < \pi/4$, it suffices to show that $I_1(x) > 0$ and $I_2(x) > 0$ for $0 < x < \pi/4$.

Elementary calculations reveal that

$$\begin{aligned} I_1(x) &= \left(x^2 + \frac{1}{4}\right)\sin x - \frac{1}{2}\sin(2x) + \frac{1}{4}\sin(3x) - \frac{3}{4}x\cos x + x\cos(2x) - \frac{1}{4}x\cos(3x) \\ &= \frac{7}{90}x^7 - \frac{41}{1890}x^9 + \sum_{n=5}^{\infty}(-1)^{n-1}W_n(x), \end{aligned} \tag{3.14}$$

where

$$W_n(x) = \frac{(n-1)9^n - 4n\cdot 4^n + 8n^2 + 7n + 1}{2\cdot(2n+1)!}x^{2n+1}.$$

Noting that $\frac{1}{2}x^2 < \frac{1}{2}(\frac{\pi}{4})^2 < 1$ holds for $0 < x < \pi/4$, we find that, for $0 < x < \pi/4$ and $n \geqslant 5$,

$$\begin{aligned} \frac{W_{n+1}(x)}{W_n(x)} &= \frac{\frac{1}{2}x^2\Big(9n\cdot 9^n - (16n+16)4^n + 8n^2 + 23n + 16\Big)}{(n+1)(2n+3)\Big((n-1)9^n - 4n\cdot 4^n + 8n^2 + 7n + 1\Big)} \\ &< \frac{9n\cdot 9^n + 8n^2 + 23n + 16}{(n+1)(2n+3)\Big((n-1)9^n - 4n\cdot 4^n\Big)} \\ &= \frac{9n + \frac{8n^2+23n+16}{9^n}}{(n+1)(2n+3)\Big((n-1) - 4n\left(\frac{4}{9}\right)^n\Big)}. \end{aligned}$$

Noting that the sequences $\left\{\frac{8n^2+23n+16}{9^n}\right\}$ and $\left\{4n\left(\frac{4}{9}\right)^n\right\}$ are both strictly decreasing for $n \geqslant 5$, we have, for $n \geqslant 5$,

$$0 < \frac{8n^2+23n+16}{9^n} \leqslant \left[\frac{8n^2+23n+16}{9^n}\right]_{n=5} = \frac{331}{59049}$$

and

$$0 < 4n\left(\frac{4}{9}\right)^n \leqslant \left[4n\left(\frac{4}{9}\right)^n\right]_{n=5} = \frac{20480}{59049}.$$

We then obtain that for $0 < x < \pi/4$ and $n \geqslant 5$,

$$\frac{W_{n+1}(x)}{W_n(x)} < \frac{9n + \frac{331}{59049}}{(n+1)(2n+3)\left((n-1) - \frac{20480}{59049}\right)} < 1.$$

Therefore, for fixed $x \in (0, \pi/4)$, the sequence $n \longmapsto W_n(x)$ is strictly decreasing for $n \geqslant 5$. We then obtain from (3.14) that, for $0 < x < \pi/4$,

$$I_1(x) > x^7\left(\frac{7}{90} - \frac{41}{1890}x^2\right) > 0.$$

Using (1.15) and (1.19), we obtain

$$\begin{aligned}\frac{I_2(x)}{\sin(2x)} &= x^2\csc(2x) - x + \frac{1}{4}\sin(2x)\\ &= \sum_{n=2}^{\infty}\left\{\frac{2(2n+1)(2^{2n-1}-1)|B_{2n}| + (-1)^n}{(2n+1)!}\right\}2^{2n-1}x^{2n+1}.\end{aligned} \tag{3.15}$$

By the first inequality in (3.5), we find that for $n \geqslant 2$,

$$2(2n+1)(2^{2n-1}-1)|B_{2n}| > 2(2n+1)(2^{2n-1}-1)\frac{2(2n)!}{(2\pi)^{2n}} > 1.$$

We see from (3.15) that

$$I_2(x) > 0, \qquad 0 < x < \frac{\pi}{4}.$$

We then obtain $J_4'(x) < 0$ for $0 < x < \pi/4$. Hence, $J_4(x)$ is strictly decreasing for $0 < x < \pi/4$. The proof is complete. □

THEOREM 3.3. *The inequalities*

$$(1-\mu_3)L + \mu_3 T < A < (1-\nu_3)L + \nu_3 T \tag{3.16}$$

and

$$(1-\mu_4)L + \mu_4 Q < T < (1-\nu_4)L + \nu_4 Q \tag{3.17}$$

hold if and only if

$$\mu_3 \leqslant \frac{1}{2}, \quad \nu_3 \geqslant \frac{\pi}{4}, \quad \mu_4 \leqslant \frac{4}{5}, \quad \nu_4 \geqslant \frac{2\sqrt{2}}{\pi}. \tag{3.18}$$

Proof. We first prove (3.16) and (3.17) with $\mu_3 = \frac{1}{2}, \nu_3 = \frac{\pi}{4}, \mu_4 = \frac{4}{5}, \nu_4 = \frac{2\sqrt{2}}{\pi}$, namely,

$$\frac{1}{2}L + \frac{1}{2}T < A < \left(1 - \frac{\pi}{4}\right)L + \frac{\pi}{4}T \tag{3.19}$$

and

$$\frac{1}{5}L + \frac{4}{5}Q < T < \left(1 - \frac{2\sqrt{2}}{\pi}\right)L + \frac{2\sqrt{2}}{\pi}Q. \tag{3.20}$$

In fact, (3.7) $\Longrightarrow$ (3.19) and (3.8) $\Longrightarrow$ (3.20). More precisely, the following inequalities are true:

$$\frac{1}{2}L + \frac{1}{2}T < \frac{1}{4}H + \frac{3}{4}T < A < \left(1 - \frac{\pi}{4}\right)H + \frac{\pi}{4}T < \left(1 - \frac{\pi}{4}\right)L + \frac{\pi}{4}T \tag{3.21}$$

and

$$\frac{1}{5}L + \frac{4}{5}Q < \frac{1}{9}H + \frac{8}{9}Q < T < \left(1 - \frac{2\sqrt{2}}{\pi}\right)H + \frac{2\sqrt{2}}{\pi}Q < \left(1 - \frac{2\sqrt{2}}{\pi}\right)L + \frac{2\sqrt{2}}{\pi}Q. \tag{3.22}$$

Obviously, the last inequalities in (3.21) and (3.22) hold. The first inequalities in (3.21) and (3.22) can be written, respectively, as

$$\frac{H+T}{2} > L \quad \text{and} \quad \frac{5H+4Q}{9} > L.$$

We now prove that

$$\frac{H+T}{2} > \frac{5H+4Q}{9} > L. \tag{3.23}$$

The first inequality in (3.23) can be written as

$$\frac{H+8Q}{9} < T,$$

which is the left-hand side of (3.8). The second inequality in (3.23) is mentioned in [9, Table 2]. It can be written, by Remark 1.1, as

$$5(1-z^2) + 4\sqrt{1+z^2} > \frac{18z}{\ln\frac{1+z}{1-z}}. \tag{3.24}$$

For $0 < z < 1$, let

$$\xi(z) = \ln\frac{1+z}{1-z} - \frac{18z}{5(1-z^2) + 4\sqrt{1+z^2}}.$$

Differentiation yields

$$\xi'(z) = \frac{2\Big((5-7z^2+52z^4)\sqrt{1+z^2}-5+45z^2-40z^4\Big)}{(1-z^2)(4-4z^2+5\sqrt{1+z^2})^2\sqrt{1+z^2}}.$$

By an elementary change of variable $z=\sqrt{y^2-1}$ $(1<y<\sqrt{2})$, we find

$$\begin{aligned}(5-7z^2+52z^4)&\sqrt{1+z^2}-5+45z^2-40z^4\\ &=52y^5-40y^4-111y^3+125y^2+64y-90\\ &=81(y-1)+72(y-1)^2+249(y-1)^3+220(y-1)^4+52(y-1)^5>0.\end{aligned}$$

We then obtain $\xi'(z)>0$ for $0<z<1$. Hence, $\xi(z)$ is strictly increasing for $0<z<1$, and we have

$$\ln\frac{1+z}{1-z}-\frac{18z}{5(1-z^2)+4\sqrt{1+z^2}}=\xi(z)>\xi(0)=0$$

for $0<z<1$. This means that (3.24) holds. Hence, the second inequality in (3.23) holds.

We then obtain (3.16) and (3.17) with $\mu_3=\frac{1}{2}, \nu_3=\frac{\pi}{4}, \mu_4=\frac{4}{5}, \nu_4=\frac{2\sqrt{2}}{\pi}$.

Conversely, if (3.16) and (3.17) are valid, then we get

$$\mu_3<\frac{1-\frac{2z}{\ln\frac{1+z}{1-z}}}{\frac{z}{\arctan z}-\frac{2z}{\ln\frac{1+z}{1-z}}}<\nu_3 \quad \text{and} \quad \mu_4<\frac{\frac{z}{\arcsin z}-\frac{2z}{\ln\frac{1+z}{1-z}}}{\sqrt{1+z^2}-\frac{2z}{\ln\frac{1+z}{1-z}}}<\nu_4.$$

The limit relations

$$\lim_{z\to0^+}\frac{1-\frac{2z}{\ln\frac{1+z}{1-z}}}{\frac{z}{\arctan z}-\frac{2z}{\ln\frac{1+z}{1-z}}}=\frac{1}{2},\quad \lim_{z\to1^-}\frac{1-\frac{2z}{\ln\frac{1+z}{1-z}}}{\frac{z}{\arctan z}-\frac{2z}{\ln\frac{1+z}{1-z}}}=\frac{\pi}{4},$$

$$\lim_{z\to0^+}\frac{\frac{z}{\arcsin z}-\frac{2z}{\ln\frac{1+z}{1-z}}}{\sqrt{1+z^2}-\frac{2z}{\ln\frac{1+z}{1-z}}}=\frac{4}{5},\quad \lim_{z\to1^-}\frac{\frac{z}{\arcsin z}-\frac{2z}{\ln\frac{1+z}{1-z}}}{\sqrt{1+z^2}-\frac{2z}{\ln\frac{1+z}{1-z}}}=\frac{2\sqrt{2}}{\pi}$$

yield

$$\mu_3\leqslant\frac{1}{2},\quad \nu_3\geqslant\frac{\pi}{4},\quad \mu_4\leqslant\frac{4}{5},\quad \nu_4\geqslant\frac{2\sqrt{2}}{\pi}.$$

The proof is complete. □

Acknowledgement. The authors thank the referees for their careful reading of the manuscript and insightful comments.

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(Received December 15, 2016)

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