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ON THE ORLICZ SYMMETRY OPERATOR

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Abstract. R. Schneider (1970) proved that if $K \in \mathbb{R}^n$ is a convex body, such that each shadow boundary of K with respect to parallel illumination halves the Euclidean surface area of K, then K is centrally symmetric. A generalization of the results of R. Schneider was given by G. Averkov, E. Makai and H. Martini (2009). In this paper, by introducing an Orlicz symmetry operator $\Delta_\phi : \mathscr{K}^n \to \mathscr{K}^n$, we show a new method to obtain the characterization of symmetry for convex bodies. As an application, we will show that there is a unique member of $\Delta_\phi\langle K\rangle$ characterized by having larger volume than that of any other member of $\Delta_\phi\langle K\rangle$, where $\Delta_\phi\langle K\rangle$ is the Orlicz symmetric equivalence class of K.

1. Introduction

Let $\mathbb{R}^n$ denote the n-dimensional Euclidean space. Let $K \in \mathbb{R}^n$ be a convex body (compact, convex set with non-empty interiors) and $x \in \mathbb{R}^n \backslash K$. Then there is a unique point of K closest to x, which we denote by $p(K,x)$, see [23]. We write $d(K,x) := |x - p(K,x)|$, and $u(K,x) := (x - p(K,x))/d(K,x)$, with $|\cdot|$ denoting the Euclidean norm. For a Borel set $B \in \mathbb{R}^n$ and $r > 0$, we consider the Lebesgue measure of the set $\{x \in \mathbb{R}^n | 0 < d(K,x) \leqslant r, p(K,x) \in B\}$. It is of the form

$$\frac{1}{n}\sum_{k=0}^{n-1}\binom{n-1}{k}C_k(K,B)r^k$$

where $C_k(K,B)$, for $0 \leqslant k \leqslant n$, is called the k-th curvature measure of K.

Let $H_u^+ := \{x \in \mathbb{R}^n | \langle x,u\rangle \geqslant 0\}$ and $H_u^- := \{x \in \mathbb{R}^n | \langle x,u\rangle \leqslant 0\}$. Using the curvature measure, R. Schneider (see [21] and [22]) proved the following theorem.

THEOREM A. *Let $K \in \mathbb{R}^n$ be a convex body with $0 \in \mathrm{int}K$, and let k be an integer with $0 \leqslant k \leqslant n$. Suppose that for each $u \in S^{n-1}$ we have $C_k(K,H_u^+) = C_k(K,H_u^-)$. Then K is 0-symmetric.*

For a Borel measure $\omega \in S^{n-1}$, G. Averkov, E. Makai and H. Martini considered the signed Borel measure $\int_\omega \varphi(u)dS_K(u)$ that satisfies the following two conditions.

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a) $\varphi : S^{n-1} \to \mathbb{R}$ is an even Borel measurable function, with $\int_\omega |\varphi(u)| dS_K(u)$, i.e., the total variation of the above signed Borel measure being finite, and

b) $\varphi(u) \neq 0$ for $dS_K(u)$ almost every $u \in S^{n-1}$.

A generalization of the result of R. Schneider was given by G. Averkov, E. Makai and H. Martini [1] as follows.

THEOREM B. *Let K be a convex body in $\mathbb{R}^n$, let $dS_K(u)$ be the Euclidean surface area measure of K, and $\varphi : S^{n-1} \to \mathbb{R}$ be a function satisfying a) and b) above. Then the following statements are equivalent.*

A) The body K is centrally symmetric.

B) The equality

$$\int_{S_u^+} \varphi(u) dS_K(u) = \int_{S_u^-} \varphi(u) dS_K(u) \tag{1}$$

holds for every direction $u \in S^{n-1}$, where $S_u^+ := \{v \in S^{n-1} | \langle u, v \rangle \geqslant 0\}$, $S_u^- := \{v \in S^{n-1} | \langle u, v \rangle \leqslant 0\}$.

C) Equality (1) holds for du-almost every direction $u \in S^{n-1}$, where du is the Lebesgue measure on S^{n-1}.

D) Equality (1) holds for du-almost every direction $u \in S^{n-1}$ among those directions u, for which the shadow boundary of K with respect to parallel illumination from direction u is sharp.

In this paper, the Orlicz symmetry operator $\Delta_\phi : \mathscr{K}^n \to \mathscr{K}^n$ is introduced to obtain analogous characterizations of symmetry for convex bodies. Motivated by recent progress in the asymmetric L_p Brunn-Minkowski theory (see, e.g., [7, 8, 9, 12, 14, 19, 20, 24, 26]), Lutwak, Yang, and Zhang introduced the Orlicz Brunn-Minkowski theory in two articles [17, 18]. More precisely, Lutwak, Yang, and Zhang [17, 18] introduced Orlicz projection bodies and Orlicz centroid bodies, and they successively established the fundamental affine inequalities for these bodies. Recently, Haberl, Lutwak, Yang, and Zhang [6] dealt with the even Orlicz Minkowski problem. For the development of the Orlicz Brunn-Minkowski theory, see [2, 4, 5, 10, 11, 28].

We consider the convex and strictly increasing function $\phi : [0, +\infty) \to [0, +\infty)$ satisfying $\phi(0) = 0$. It is not hard to conclude from [23] that ϕ is continuous on $[0, +\infty)$.

DEFINITION 1. Let $K \subset \mathbb{R}^n$ be a convex body with origin in its interior, and $\phi \in \mathscr{C}$. For $x \in \mathbb{R}^n$, we define the Orlicz symmetry operator $\Delta_\phi : \mathscr{K}^n \to \mathscr{K}^n$ by

$$h_{\Delta_\phi K}(x) = \inf\Big\{\lambda > 0 : \phi\Big(\frac{h_K(x)}{2\lambda}\Big) + \phi\Big(\frac{h_K(-x)}{2\lambda}\Big) \leqslant 1\Big\}. \tag{2}$$

Using the Orlicz symmetry operator $\Delta_\phi : \mathscr{K}^n \to \mathscr{K}^n$, we obtain the following characterizations of symmetry for convex bodies.

THEOREM 1. (Main) *Let $\phi \in C$ and $K \in \mathbb{R}^n$ be a convex body containing the origin in its interior. Then we have*

$$V(\Delta_\phi K) \geqslant r_1^n V(K), \tag{3}$$

where $r_1 = \frac{1}{2\phi^{-1}(\frac{1}{2})}$. *Equality holds if* K *is origin-symmetric. Furthermore, when* ϕ *is strictly convex, equality holds if and only if* K *is origin-symmetric.*

As an application, we obtain the following conclusion.

COROLLARY 1. *Suppose* $K \in \mathcal{K}_0^n$ *(the class of convex bodies containing the origin in their interiors). Then* $\Delta_\phi\langle K\rangle$ *contains a unique member characterized by having larger volume than that of any other member of* $\Delta_\phi\langle K\rangle$*, where*

$$\Delta_\phi\langle K\rangle = \{L \in \mathcal{K}_0^n : \Delta_\phi L = \Delta_\phi K\}.$$

For later reference, we list in Section 2 some basic facts regarding convex bodies. The basic properties of the Orlicz symmetry operator are introduced in Section 3. In Section 4 we prove, by using symmetrization, the Theorem.

2. Notations and preliminaries

Let S^{n-1} denote the *unit sphere*, B^n the *unit* n*-ball*, ω_n the *volume of* B^n, and 0 the *origin* in the Euclidean n-dimensional space $\mathbb{R}^n$. We write $x \cdot y$ for the standard *inner product* of x, y in $\mathbb{R}^n$.

A *convex body* $K \in \mathbb{R}^n$ is a compact, convex set with nonempty interior. The *volume of* K will be denoted by $V(K)$. A real normed linear space of dimension n is called a *Minkowski space* and denoted by $\mathbb{M}^n$ (i.e., $\mathbb{R}^n$, endowed with some Minkowski metric), whose unit ball is a convex body centred at the origin.

Denote by $\mathcal{K}^n$ the class of convex bodies in $\mathbb{R}^n$, and let $\mathcal{K}_0^n$ be the class of members of $\mathcal{K}^n$ containing the origin in their interiors.

Let $\nu_K : \partial K \to S^{n-1}$ be the *Gauss map* of K, defined on ∂K (the *boundary* of K), the set of points of ∂K that have a unique outer unit normal.

Let C be the class of convex, strictly increasing function $\phi : [0, +\infty) \to [0, +\infty)$ satisfying $\phi(0) = 0$. We say a sequence $\phi_i \in C$ is such that $\phi_i \to \phi \in C$, provided

$$|\phi_i - \phi|_I = \max_{x \in I} |\phi_i(x) - \phi(x)| \to 0,$$

for every compact interval $I \subset [0, +\infty)$.

The *support function* $h_K : \mathbb{R}^n \to \mathbb{R}$ of a compact convex set $K \in \mathbb{R}^n$ is defined, for $x \in \mathbb{R}^n$, by

$$h_K(x) = \max\{x \cdot y : y \in K\},$$

and it uniquely determines this compact convex set.

A function is a support function of a compact convex set if and only if it is positively homogeneous of degree one and subadditive. Obviously, for a pair of compact convex sets $K, L \in \mathbb{R}^n$, we have $h_K \leqslant h_L$ if and only if $K \subset L$.

The *Hausdorff distance* between convex bodies K, L is defined by

$$\delta(K, L) := \min\{\lambda \geqslant 0 | K \subset L + \lambda B^d, L \subset K + \lambda B^d\}.$$

In terms of the support function, the Hausdorff distance between two convex bodies K, L can also be expressed as follows (cf. [23]):

$$\delta(K, L) = \max_{u \in S^{d-1}} |h_K(u) - h_L(u)|. \tag{4}$$

A class of convex bodies $\{K_i\}$ is said to *converge* to a convex body K if $\delta(K_i,K)\to 0$, as $i\to+\infty$.

For a convex body K and a direction $u\in S^{n-1}$, let K_u denote the image of the orthogonal projection of K onto $u^\perp$, the subspace of $\mathbb{R}^n$ orthogonal to u. We write $\underline{\ell}_u(K,\cdot):K_u\to\mathbb{R}$ and $\overline{\ell}_u(K,\cdot):K_u\to\mathbb{R}$ for the undergraph and overgraph functions of K in the direction u; i.e., $K=\{y'+tu:-\underline{\ell}_u(K,y')\leqslant t\leqslant\overline{\ell}_u(K,y')\ for\ y'\in K_u\}$. Thus the *Steiner symmetral* S_uK of $K\in\mathscr{K}_0^n$ in direction u can be defined as the body whose orthogonal projection onto $u^\perp$ is identical to that of K and whose undergraph and overgraph functions are given by

$$\underline{\ell}_u(S_uK,y')=\frac{1}{2}[\underline{\ell}_u(K,y')+\overline{\ell}_u(K,y')] \tag{5}$$

and

$$\overline{\ell}_u(S_uK,y')=\frac{1}{2}[\underline{\ell}_u(K,y')+\overline{\ell}_u(K,y')]. \tag{6}$$

The following two well known propositions will be used to prove our results.

PROPOSITION 2.1. (see [17], Lemma 1.2) *Suppose $K\in\mathscr{K}_0^n$ and $u\in S^{n-1}$. For $y'\in\mathrm{relint}K_u$, the overgraph and undergraph functions of K in direction u are given by*

$$\overline{\ell}_u(K,y')=\min_{x'\in u^\perp}\{h_K(x',1)-x'\cdot y'\}, \tag{7}$$

and

$$\underline{\ell}_u(K,y')=\min_{x'\in u^\perp}\{h_K(x',-1)-x'\cdot y'\}. \tag{8}$$

PROPOSITION 2.2. (see [27], Lemma 4.2) *Suppose $K\in\mathscr{K}_0^n$ and $u\in S^{n-1}$. For any $x'_1,x'_2\in u^\perp$ we have*

$$h_K(x'_1,1)+h_K(x'_2,-1)\geqslant 2\max\Big\{h_{S_uK}\Big(\frac{x'_1+x'_2}{2},1\Big),h_{S_uK}\Big(\frac{x'_1+x'_2}{2},-1\Big)\Big\}. \tag{9}$$

3. Properties of the Orlicz symmetric operator

Since the convex function ϕ is strictly increasing on $[0,\infty)$, it follows that the function

$$\lambda\to\phi\Big(\frac{h_K(x)}{2\lambda}\Big)+\phi\Big(\frac{h_K(-x)}{2\lambda}\Big)$$

is strictly decreasing on $[0,\infty)$ and continuous. From this observation and (2), we obtain the following proposition.

PROPOSITION 3.1. *Suppose $K\in\mathscr{K}^n$ and $\phi\in\mathscr{C}$. Then*
1) $h_{\Delta_\phi K}(x)\leqslant\lambda_0$ if and only if $\phi\Big(\frac{h_K(x)}{2\lambda_0}\Big)+\phi\Big(\frac{h_K(-x)}{2\lambda_0}\Big)\leqslant 1$;
2) $h_{\Delta_\phi K}(x)=\lambda_0$ if and only if $\phi\Big(\frac{h_K(x)}{2\lambda_0}\Big)+\phi\Big(\frac{h_K(-x)}{2\lambda_0}\Big)=1$;
3) $h_{\Delta_\phi K}(x)\geqslant\lambda_0$ if and only if $\phi\Big(\frac{h_K(x)}{2\lambda_0}\Big)+\phi\Big(\frac{h_K(-x)}{2\lambda_0}\Big)\geqslant 1$.

PROPOSITION 3.2. *Suppose $K \in \mathscr{K}_0^n$ and $\phi \in \mathscr{C}$. Then $\Delta_\phi K$ is a convex body symmetric with respect to origin.*

Proof. For $r > 0$, we have

$$\begin{aligned} h_{\Delta_\phi K}(rx) &= \inf\left\{\lambda > 0 : \phi\left(\frac{h_K(rx)}{2\lambda}\right) + \phi\left(\frac{h_K(-rx)}{2\lambda}\right) \leqslant 1\right\} \\ &= r\inf\left\{\frac{\lambda}{r} > 0 : \phi\left(\frac{h_K(x)}{2\frac{\lambda}{r}}\right) + \phi\left(\frac{h_K(-rx)}{2\frac{\lambda}{r}}\right) \leqslant 1\right\} \\ &= rh_{\Delta_\phi K}(x). \end{aligned} \tag{10}$$

We claim that for $x_1, x_2 \in \mathbb{R}^n$,

$$h_{\Delta_\phi K}(x_1 + x_2) \leqslant h_{\Delta_\phi K}(x_1) + h_{\Delta_\phi K}(x_2).$$

Set $h_{\Delta_\phi K}(x_1) = r_1$ and $h_{\Delta_\phi K}(x_2) = r_2$, from (2) and Proposition (3.1), then we have

$$\begin{aligned} 1 &= \frac{r_1}{r_1+r_2}\phi\left(\frac{h_K(x_1)}{2r_1}\right) + \frac{r_2}{r_1+r_2}\phi\left(\frac{h_K(x_2)}{2r_2}\right) \\ &\quad + \frac{r_1}{r_1+r_2}\phi\left(\frac{h_K(-x_1)}{2r_1}\right) + \frac{r_2}{r_1+r_2}\phi\left(\frac{h_K(-x_2)}{2r_2}\right) \\ &\geqslant \phi\left(\frac{h_K(x_1)+h_K(x_2)}{2(r_1+r_2)}\right) + \phi\left(\frac{h_K(-x_1)+h_K(-x_2)}{2(r_1+r_2)}\right) \\ &\geqslant \phi\left(\frac{h_K(x_1+x_2)}{2(r_1+r_2)}\right) + \phi\left(\frac{h_K(-(x_1+x_2))}{2(r_1+r_2)}\right), \end{aligned}$$

which implies that

$$h_{\Delta_\phi K}(x_1 + x_2) \leqslant r_1 + r_2 = h_{\Delta_\phi K}(x_1) + h_{\Delta_\phi K}(x_2). \tag{11}$$

The positively homogeneous of degree one property of (10) and subadditivity (11) of $h_{\Delta_\phi K}$ show that $\Delta_\phi K$ is convex.

From the definition (2), for any $x \in \mathbb{R}^n$, it is obvious that

$$h_{\Delta_\phi K}(x) = \inf\left\{\lambda > 0 : \phi\left(\frac{h_K(x)}{2\lambda}\right) + \phi\left(\frac{h_K(-x)}{2\lambda}\right) \leqslant 1\right\} = h_{\Delta_\phi K}(-x),$$

which shows that $\Delta_\phi K$ is symmetric with respect to origin.

Thus, $\Delta_\phi K$ is a convex body symmetric with respect to origin. □

PROPOSITION 3.3. *Suppose $\phi \in C$ and $K \in \mathscr{K}_0^n$.*
(i) If $r > 0$, then $\Delta_\phi rK = r\Delta_\phi K$.
(ii) For $A \in GL(n)$, $\Delta_\phi AK = A\Delta_\phi K$.
(iii) Suppose $K_i \in \mathscr{K}_0^n$ are such that $K_i \to K$. Then $\Delta_\phi K_i \to \Delta_\phi K$.
(iv) Suppose $\phi_i \in C$ are such that $\phi_i \to \phi$. Then $\Delta_{\phi_i} K \to \Delta_\phi K$.

Proof. (i) Let $r>0$. For any $x\in\mathbb{R}^n$,

$$\begin{aligned} h_{\Delta_\phi rK}(x) &= \inf\left\{\lambda>0:\phi\left(\frac{h_{rK}(x)}{2\lambda}\right)+\phi\left(\frac{h_{rK}(-x)}{2\lambda}\right)\leqslant 1\right\} \\ &= r\inf\left\{\frac{\lambda}{r}>0:\phi\left(\frac{h_K(x)}{2\frac{\lambda}{r}}\right)+\phi\left(\frac{h_K(-rx)}{2\frac{\lambda}{r}}\right)\leqslant 1\right\} \\ &= rh_{\Delta_\phi K}(x) \\ &= h_{r\Delta_\phi K}(x). \end{aligned}$$

Thus we have $\Delta_\phi rK = r\Delta_\phi K$.

(ii) Let $A\in GL(n)$. For any $x\in\mathbb{R}^n$,

$$\begin{aligned} h_{\Delta_\phi AK}(x) &= \inf\left\{\lambda>0:\phi\left(\frac{h_{AK}(x)}{2\lambda}\right)+\phi\left(\frac{h_{AK}(-x)}{2\lambda}\right)\leqslant 1\right\} \\ &= \inf\left\{\lambda>0:\phi\left(\frac{h_K(A^Tx)}{2\lambda}\right)+\phi\left(\frac{h_K(-A^Tx)}{2\lambda}\right)\leqslant 1\right\} \\ &= h_{\Delta_\phi K}(A^Tx) \\ &= h_{A\Delta_\phi K}(x). \end{aligned}$$

Thus we have $\Delta_\phi AK = A\Delta_\phi K$.

(iii) Suppose $u_0\in S^{n-1}$. We will show that for the support functions of the convex bodies $\Delta_\phi K_i$, we have

$$h_{\Delta_\phi K_i}(u_0)\to h_{\Delta_\phi K}(u_0).$$

Let $h_{\Delta_\phi K_i}(u_0)=r_i$. From Proposition 3.1, we have

$$1=\phi\left(\frac{h_{K_i}(u_0)}{2r_i}\right)+\phi\left(\frac{h_{K_i}(-u_0)}{2r_i}\right)<2\phi\left(\frac{h_{K_i}(u_0)+h_{K_i}(-u_0)}{2r_i}\right),$$

which implies $r_i<\frac{h_{K_i}(u_0)+h_{K_i}(-u_0)}{2\phi^{-1}(\frac{1}{2})}$. On the other hand,

$$1=\phi\left(\frac{h_{K_i}(u_0)}{2r_i}\right)+\phi\left(\frac{h_{K_i}(-u_0)}{2r_i}\right)\geqslant 2\phi\left(\frac{h_{K_i}(u_0)+h_{K_i}(-u_0)}{4r_i}\right),$$

which implies $r_i\geqslant\frac{h_{K_i}(u_0)+h_{K_i}(-u_0)}{4\phi^{-1}(\frac{1}{2})}$. Thus,

$$\frac{h_{K_i}(u_0)+h_{K_i}(-u_0)}{4\phi^{-1}(\frac{1}{2})}\leqslant r_i<\frac{h_{K_i}(u_0)+h_{K_i}(-u_0)}{2\phi^{-1}(\frac{1}{2})}.$$

Since $K_i\to K$, we have $h_{K_i}(u_0)+h_{K_i}(-u_0)\to h_K(u_0)+h_K(-u_0)$. Thus there are constants $r,R>0$ such that $0<r\leqslant r_i<R$, $i=1,2,3,\dots$, which means that the sequence r_i is bounded.

To show that the bounded sequence r_i converges to r_0, we show that every convergent subsequence of r_i converges to r_0. Denote an arbitrary convergent subsequence of r_i by r_i as well, and suppose that for this subsequence we have $r_i\to r_0$. Thus $0<r\leqslant r_0<R$, and from the continuity of ϕ, we have

$$1=\lim_{i\to+\infty}\left[\phi\left(\frac{h_{K_i}(u_0)}{2r_i}\right)+\phi\left(\frac{h_{K_i}(-u_0)}{2r_i}\right)\right]=\phi\left(\frac{h_K(u_0)}{2r_0}\right)+\phi\left(\frac{h_K(-u_0)}{2r_0}\right).$$

This and Proposition 3.1 give

$$h_{\Delta_\phi K_i}(u_0) = r_i \to r_0 = h_{\Delta_\phi K}(u_0).$$

But for support functions on S^{n-1} pointwise and uniform convergence are equivalent (see, e.g., Schneider [23]). Thus, the pointwise convergence $h_{\Delta_\phi K_i}(u_0) = h_{\Delta_\phi K}(u_0)$ shows that $\delta(\Delta_\phi K_i, \Delta_\phi K) \to 0$, as $i \to +\infty$. Hence $\Delta_\phi K_i \to \Delta_\phi K$.

(iv) Suppose $u_0 \in S^{n-1}$. We will show that for the support functions of the convex bodies $\Delta_{\phi_i} K$, we have

$$h_{\Delta_{\phi_i} K}(u_0) \to h_{\Delta_\phi K}(u_0).$$

Let $h_{\Delta_\phi K_i}(u_0) = r_i$.

As in the proof of (ii), we have

$$\frac{h_K(u_0) + h_K(-u_0)}{2\phi_i^{-1}(\frac{1}{2})} \geqslant r_i \geqslant \frac{h_K(u_0) + h_K(-u_0)}{4\phi_i^{-1}(\frac{1}{2})}.$$

From the fact that $\phi_i \to \phi$, it is easy to show that $\phi_i^{-1} \to \phi^{-1}$. Thus, there are constants $r, R > 0$ such that $0 < r \leqslant r_i < R$, $i = 1, 2, 3, \ldots$, which means that the sequence r_i is bounded.

Denote an arbitrary convergent subsequence of r_i by r_i as well, and suppose that for this subsequence we have $r_i \to r_0$. Thus $0 < r \leqslant r_0 < R$ and from the continuity of ϕ, we have

$$1 = \lim_{i \to +\infty} \left[\phi_i \left(\frac{h_K(u_0)}{2r_i} \right) + \phi_i \left(\frac{h_K(-u_0)}{2r_i} \right) \right] = \phi \left(\frac{h_K(u_0)}{2r_0} \right) + \phi \left(\frac{h_K(-u_0)}{2r_0} \right).$$

This and Proposition 3.1 give

$$h_{\Delta_{\phi_i} K}(u_0) = r_i \to r_0 = h_{\Delta_\phi K}(u_0).$$

As in the proof of (ii), we have shown $\Delta_{\phi_i} K \to \Delta_\phi K$. $\square$

4. The characterization of symmetry for convex bodies

LEMMA 4.1. *Suppose $\phi \in C$ and $K \in \mathscr{K}_0^n$. For any $u \in S^{n-1}$, we have*

$$\Delta_\phi S_u K \subseteq S_u \Delta_\phi K. \tag{12}$$

If ϕ is strictly convex and the inclusion is an identity, then K is origin symmetric.

Proof. From Proposition 2.1,for any $y' \in (\Delta_\phi K)_u$,there exist points $x_1', x_2' \in u^\perp$ such that

$$\overline{\ell}_u(\Delta_\phi K, y') = h_{\Delta_\phi K}(x_1', 1) - x_1' \cdot y', \tag{13}$$

and

$$\underline{\ell}_u(\Delta_\phi K, y') = h_{\Delta_\phi K}(x_2', -1) - x_2' \cdot y'. \tag{14}$$

Let $\lambda_1 = h_{\Delta_\phi K}(x_1',1)$ and $\lambda_2 = h_{\Delta_\phi K}(x_2',-1)$. From Proposition 3.1, the convexity of ϕ and Proposition 2.2, we have

$$\begin{aligned}
1 &= \frac{\lambda_1}{\lambda_1+\lambda_2}\Big[\phi\Big(\frac{h_K(x_1',1)}{2\lambda_1}\Big)+\phi\Big(\frac{h_K(-x_1',-1)}{2\lambda_1}\Big)\Big] \\
&\quad + \frac{\lambda_2}{\lambda_1+\lambda_2}\Big[\phi\Big(\frac{h_K(x_2',-1)}{2\lambda_2}\Big)+\phi\Big(\frac{h_K(-x_2',1)}{2\lambda_2}\Big)\Big] \\
&= \Big[\frac{\lambda_1}{\lambda_1+\lambda_2}\phi\Big(\frac{h_K(x_1',1)}{2\lambda_1}\Big)+\frac{\lambda_2}{\lambda_1+\lambda_2}\phi\Big(\frac{h_K(x_2',-1)}{2\lambda_2}\Big)\Big] \\
&\quad + \Big[\frac{\lambda_1}{\lambda_1+\lambda_2}\phi\Big(\frac{h_K(-x_1',-1)}{2\lambda_1}\Big)+\frac{\lambda_2}{\lambda_1+\lambda_2}\phi\Big(\frac{h_K(-x_2',1)}{2\lambda_2}\Big)\Big] \\
&\geqslant \phi\Big(\frac{h_K(x_1',1)+h_K(x_2'-1)}{2(\lambda_1+\lambda_2)}\Big)+\phi\Big(\frac{h_K(-x_1',-1)+h_K(-x_2'1)}{2(\lambda_1+\lambda_2)}\Big) \qquad (15) \\
&\geqslant \phi\Big(\frac{h_{S_uK}(\frac{x_1'+x_2'}{2},1)}{\lambda_1+\lambda_2}\Big)+\phi\Big(\frac{h_{S_uK}(-\frac{x_1'+x_2'}{2},-1)}{\lambda_1+\lambda_2}\Big).
\end{aligned}$$

From Proposition 3.1, we have

$$h_{\Delta_\phi S_uK}\Big(\frac{x_1'+x_2'}{2},1\Big) \leqslant \frac{\lambda_1+\lambda_2}{2} = \frac{h_{\Delta_\phi K}(x_1',1)+h_{\Delta_\phi K}(x_2',-1)}{2}. \qquad (16)$$

From (6), (14), (13), Proposition 2.2 and (7), we have

$$\begin{aligned}
\overline{\ell}_u(S_u\Delta_\phi K,y') &= \frac{1}{2}\Big[\overline{\ell}_u(\Delta_\phi K,y')+\underline{\ell}_u(\Delta_\phi K,y')\Big] \\
&= \frac{1}{2}\Big[h_{\Delta_\phi K}(x_1',1)+h_{\Delta_\phi K}(x_2',-1)-(x_1'+x_2')\cdot y'\Big] \\
&\geqslant h_{\Delta_\phi S_uK}\Big(\frac{x_1'+x_2'}{2},1\Big)-\frac{x_1'+x_2'}{2}\cdot y' \\
&\geqslant \min_{x'\in u^\perp}\{h_{\Delta_\phi S_uK}(x',1)-x'\cdot y'\} \\
&= \overline{\ell}_u(\Delta_\phi S_uK,y'). \qquad (17)
\end{aligned}$$

In the same way, we can also have $\underline{\ell}_u(S_u\Delta_\phi K,y') \geqslant \underline{\ell}_u(\Delta_\phi S_uK,y')$.

Since $y' \in \operatorname{relint}(\Delta_\phi K)_u$ is arbitrary, we get $\Delta_\phi S_uK \subseteq S_u\Delta_\phi K$.

If the inclusion (12) is an identity, then (4.4) is also an equality. Since ϕ is strictly convex, (4.4) is an equality if and only if

$$\frac{h_K(x_1',1)}{2\lambda_1} = \frac{h_K(x_2',-1)}{2\lambda_2} \quad \text{and} \quad \frac{h_K(-x_1',-1)}{2\lambda_1} = \frac{h_K(-x_2',1)}{2\lambda_2}.$$

Due to the facts $K \in \mathscr{K}_0^n$ and $h_K(-u) = h_{-K}(u)$, there is a positive constant r_0 such that

$$r_0 = \frac{h_K(x_1',1)}{h_{-K}(x_1',1)} = \frac{h_K(x_2',-1)}{h_{-K}(x_2',-1)}. \qquad (18)$$

For any $y' \in (\Delta_\phi K)_u$, there are $x_1', x_2' \in u^\perp$ such that $(x_1',1),(x_2',-1)$ are the outer normal vectors of $\Delta_\phi K$ at the boundary points $(y',\overline{\ell}_u(\Delta_\phi K,y'))$ and $(y',\underline{\ell}_u(\Delta_\phi K,y'))$, respectively.

For any $v \in S^{n-1}$, since $\partial(\Delta_\phi K) \cap \{(\Delta_\phi K)_u + tu | t \in \mathbb{R}\} \cap \{(\Delta_\phi K)_v + tv | t \in \mathbb{R}\} \neq \emptyset$, there always exists $y' \in (\Delta_\phi K)_u$ such that v is the outer normal vector of $\Delta_\phi K$ at the boundary point $(y', \overline{\ell}_u(\Delta_\phi K, y'))$ or $(y', \underline{\ell}_u(\Delta_\phi K, y'))$. Hence, by the same argument as with (4.7), we always have

$$\frac{h_K(v)}{h_{-K}(v)} = r_0,$$

which shows that K and $-K$ are dilates. From this and the fact $V(K) = V(-K)$, we have $K = -K$, i.e., K is origin-symmetric. □

THEOREM 4.1. *Suppose $\phi \in C$ and $K \in \mathscr{K}_0^n$. Then*

$$V(\Delta_\phi K) \geqslant r_1^n V(K), \tag{19}$$

where $r_1 = \frac{1}{2\phi^{-1}(\frac{1}{2})}$. Equality holds if K is origin-symmetric. Furthermore, when ϕ is strictly convex, equality holds if and only if K is origin-symmetric.

Proof. Let $V(K) = a^n \omega_n$. From the Steiner Symmetrization argument and Lemma 4.1, for any $u \in S^{n-1}$ we have

$$V(\Delta_\phi K) = V(S_u \Delta_\phi K) \geqslant V(\Delta_\phi S_u K) = V(\Delta_\phi a B_2^n) = r_1^n a^n \omega_n = r_1^n V(K). \tag{20}$$

If K is origin symmetric, i.e., $h_K(u) = h_K(-u)$ for all $u \in S^{n-1}$, then

$$h_{\Delta_\phi K}(x) = \inf\left\{\lambda > 0 : \phi\left(\frac{h_K(x)}{2\lambda}\right) + \phi\left(\frac{h_K(-x)}{2\lambda}\right) \leqslant 1\right\} = \frac{h_K(x)}{2\phi^{-1}(\frac{1}{2})}, \tag{21}$$

for all $x \in \mathbb{R}^n$. Therefore,

$$V(\Delta_\phi K) = r_1^n V(K).$$

Suppose ϕ is strictly convex. Due to (20), equality holds in (19) if and only if $\Delta_\phi S_u K = S_u \Delta_\phi K$. By Lemma 4.1, this holds if and only if K is origin-symmetric. □

If $\phi(t) = t^p$, for $p \geqslant 1$, we obtain the *p-difference body* $\Delta_p K$ (see e.g., [16]), whose support function is given by

$$h_{\Delta_p K}(x)^p = \frac{h_K(x)^p + h_K(-x)^p}{2}. \tag{22}$$

Then the corresponding result of the Theorem in the L_p Brunn-Minkowski theory is as follows.

COROLLARY 4.1. *Suppose $K \in \mathscr{K}_0^n$, for $p \geqslant 1$. Then*

$$V(\Delta_p K) \geqslant 2^{\frac{1}{p}-1} V(K), \tag{23}$$

and equality holds if and only if K is origin-symmetric.

Using the Theorem, we obtain the following conclusion about the Orlicz symmetric equivalence class.

COROLLARY 4.2. *Suppose $K \in \mathscr{K}_0^n$. Then $\Delta_\phi \langle K \rangle$ contains a unique member characterized by having larger volume than any other member of $\Delta_\phi \langle K \rangle$, where*

$$\Delta_\phi \langle K \rangle = \{L \in \mathscr{K}_0^n : \Delta_\phi L = \Delta_\phi K\}.$$

Proof. We first suppose that K is origin-symmetric.

From Theorem 4.1, for any $L \in \Delta_\phi \langle K \rangle$ we get

$$V(K) = \frac{1}{r_1^n} V(\Delta_\phi K) = \frac{1}{r_1^n} V(\Delta_\phi L) \geqslant V(L), \tag{24}$$

which shows that the volume of K is larger than that of any other member of $\Delta_\phi \langle K \rangle$.

Next, we prove that K is unique. Suppose that there is another $L \in \Delta_\phi \langle K \rangle$ having larger volume than any other member.

From (21), for any $x \in \mathbb{R}^n$, we have

$$h_K(x) = 2\phi^{-1}\left(\frac{1}{2}\right) h_{\Delta_\phi K}(x) = 2\phi^{-1}\left(\frac{1}{2}\right) h_{\Delta_\phi L}(x) = h_L(x), \tag{25}$$

which yields the result. □

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