Mathematical
Inequalities
& Applications
Volume 21, Number 1 (2018), 133–138
doi:10.7153/mia-2018-21-11

ON SYMMETRIC NORM INEQUALITIES AND POSITIVE DEFINITE BLOCK–MATRICES

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(*Communicated by J.-C. Bourin*)

Abstract. The main purpose of this paper is to englobe some new and known types of positive semi definite matrices $M = \begin{pmatrix} A & X \\ X^* & B \end{pmatrix}$ satisfying or not the inequality $\|M\| \leqslant \|A+B\|$ for all symmetric norms.

1. Introduction and preliminaries

The first section presents some known results related to the inequality together with some preliminaries we used in the second section to derive some new and generalization results. Let $\mathbb{M}_n^+$ denote the set of positive and semi definite part of the space of $n \times n$ complex matrices. For positive semi definite block-matrix M, we say that M is P.S.D. and we write $M = \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \in \mathbb{M}_{n+m}^+$, with $A \in \mathbb{M}_n^+$, $B \in \mathbb{M}_m^+$.

The modulus of a matrix X stands for $(X^*X)^{\frac{1}{2}}$ and is denoted by $|X|$. A norm $\|.\|$ over the space of matrices is a symmetric norm if $\|UAV\| = \|A\|$ for all A and all unitaries U and V. Let A be an $n \times n$ matrix and F an $m \times m$ matrix, $(m > n)$ written by blocks such that A is a diagonal block and all entries other than those of A are zeros, then the two matrices have the same singular values and $\|A\| = \|F\| = \|A \oplus 0\|$ for all symmetric norms, we say then that the symmetric norm on $\mathbb{M}_m$ induces a symmetric norm on $\mathbb{M}_n$, so for square matrices we may assume that our norms are defined on all spaces $\mathbb{M}_n$, $n \geqslant 1$. The spectral norm is denoted by $\|.\|_s$, the Frobenius norm by $\|.\|_{(2)}$, and the Ky Fan $k-$norm by $\|.\|_k$.

A positive partial transpose matrix denoted by P.P.T. is a P.S.D. block matrix M such that both $M = \begin{pmatrix} A & X \\ X^* & B \end{pmatrix}$ and $M' = \begin{pmatrix} A & X^* \\ X & B \end{pmatrix}$ are positive semi definite. Let $Im(X) := \dfrac{X - X^*}{2i}$ respectively $Re(X) := \dfrac{X + X^*}{2}$ be the imaginary part respectively the real part of a matrix X, if $W(X)$ denotes the field of values of X then $W(Re(X)) = \Re(W(X))$ and $W(Im(X)) = \Im(W(X))$ see [1].

Mathematics subject classification (2010): 15A60, 15A42, 47A30.
Keywords and phrases: Symmetric norm, positive semi definite block-matrices, positive partial transpose.

LEMMA 1.1. [2] *For every matrix in* $\mathbb{M}_{2n}^{+}$ *written in blocks of the same size, we have the decomposition:*

$$\begin{pmatrix} A & X \\ X^* & B \end{pmatrix} = U \begin{pmatrix} \frac{A+B}{2} + Im(X) & 0 \\ 0 & 0 \end{pmatrix} U^* + V \begin{pmatrix} 0 & 0 \\ 0 & \frac{A+B}{2} - Im(X) \end{pmatrix} V^*$$

for some unitaries $U, V \in \mathbb{M}_{2n}$.

LEMMA 1.2. [2] *For every matrix in* $\mathbb{M}_{2n}^{+}$ *written in blocks of the same size, we have the decomposition:*

$$\begin{pmatrix} A & X \\ X^* & B \end{pmatrix} = U \begin{pmatrix} \frac{A+B}{2} + Re(X) & 0 \\ 0 & 0 \end{pmatrix} U^* + V \begin{pmatrix} 0 & 0 \\ 0 & \frac{A+B}{2} - Re(X) \end{pmatrix} V^*$$

for some unitaries $U, V \in \mathbb{M}_{2n}$.

REMARK 1.3. The proofs of Lemma 1.1 respectively Lemma 1.2 suggests that we have $A+B \geqslant -\dfrac{(X-X^*)}{i}$ and $A+B \geqslant \dfrac{(X-X^*)}{i}$, respectively $A+B \geqslant -(X+X^*)$ and $A+B \geqslant (X+X^*)$ since if we let hereafter $M_1 := \dfrac{A+B}{2} + Im(X)$, $M_2 := \dfrac{A+B}{2} - Im(X)$, $N_1 := \dfrac{A+B}{2} + Re(X)$ and $N_2 := \dfrac{A+B}{2} - Re(X)$ then N_1, N_2, M_1, M_2 are P.S.D as diagonal blocks of the P.S.D matrix JMJ^* for some unitary matrix J ([2]).

2. Main results

2.1. Symmetric norm inequality

It is well known that if $M \in \mathbb{M}_{n+m}^{+}$ with $M = \begin{pmatrix} A & X \\ X^* & B \end{pmatrix}$ then

$$\|M\| \leqslant \|A\| + \|B\| \tag{2.1}$$

for all symmetric norms (see [3]). Hereafter our block matrices are such their diagonal blocks are of equal size.

LEMMA 2.1. [2] *Let* $M = \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \in \mathbb{M}_{2n}^{+}$, *if* X *is Hermitian or Skew-Hermitian then*

$$\|M\| \leqslant \|A+B\| \tag{2.2}$$

for all symmetric norms.

See [5] for another proof of Lemma 2.1 (the case X is Hermitian).

LEMMA 2.2. [4] *Let* $M = \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \in \mathbb{M}_{2n}^{+}$ *be a positive partial transpose matrix then*

$$\|M\| \leqslant \|A+B\| \tag{2.3}$$

for all symmetric norms.

PROPOSITION 2.3. *Let* $M = \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \in \mathbb{M}_{2n}^{+}$ *be a given positive semi definite matrix. If* X^* *commute with* A *or* B, *then* M *is unitarily congruent to a P.P.T. matrix and*

$$\|M\| \leqslant \|A+B\|$$

for all symmetric norms. In addition if X *is normal then* M *is a positive partial transpose matrix.*

Proof. We will assume without loss of generality that X^* commute with A (up to a permutation congruence) and that X is invertible (by a continuity argument). Take the polar decomposition of X so $X = U|X|$ and $X^* = |X|U^*$. Since U^* is unitary and X^* commute with A, X and $|X|$ commute with A thus $AU^* = U^*A$. If I_n is the identity matrix of order n, a direct computation shows that

$$\begin{bmatrix} U^* & 0 \\ 0 & I_n \end{bmatrix} \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \begin{bmatrix} U & 0 \\ 0 & I_n \end{bmatrix} = \begin{bmatrix} A & |X| \\ |X| & B \end{bmatrix},$$

consequently we have $\|M\| \leqslant \|A+B\|$ for all symmetric norms and that completes the proof. If X is normal then $|X| = |X^*|$. The polar decomposition discussed above and the following known decomposition: $X = U|X| = |X^*|U$, let us write

$$\begin{bmatrix} U^* & 0 \\ 0 & I_n \end{bmatrix} \begin{bmatrix} A & |X| \\ |X| & B \end{bmatrix} \begin{bmatrix} U & 0 \\ 0 & I_n \end{bmatrix} = \begin{bmatrix} A & X^* \\ X & B \end{bmatrix},$$

which implies that M is a P.P.T. matrix. $\square$

REMARK 2.4. It is easily seen that if X commute with the Hermitian matrix A so is X^* and conversely.

The following is a generalization result:

LEMMA 2.5. *Let* $M = \begin{bmatrix} A & X \\ X^* & B \end{bmatrix}$ *be a positive semi definite matrix, if* $Im(X) = rI_n$ *or* $Re(X) = rI_n$ *for some* r, *then* $\|M\| \leqslant \|A+B\|$ *for all symmetric norms.*

Proof. Let $\sigma_i(H)$ denote the singular values of a matrix H ordered in decreasing order, by Remark 1.3 the matrices $M_1 = \frac{A+B}{2} + Im(X)$ and $M_2 = \frac{A+B}{2} - Im(X)$ are positive semi definite since $Im(X) = rI_n$ we have:

$$\sum_{i=1}^{k} \sigma_i\left(\frac{A+B}{2} + Im(X)\right) + \sum_{i=1}^{k} \sigma_i\left(\frac{A+B}{2} - Im(X)\right) = \sum_{i=1}^{k} \sigma_i(A+B).$$

In other words by Lemma 1.1 $\|M\|_k \leqslant \|M_1\|_k + \|M_2\|_k = \|A+B\|_k$ for all Ky-Fan $k-$norms which from the Ky-Fan dominance theorem (see [1] -Sec 10.7-) completes the proof. Using Lemma 1.2 the other case is similarly proven. □

THEOREM 2.6. *Let* $M = \begin{bmatrix} A & X \\ X^* & B \end{bmatrix}$ *be a positive semi definite matrix. If* $W(X)$ *is a line segment in the complex plane then* $\|M\| \leqslant \|A+B\|$ *for all symmetric norms.*

Proof. $M_\theta = \begin{bmatrix} A & e^{i\theta}X \\ e^{-i\theta}X^* & B \end{bmatrix}$ is unitarily congruent to M by the unitary matrix $U = \begin{bmatrix} I & 0 \\ 0 & e^{-i\theta}I \end{bmatrix}$, we may choose θ in such a way that $W(e^{i\theta}X)$ is a line segment parallel to the imaginary axis, that is $\Re(W(e^{i\theta}X)) = W(Re(e^{i\theta}X)) = r$ for some real scalar r which implies that $Re(e^{i\theta}X) = rI_n$. Applying Lemma 2.5 to M_θ completes the proof. □

REMARK 2.7. From Theorem 2.6 if A, B, X are 2×2 complex matrices with X normal then $\|M\| \leqslant \|A+B\|$ for all symmetric norms.

The next example shows that (2.3) is a sharp inequality for P.S.D. block-matrices.

EXAMPLE 2.8. Let

$$M = \begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} A & X \\ X^* & B \end{bmatrix},$$

where $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$. Then we have

$$\|A+B\|_s = 3 = \|M\|_s < \|A\|_s + \|B\|_s = 4.$$

Matrices verifying the condition of Theorem 2.6 are not necessarily positive partial transpose considering the matrix $M = \begin{pmatrix} 6 & i & i & 1 \\ -i & 1 & 1 & i \\ -i & 1 & 1 & i \\ 1 & -i & -i & 6 \end{pmatrix}$.

THEOREM 2.9. *Let* $M = \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \geqslant 0$ *and let* $r_1,\ r_2$ *be two nonnegative numbers, if* $M_1 \geqslant r_1 I_n$ *and* $M_2 \geqslant r_2 I_n$ *or* $N_1 \geqslant r_1 I_n$ *and* $N_2 \geqslant r_2 I_n$ *then*

$$\|M\| \leqslant \|2(A+B) - (r_1+r_2)I_n\| \tag{2.4}$$

for all symmetric norms. In particular if $M \geqslant rI_n$ *for some* $r \geqslant 0$ *then*

$$\|M\| \leqslant 2\|(A+B) - rI_n\| \tag{2.5}$$

for all symmetric norms.

Proof. If we have the case

$$\frac{A+B}{2}+Im(X)\geqslant r_1I_n \tag{2.6}$$

$$\frac{A+B}{2}-Im(X)\geqslant r_2I_n \tag{2.7}$$

or the case

$$\frac{A+B}{2}+Re(X)\geqslant r_1I_n \tag{2.8}$$

$$\frac{A+B}{2}-Re(X)\geqslant r_2I_n \tag{2.9}$$

then summing both equations in each case gives $A+B\geqslant(r_1+r_2)I_n$ so first $\|A+B\|\leqslant\|2(A+B)-(r_1+r_2)I_n\|$. Since $\frac{A+B}{2}-r_2I_n\geqslant Im(X)$ respectively $\frac{A+B}{2}-r_1I_n\geqslant -Im(X)$; $\|M_1\|_k\leqslant\|A+B-r_2I_n\|_k$ respectively $\|M_2\|_k\leqslant\|A+B-r_1I_n\|_k$ for all $k\leqslant n$, in consequence we derive the following inequality:

$$\|M\|_k\leqslant\|M_1\|_k+\|M_2\|_k=\|2(A+B)-(r_1+r_2)I_n\|_k$$

for all Ky-Fan $k-$norms. By replacing M_i by N_i for $i=1,2$ Lemma 1.2 gives the same inequality. The particular case can be easily concluded since by the decompositions in Lemma 1.1 and Lemma 1.2, if $M\geqslant rI_n$ then all of M_1-rI_n, N_1-rI_n, M_2-rI_n and N_2-rI_n are positive semi definite matrices. $\square$

Inequality (2.4) can be sharper than (2.1) as these examples show:

EXAMPLE 2.10. Let

$$M=\begin{bmatrix}4&0&0&-3\\0&2&2&0\\0&2&2&0\\-3&0&0&4\end{bmatrix}=\begin{bmatrix}A&X\\X^*&B\end{bmatrix},$$

where $A=\begin{bmatrix}4&0\\0&2\end{bmatrix}$ and $B=\begin{bmatrix}2&0\\0&4\end{bmatrix}$. M is positive semi definite, $r_1=r_2=2.5$ with

$$8=\|A\|_s+\|B\|_s>\|2(A+B)-(r_1+r_2)I_n\|_s=7=\|M\|_s>\|A+B\|_s=6.$$

EXAMPLE 2.11. Let

$$M=\begin{bmatrix}1&0&0&0.25\\0&0&0&0\\0&0&0&0\\0.25&0&0&1\end{bmatrix}=\begin{bmatrix}A&X\\X^*&B\end{bmatrix},$$

where $A=\begin{bmatrix}1&0\\0&0\end{bmatrix}$ and $B=\begin{bmatrix}0&0\\0&1\end{bmatrix}$. It can be verified that M is positive semi definite, $r_1=r_2=0.375$ and we have $\|M\|_{(2)}=\sqrt{2.125}>\|A+B\|_{(2)}=\sqrt{2}$ with

$$2=\|A\|_{(2)}+\|B\|_{(2)}>\|2(A+B)-(r_1+r_2)I_n\|_{(2)}=\sqrt{3.125}>\|M\|_{(2)}>\|A+B\|_{(2)}.$$

Acknowledgements. I want to thank Dr. Minghwa Lin for offering me references, Dr. Ajit Iqbal Singh for her helpful comments; an anonymous referee and J.-C. Bourin for his enlightening remarks.

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(Received May 31, 2017)

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