Mathematical
Inequalities
& Applications
Volume 21, Number 1 (2018), 201–215 doi:10.7153/mia-2018-21-16

HARDY TYPE INEQUALITIES AND COMPACTNESS OF A CLASS OF INTEGRAL OPERATORS WITH LOGARITHMIC SINGULARITIES

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(*Communicated by M. Praljak*)

Abstract. We establish criteria for both boundedness and compactness for some classes of integral operators with logarithmic singularities in weighted Lebesgue spaces for cases $1 < p \leqslant q < \infty$ and $1 < q < p < \infty$. As corollaries some corresponding new Hardy inequalities are pointed out.

1. Introduction

Let $0 < q < \infty$, $1 < p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$, $R_+ = (0,\infty)$. Moreover, let $u : R_+ \to R$ and $v : R_+ \to R$ be weight functions, i.e. non-negative measurable functions on R_+.

Since the 70-s of the last century weighted estimates of the form

$$\|vKf\|_q \leqslant C\|uf\|_p \tag{1}$$

are intensively studied in the literature for different classes of the operators K, where $\|\cdot\|_p$ is the usual norm of the space $L_p \equiv L_p(R_+)$. Review of research in the period $1970-1982$, where estimates of the form (1) are given, can be found in [5]. Some directions of research of the estimate (1) until 2009 for integral operators are summarized in the books [6, 11, 12, 14]. Estimates of the form (1) are considered not only in Lebesgue spaces but also in other function spaces (see. e.g. [4, 8, 17] and Chapter 11 of the book [11]). Moreover, in [18] a sequence of classes of non-negative functions $K(\cdot,\cdot)$ was considered and when the kernels $K(x,s)$ of an integral operator

$$Kf(x) = \int_0^x K(x,s)f(s)ds, \tag{2}$$

belong to these classes, a full description of weights v and u was given, so that, the estimate (1) holds for the operator K defined by (2). However, these results do not

Mathematics subject classification (2010): 47G10, 47B38, 45P05.
Keywords and phrases: Inequalities, Hardy type inequalities, singular integral operator, logarithmic singularity, fractional integration operator, boundedness, compactness.

include the operator in the form of (2), when the kernel $K(\cdot,\cdot)$ have a singularity, for example, the Riemann-Liouville operator

$$R_\alpha f(x) = \int_0^x \frac{f(s)ds}{(x-s)^{1-\alpha}}, \tag{3}$$

when $0 < \alpha < 1$. The estimate of the form (1) remains open for the operator (3) in the general case. However, the following cases are studied: $v \equiv u$ in [3], $u \equiv 1$ in [15, 20] and u is non-decreasing in [7] and when one of the weighted functions v, u is non-increasing in [21].

The estimate (1) for a singular operator in a form

$$Kf(x) = \int_0^x s^{\gamma-1} \ln \frac{x}{x-s} f(s)ds, \tag{4}$$

is equivalent to an estimate

$$\|K_\gamma f\|_q \leqslant C\|f\|_p \tag{5}$$

for the operator

$$K_\gamma f(x) = v(x) \int_0^x u(s)s^{\gamma-1} \ln \frac{x}{x-s} f(s)ds. \tag{6}$$

The estimate (5) is equivalent to the boundedness of the operator (6) from L_p to L_q with the norm $\|K_\gamma\| = C$, where C is the best constant in (5). The operator (4) in the case $\gamma = 0$ is called a fractional integration operator of infinitesimal order [16].

The operator

$$K_\gamma^* f(s) = u(s)s^{\gamma-1} \int_s^\infty v(x) \ln \frac{x}{x-s} f(x)dx, \quad s > 0, \tag{7}$$

is dual to the operator K_γ with respect to the scalar product $\int_0^\infty f(x)g(x)dx$.

The main purpose of this paper is to establish the boundedness and compactness of the operator (6) and the dual operator (7) from L_p to L_q.

In the case $u(x) \equiv 1$ of boundedness and of compactness from L_p to L_q of the operator (6) was studied in [1] and [2], respectively.

The main results (Theorems 1–4) are presented in Section 3. As corollaries some corresponding new Hardy type inequalities (Corollaries 1–4) are pointed out. The detailed proofs are given in Section 4 and in order not to disturb the argumentations in these proofs some auxiliary results are collected in Section 2.

CONVENTIONS. *Uncertainties of the type* $0 \cdot \infty$, $\frac{0}{0}$, $\frac{\infty}{\infty}$ *are assumed to be zero. The inequality of the form* $A \leqslant \beta B$ *is written in the form* $A \ll B$, *where the positive constant* β *may be dependent on the parameters* p, q, γ, *and the relation* $A \approx B$ *means that* $A \ll B \ll A$. $\chi_{(a,b)}(\cdot)$ *denotes a characteristic function of the interval* (a,b), Z *is the set of integer numbers. The notations* $\sum_k$, $\sup_k$ *mean* $\sum_{k\in Z}$, $\sup_{k\in Z}$, *respectively.*

2. Auxiliary results

Since

$$\ln\frac{x}{x-s}=\int_0^s\frac{dt}{x-t}\quad\text{for } x>s\geqslant 0, \tag{8}$$

the following inequalities

$$\frac{s}{x-s}>\ln\frac{x}{x-s}>\frac{s}{x},\quad x>s>0 \tag{9}$$

hold. The function $\ln\frac{x}{x-s}$ decreases with respect to x and increases with respects to s when $x>s\geqslant 0$, and from the inequality (9) it follows that the functions $x\ln\frac{x}{x-s}$, $\frac{1}{s}\ln\frac{x}{x-s}$ also decreases with respect to x and increases with respects to s when $x>s>0$. Indeed,

$$\frac{\partial}{\partial x}\left(x\ln\frac{x}{x-s}\right)=\ln\frac{x}{x-s}-\frac{s}{x-s}<0,$$

and

$$\frac{\partial}{\partial s}\left(\frac{1}{s}\ln\frac{x}{x-s}\right)=\frac{1}{s^2}\left(\frac{s}{x-s}-\ln\frac{x}{x-s}\right)>0$$

for $x>s>0$.

From (8) we have

$$\int_0^x\ln\frac{x}{x-s}f(s)ds=\int_0^x\int_0^s\frac{dt}{x-t}f(s)ds=\int_0^x\frac{1}{x-t}\int_t^x f(s)dsdt. \tag{10}$$

In the case when the function u is positive a.e. in R_+ we put $u(s)s^{\gamma-1}f(s)=g'(s)$. Then from (10) and (6) it follows that the inequality (5) is equivalent to the inequality

$$\left(\int_0^\infty\left|v(x)\int_0^x\frac{g(x)-g(s)}{x-s}ds\right|^q dx\right)^{\frac{1}{q}}\leqslant C\left(\int_0^\infty|g'(x)u^{-1}(x)x^{1-\gamma}|^p dx\right)^{\frac{1}{p}} \tag{11}$$

for the differentiable functions g.

Similarly, if the function v is positive a.e. in R_+, then the inequality (5) for the operator (7) is equivalent to the inequality

$$\left(\int_0^\infty\left|u(s)s^\gamma\int_s^\infty\frac{f(x)-f(s)}{x-s}\frac{dx}{x}\right|^q ds\right)^{\frac{1}{q}}\leqslant C\left(\int_0^\infty|f'(x)v^{-1}(x)|^p dx\right)^{\frac{1}{p}} \tag{12}$$

for any differentiable functions f. In this case we have that

$$\int_s^\infty\ln\frac{x}{x-s}f(x)dx=\int_s^\infty f(x)\int_x^\infty\frac{sdt}{t(t-s)}dx=s\int_s^\infty\frac{1}{t-s}\int_t^x f(s)ds\frac{dt}{t}.$$

Along with the operator K_γ defined by (6) we consider the operator H_γ defined by

$$H_\gamma f(x) = \frac{v(x)}{x} \int_0^x u(s) s^\gamma f(s) ds, \;\; x > 0.$$

It is easy to see that

$$K_\gamma f \geqslant H_\gamma f \tag{13}$$

for $f \geqslant 0$. Let

$$A(x) = \left(\int_0^x u^{p'}(s) s^{\gamma p'} ds \right)^{\frac{1}{p'}} \left(\int_x^\infty \frac{v^q(t)}{t^q} dt \right)^{\frac{1}{q}}, \quad A = \sup_{x>0} A(x).$$

For the operator H_γ the following theorem holds [11, 12, 19]:

THEOREM A. *Let $1 < p \leqslant q < \infty$. Then the operator H_γ is bounded from L_p to L_q if and only if $A < \infty$. Moreover, $\|H_\gamma\| \approx A$.*

REMARK 1. Here and below for any operator T the value $\|T\|$ denotes the norm of the operator T from L_p to L_q.

The corresponding result for the case $q < p$ reads:

THEOREM B. *Let $0 < q < p < \infty$, $p > 1$. The operator H_γ is bounded from L_p to L_q if and only if*

$$B = \left(\int_0^\infty \left(\int_x^\infty \frac{v^q(t)}{t^q} dt \right)^{\frac{p}{p-q}} \left(\int_0^x u^{p'}(s) s^{p'\gamma} ds \right)^{\frac{p(q-1)}{p-q}} u^{p'}(x) x^{p'\gamma} dx \right)^{\frac{p-q}{pq}} < \infty.$$

Moreover, $\|H_\gamma\| \approx B$.

REMARK 2. In the case $1 < q < p < \infty$, the constant B is equivalent to the constant

$$\widetilde{B} = \left(\int_0^\infty \left(\int_x^\infty \frac{v^q(t)}{t^q} dt \right)^{\frac{q}{p-q}} \left(\int_0^x u^{p'}(s) s^{p'\gamma} ds \right)^{\frac{q(p-1)}{p-q}} \frac{v^q(x)}{x^q} dx \right)^{\frac{p-q}{pq}}.$$

3. The main results

Our first main result reads:

THEOREM 1. *Let* $1<p\leqslant q<\infty$, $\gamma>\frac{1}{p}$, *and* $u(x)$ *be a non-increasing function. Then the operator* K_γ *defined by* (6)

i) is bounded from L_p *to* L_q *if and only if* $A<\infty$ *and, moreover,* $\|K_\gamma\|\approx A$,

ii) is compact from L_p *to* L_q *if and only if* $A<\infty$ *and* $\lim\limits_{x\to 0^+}A(x)=\lim\limits_{x\to\infty}A(x)=0$.

COROLLARY 1. *Let the function* u *be positive a.e. on* R_+ *and the conditions of Theorem* 1 *be fulfilled. Then the Hardy type inequality* (11) *holds if and only if* $A<\infty$. *Moreover,* $A\approx C$, *where* C *is the best constant in* (11).

The corresponding result for the case $q<p$ reads:

THEOREM 2. *Let* $p>1$, $0<q<p<\infty$ *and* $\gamma>\frac{1}{p}$. *Let* u *be a non-increasing function on* R_+. *Then the operator* K_γ *defined by* (6)

i) is bounded from L_p *to* L_q *if and only if* $B<\infty$ *and, moreover,* $\|K_\gamma\|\approx B$,

ii) is compact from L_p *to* L_q *if and only if* $B<\infty$ *when ever* $q>1$.

COROLLARY 2. *Let* $0<q<p<\infty$. *Let the function* u *be positive a.e. in* R_+ *and the conditions of Theorem* 2 *be fulfilled. Then the Hardy type inequality* (11) *holds if and only if* $B<\infty$. *Moreover,* $B\approx C$ *for the best constant* C *in* (11).

We define

$$A^*(x)=\left(\int_x^\infty \frac{v^{p'}(t)}{t^{p'}}dt\right)^{\frac{1}{p'}}\left(\int_o^x s^{q\gamma}u^q(s)ds\right)^{\frac{1}{q}},\quad A^*=\sup_{x>0}A^*(x),$$

and

$$B^*=\left(\int_0^\infty\left(\int_x^\infty \frac{v^q(t)}{t^{p'}}dt\right)^{\frac{q(p-1)}{p-q}}\left(\int_0^x s^{q\gamma}u^q(s)ds\right)^{\frac{q}{p-q}}x^{q\gamma}u^q(x)dx\right)^{\frac{p-q}{pq}}.$$

We consider the operator K_γ^* (defined by (7)) and its action from L_p to L_q. If $1<p,q<\infty$, then the operator K_γ^* is bounded (compact) from L_p to L_q if and only if the operator K_γ is bounded (compact) from $L_{q'}$ to $L_{p'}$. In this case the conditions $1<p\leqslant q<\infty$ and $1<q<p<\infty$ are equivalent to the conditions $1<q'\leqslant p'<\infty$ and $1<p'<q'<\infty$, respectively. Therefore from Theorems 1 and 2, we have the following:

THEOREM 3. *Let* $1 < p \leqslant q < \infty$ *and* $\gamma > \frac{1}{p}$. *Then the operator* K_γ^* *defined by* (7)

i) is bounded from L_p *to* L_q *if only if* $A^* < \infty$ *and, moreover,* $\|K_\gamma^*\| \approx A^*$,

ii) is compact from L_p *to* L_q *if only if* $A^* < \infty$ *and* $\lim_{x\to 0^+} A^*(x) = \lim_{x\to\infty} A^*(x) = 0$.

COROLLARY 3. *Let the function* v *be positive a.e. on* R_+ *and the conditions of Theorem* 3 *be fulfilled. Then the Hardy type inequality* (12) *holds if and only if* $A^* < \infty$. *Moreover,* $A^* \approx C$, *where* C *is the best constant in* (12).

THEOREM 4. *Let* $1 < q < p < \infty$ *and* $\gamma > \frac{1}{p}$. *Then the operator* K_γ^* *defined by* (7) *is bounded and compact from* L_p *to* L_q *if only if* $B^* < \infty$ *and, moreover,* $\|K_\gamma^*\| \approx B^*$.

COROLLARY 4. *Let the function* v *be positive a.e. on* R_+ *and the conditions of Theorem* 4 *be fulfilled. Then the Hardy type inequality* (12) *holds if and only if* $B^* < \infty$. *Moreover,* $B^* \approx C$ *for the best constant* C *in* (12).

4. Proofs of the main results

Proof of Theorem 1. *Proof of* $i)$. *Necessity.* Let the operator (6) be bounded from L_p to L_q. Then, in view of (13), the operator H_γ is bounded from L_p to L_q and $\|K_\gamma\| \geqslant \|H_\gamma\|$. Therefore, by Theorem A the value $A < \infty$ and

$$\|K_\gamma\| \gg A. \tag{14}$$

Sufficiency. Let $A < \infty$. Since $\ln\frac{x}{x-s} \geqslant 0$ when $x > s \geqslant 0$, then it is enough to prove the inequality (5) for $f \geqslant 0$. Let $0 \leqslant f \in L_p$. Then we have

$$\begin{aligned}
\|K_\gamma f\|_q^q &= \sum_k \int_{2^k}^{2^{k+1}} v^q(x) \left(\int_0^x u(s)s^{\gamma-1} \ln\frac{x}{x-s} f(s)ds \right)^q dx \\
&\ll \sum_k \int_{2^k}^{2^{k+1}} v^q(x) \left(\int_0^{2^{k-1}} u(s)s^{\gamma-1} \ln\frac{x}{x-s} f(s)ds \right)^q dx \\
&\quad + \sum_k \int_{2^k}^{2^{k+1}} v^q(x) \left(\int_{2^{k-1}}^x u(s)s^{\gamma-1} \ln\frac{x}{x-s} f(s)ds \right)^q dx := I_1 + I_2.
\end{aligned} \tag{15}$$

We estimate I_1 and I_2 separately. Using the monotonicity of the function $\frac{1}{s}\ln\frac{x}{x-s}$ with respect to the variables x and s, we obtain that for $x > s \geqslant 0$

$$
\begin{aligned}
I_1 &\leqslant \sum_k \int_{2^k}^{2^{k+1}} v^q(x)\left(\int_0^{2^{k-1}} u(s)s^\gamma \frac{1}{2^{k-1}}\ln\frac{2^k}{2^k-2^{k-1}} f(s)ds\right)^q dx \\
&\leqslant (\ln 2)^q \sum_k \int_{2^k}^{2^{k+1}} \frac{v^q(x)}{(2^{k-1})^q}\left(\int_0^{2^{k-1}} u(s)s^\gamma f(s)ds\right)^q dx \\
&\ll \int_0^\infty \frac{v^q(x)}{x^q}\left(\int_0^x u(s)s^\gamma f(s)ds\right)^q dx = \|H_\gamma f\|_q^q.
\end{aligned} \tag{16}
$$

In view of Theorem A from (16) it follows that

$$
I_1 \ll A^q \|f\|_q^q. \tag{17}
$$

By now using the fact that the function u is increasing, applying Hölder's and Jensen's inequalities and making the change of the variable $s = xt$ in the integral below, we have

$$
\begin{aligned}
I_2 &\leqslant \sum_k u^q(2^{k-1}) \int_{2^k}^{2^{k+1}} v^q(x)\left(\int_0^x s^{p'(\gamma-1)}\ln^{p'}\frac{x}{x-s}ds\right)^{\frac{q}{p'}} dx \left(\int_{2^{k-1}}^{2^{k+1}} f^p(t)dt\right)^{\frac{q}{p}} \\
&\leqslant \sum_k \left(\int_{2^{k-1}}^{2^{k+1}} f^p(t)dt\right)^{\frac{q}{p}} u^q(2^{k-1}) \int_{2^k}^{2^{k+1}} v^q(x)x^{q(\gamma-1)}\left(\int_0^x \ln^{p'}\frac{x}{x-s}ds\right)^{\frac{q}{p'}} dx \\
&= \beta^{\frac{q}{p'}} \sum_k \left(\int_{2^{k-1}}^{2^{k+1}} f^p(t)dt\right)^{\frac{q}{p}} u^q(2^{k-1}) \int_{2^k}^{2^{k+1}} v^q(x)x^{q(\gamma-1)+\frac{q}{p'}}dx \\
&\ll \sum_k \left(\int_{2^{k-1}}^{2^{k+1}} f^p(t)dt\right)^{\frac{q}{p}} u^q(2^{k-1}) \left[2^{(k-1)(\gamma+\frac{1}{p'})}\left(\int_{2^k}^{2^{k+1}} \frac{v^q(x)}{x^q}dx\right)^{\frac{1}{q}}\right]^q \\
&\ll \sum_k \left(\int_{2^{k-1}}^{2^{k+1}} f^p(t)dt\right)^{\frac{q}{p}} \left[u(2^{k-1})\left(\int_0^{2^{k-1}} s^{p'\gamma}ds\right)^{\frac{1}{p'}}\left(\int_{2^k}^{2^{k+1}} \frac{v^q(x)}{x^q}dx\right)^{\frac{1}{q}}\right]^q \\
&\ll \sum_k \left(\int_{2^{k-1}}^{2^{k+1}} f^p(t)dt\right)^{\frac{q}{p}} \left[\left(\int_0^{2^{k-1}} s^{p'\gamma}u^{p'}(s)ds\right)^{\frac{1}{p'}}\left(\int_{2^k}^{2^{k+1}} \frac{v^q(x)}{x^q}dx\right)^{\frac{1}{q}}\right]^q \\
&\leqslant A^q \left(\sum_k \int_{2^{k-1}}^{2^{k+1}} f^p(t)dt\right)^{\frac{q}{p}} \ll A^q \|f\|_p^q,
\end{aligned} \tag{18}
$$

where $\beta = \int_0^1 t^{p'(\gamma-1)} \ln^{p'} \frac{1}{1-t} dt$. The finiteness of β follows from the estimate

$$\beta \leqslant \ln^{p'} 2 \int_0^{\frac{1}{2}} s^{p'(\gamma-1)} ds + \max\{1, 2^{-p'(\gamma-1)}\} \int_{\ln 2}^{\infty} t^{p'} e^{-t} dt$$

and from the condition $\gamma > \frac{1}{p}$.

From (15), (17) and (18) it follows that

$$\|K_\gamma f\|_q \ll A \|f\|_p.$$

Hence, $\|K_\gamma\| \ll A$. This relation together with (14) gives $\|K_\gamma\| \approx A$. The statement $i)$ of Theorem 1 is proved.

Proof of ii). Necessity. Let the operator K_γ be compact from L_p to L_q. Then the operator is bounded and therefore, by assertion $i)$, $A < \infty$. First, we prove that $\lim_{z \to 0^+} A(z) = 0$.

Consider the family of functions $\{f_t\}_{t \in I}$, where

$$f_t(x) = \chi_{(0,t)}(x) u^{p'-1}(x) x^{(p'-1)\gamma} \left(\int_0^t u^{p'}(s) s^{p'\gamma} ds \right)^{-\frac{1}{p}}. \tag{19}$$

Then

$$\int_0^\infty |f_t(x)|^p dx = \left(\int_0^t u^{p'}(s) s^{p'\gamma} ds \right)^{-1} \int_0^t u^{p'}(x) x^{p'\gamma} dx \equiv 1. \tag{20}$$

Next we show that the family of functions $\{f_t\}$ converges weakly to zero in L_p. Let $g \in L_{p'} = (L_p)^*$.

Applying the Hölder inequality and using (20) we have that

$$\int_0^\infty f_t(x) g(x) dx \leqslant \left(\int_0^t |f_t(x)|^p dx \right)^{\frac{1}{p}} \left(\int_0^t |g(x)|^{p'} dx \right)^{\frac{1}{p'}} = \left(\int_0^t |g(x)|^{p'} dx \right)^{\frac{1}{p'}}.$$

Since $g \in L_{p'}$, then the last integral converges to zero as $t \to 0^+$, which means the weak convergence to zero for the family of functions $\{f_t\}$. Then, by the compactness of the operator K_γ from L_p to L_q

$$\lim_{t \to 0^+} \|K_\gamma f_t\|_q = 0. \tag{21}$$

Since $\ln \frac{x}{x-s} \geqslant \frac{s}{x}$ for $x > s > 0$ we find that

$$\|K_\gamma f_t\|_q^q = \int_0^\infty v^q(x) \left(\int_0^x u(s) s^{\gamma-1} \ln \frac{x}{x-s} f_t(s) ds \right)^q dx$$
$$\geqslant \int_t^\infty \frac{v^q(x)}{x^q} \left(\int_0^t u(s) s^\gamma f_t(s) ds \right)^q dx$$
$$= \left(\int_0^t u^{p'}(s) s^{p'\gamma} ds \right)^{-\frac{q}{p}} \left(\int_0^t u^{p'}(s) s^{p'\gamma} ds \right)^q \int_t^\infty \frac{v^q(x)}{x^q} dx = (A(t))^q. \tag{22}$$

By combining (21) and (22) we obtain that $\lim_{t\to 0^+} A(t) = 0$.

Now we prove that $\lim_{t\to\infty} A(t) = 0$.

The compactness of the operator $K_\gamma : L_p \to L_q$ implies the compactness of the dual operator (7) from $L_{q'}$ to $L_{p'}$.

We introduce the family of functions $\{g_t\}_{t\in I}$, where

$$g_t(x) = \chi_{(t,\infty)}(x) \left(\int_t^\infty \frac{v^q(x)}{x^q} dx \right)^{-\frac{1}{q'}} \frac{v^{q-1}(x)}{x^{q-1}}.$$

Since $A < \infty$, then the function g_t is well defined.

In view of the equality

$$\int_0^\infty |g_t(x)|^{q'} dx = \left(\int_t^\infty \frac{v^q(x)}{x^q} dx \right)^{-1} \left(\int_t^\infty \frac{v^q(x)}{x^q} dx \right) = 1$$

for $f \in L_q = (L_{q'})^*$ we see that

$$\int_0^\infty f(x) g_t(x) dx \leqslant \left(\int_t^\infty |f(x)|^q dx \right)^{\frac{1}{q}} \left(\int_t^\infty |g_t(x)|^{q'} dx \right)^{\frac{1}{q'}} = \left(\int_t^\infty |f(x)|^q dx \right)^{\frac{1}{q}}.$$

Consequently, $\lim_{t\to\infty} \int_0^\infty f(x) g_t(x) dx = 0$ for any $f \in L_q$, which means the weak convergence to zero of the family of functions $\{g_t\}$. Then, by the compactness of the operator K_γ^* from $L_{q'}$ to $L_{p'}$, it follows that

$$\lim_{t\to\infty} \|K_\gamma^* g_t\|_{p'} = 0. \tag{23}$$

Again using that $\ln \frac{x}{x-s} \geqslant \frac{s}{x}$ for $x > s > 0$, we obtain that

$$\|K_\gamma^* g_t\|_{p'}^{p'} \geqslant \int_0^t |u(s)s^{\gamma-1}|^{p'} \left(\int_t^\infty v(x) \ln \frac{x}{x-s} g_t(x)dx \right)^{p'} ds$$

$$\geqslant \int_0^t u^{p'}(s) s^{p'\gamma} ds \left(\int_t^\infty \frac{v^q(x)}{x^q} dx \right)^{-\frac{p'}{q'}} \left(\int_a^t \frac{v^q(x)}{x^q} dx \right)^{p'} = A^{p'}(t). \quad (24)$$

By combining (23) and (24) it follows that $\lim_{t\to\infty} A(t) = 0$. The necessity of statement *ii*) is proved.

Sufficiency. Let $A < \infty$ and $\lim_{z\to 0^+} A(z) = \lim_{z\to\infty} A(z) = 0$.

For $0 < c < d < \infty$ we define

$$P_c f = \chi_{(0,c]} f, \qquad P_{cd} f = \chi_{(c,d]} f, \qquad Q_d f = \chi_{(d,\infty)} f.$$

Then $f = P_c f + P_{cd} f + Q_d f$ and since $P_c K_\gamma P_{cd} \equiv 0$, $P_c K_\gamma Q_d \equiv 0$, $P_{cd} K_\gamma Q_d \equiv 0$, we have that

$$K_\gamma f = P_{cd} K_\gamma P_{cd} f + P_c K_\gamma P_c f + P_{cd} K_\gamma P_c f + Q_d K_\gamma f. \quad (25)$$

We show that the operator $P_{cd} K_\gamma P_{cd}$ is compact from L_p to L_q. Since $P_{cd} K_\gamma P_{cd} f(x) = 0$ for $x \in I \setminus (c,d)$, then it is enough to show that the operator $P_{cd} K_\gamma P_{cd}$ is compact from $L_p(c,d)$ to $L_q(c,d)$. This, in turn, is equivalent to compactness of the operator

$$Tf(x) = \int_c^d K(x,s) f(s) ds$$

from $L_p(c,d)$ to $L_q(c,d)$ with the kernel

$$K(x,s) = u(s) s^{\gamma-1} v(x) \chi_{(c,d)}(x-s) \ln \frac{x}{x-s}.$$

Next we note that there are the points $2^i, 2^n$, $n > i$ such that $2^i \leqslant c < 2^{i+1}$, $2^{n-1} < d \leqslant 2^n$. We assume that the numbers c and d are chosen so that $2^{i+1} < 2^{n-1}$. Then arguing as in the estimates of I_1 and I_2 in Theorem 1, we find that

$$\int_c^d \left(\int_c^d |K(x,s)|^{p'} ds \right)^{\frac{q}{p'}} dx = \int_c^d v^q(x) \left(\int_c^x u^{p'}(s) s^{p'(\gamma-1)} \left(\ln \frac{x}{x-s} \right)^{p'} ds \right)^{\frac{q}{p'}} dx$$

$$\ll \sum_{k=i}^{n-1} \int_{2^k}^{2^{k+1}} v^q(x) \left(\int_0^{2^{k-1}} u^{p'}(s) s^{p'(\gamma-1)} \left(\ln \frac{x}{x-s} \right)^{p'} ds \right)^{\frac{q}{p'}} dx$$

$$+ \sum_{k=i}^{n-1} \int_{2^k}^{2^{k+1}} v^q(x) \left(\int_{2^{k-1}}^x u^{p'}(s) s^{p'(\gamma-1)} \left(\ln \frac{x}{x-s} \right)^{p'} ds \right)^{\frac{q}{p'}} dx$$

$$\leqslant \mu(n-i+1) A < \infty,$$

where the constant μ does not depend on i and n. Therefore, on the basis of Kantorovich condition [9] (page 589), the operator T is compact from $L_p(c,d)$ to $L_q(c,d)$, which is equivalent to the compactness of the operator $P_{cd}K_\gamma P_{cd}$ from L_p to L_q .

From (25) it follows that

$$\|K_\gamma - P_{cd}K_\gamma P_{cd}\| \leqslant \|P_cK_\gamma P_c\| + \|P_{cd}K_\gamma P_c\| + \|Q_dK_\gamma\|. \tag{26}$$

We show that the right side of (26) tends to zero at $c \to 0^+$ and $d \to \infty$. Then it follows that the operator K_γ as the uniform limit of compact operators is compact from L_p to L_q.

By statement $i)$ we have that

$$\begin{aligned}
\|P_cK_\gamma P_c f\|_q &= \left(\int_0^c v^q(x)\left|\int_0^x u(s)s^{\gamma-1}\ln\frac{x}{x-s}f(s)ds\right|^q dx\right)^{\frac{1}{q}}\\
&\ll \sup_{0<z<c}\left(\int_0^z u^{p'}(s)s^{p'\gamma}ds\right)^{\frac{1}{p'}}\left(\int_z^c v^q(x)x^{-q}dx\right)^{\frac{1}{q}}\|f\|_p\\
&\leqslant \sup_{0<z<c} A(z)\|f\|_p.
\end{aligned}$$

Consequently, $\|P_cK_\gamma P_c\| \ll \sup\limits_{0<z<c} A(z)$. Hence,

$$\lim_{c\to 0^+}\|P_cK_\gamma P_c\| \ll \lim_{c\to 0^+}\sup_{0<z<c} A(z) = \lim_{c\to 0^+} A(c) = 0. \tag{27}$$

Let $v_d = Q_d v$. Then, by using statement $i)$, we find that

$$\begin{aligned}
\|Q_dK_\gamma f\|_q &= \left(\int_0^\infty v_d^q(x)\left|\int_0^x u(s)s^{\gamma-1}\ln\frac{x}{x-s}f(s)ds\right|^q dx\right)^{\frac{1}{q}}\\
&\ll \sup_{0<z}\left(\int_0^z u^{p'}(s)s^{p'\gamma}ds\right)^{\frac{1}{p'}}\left(\int_z^\infty v_d^q(x)x^{-q}dx\right)^{\frac{1}{q}}\|f\|_p\\
&\leqslant \sup_{d<z} A(z)\|f\|_p.
\end{aligned}$$

Therefore,

$$\lim_{d\to\infty}\|Q_dK_\gamma\| \ll \lim_{d\to\infty} A(d) = 0. \tag{28}$$

Now we will prove that

$$\lim_{c\to 0^+}\|P_{cd}K_\gamma P_c\| = 0. \tag{29}$$

We put $v_{cd} = P_{cd}v$ and $u_c = P_c u$. It is obvious that the function u_c is non-increasing. Therefore, according to statement $i)$, we get that

$$\begin{aligned}\|P_{cd}K_\gamma P_c f\|_q &= \left(\int_0^\infty v_{cd}^q(x)\left|\int_0^x u_c(s)s^{\gamma-1}\ln\frac{x}{x-s}f(s)ds\right|^q dx\right)^{\frac{1}{q}}\\ &\ll \sup_{0<z}\left(\int_0^z u_c^{p'}(s)s^{p'\gamma}ds\right)^{\frac{1}{p'}}\left(\int_z^\infty v_{cd}^q(x)x^{-q}dx\right)^{\frac{1}{q}}\|f\|_p\\ &\leqslant A(c)\|f\|_p.\end{aligned}$$

and we conclude that equality (29) holds.

From (27), (28) and (29) it follows that the right side of (26) tends to zero at $c\to 0^+$ and $d\to\infty$. Hence, also the sufficiency of $ii)$ is proved. The proof is complete. □

Proof of Theorem 2. Proof of statement $i)$. Necessity. Let the operator (6) be bounded from L_p to L_q. Then, in view of (13), the operator H_γ is bounded from L_p to L_q and $\|K_\gamma\| \geqslant \|H_\gamma\|$. Therefore, by Theorem B the value $B<\infty$ and

$$\|K_\gamma\| \gg B. \tag{30}$$

Sufficiency. Let $B<\infty$. We have the estimate (15) for $0\leqslant f\in L_p$. In view of Theorem B and from (16) we have that

$$I_1 \ll B^q\|f\|_q^q. \tag{31}$$

Moreover, from the estimate I_2 in the proof of $i)$ of Theorem 1 it follows that

$$\begin{aligned}I_2 &\ll \sum_k\left(\int_{2^{k-1}}^{2^{k+1}} f^p(t)dt\right)^{\frac{q}{p}} u^q(2^{k-1})2^{k\frac{q}{p'}(p'\gamma+1)}\int_{2^k}^{2^{k+1}}\frac{v^q(x)}{x^q}dx\\ &\ll \sum_k\left(\int_{2^{k-1}}^{2^{k+1}} f^p(t)dt\right)^{\frac{q}{p}}\left(u^{p'}(2^{k-1})\int_{2^{k-2}}^{2^{k-1}} t^{p'\gamma}dt\right)^{\frac{q}{p'}}\int_{2^k}^{2^{k+1}}\frac{v^q(x)}{x^q}dx\\ &\leqslant \sum_k\left(\int_{2^{k-1}}^{2^{k+1}} f^p(t)dt\right)^{\frac{q}{p}}\left(\int_{2^{k-2}}^{2^{k-1}} u^{p'}(t)t^{p'\gamma}dt\right)^{\frac{q}{p'}}\int_{2^k}^{2^{k+1}}\frac{v^q(x)}{x^q}dx.\end{aligned} \tag{32}$$

By now using the Hölder inequality with exponents $\frac{p}{q}$, $\frac{p}{p-q}$ and the estimate

$$\left(\int_{2^{k-2}}^{2^{k-1}} u^{p'}(t)t^{\gamma p'}dt\right)^{\frac{q(p-1)}{p-q}} \ll \int_{2^{k-2}}^{2^{k-1}}\left(\int_{2^{k-2}}^{x} u^{p'}(s)s^{\gamma p'}ds\right)^{\frac{p(q-1)}{p-q}} u^{p'}(x)x^{\gamma p'}dx$$

in (32) we find that

$$I_2 \ll \left(\sum_k \left(\int_{2^{k-2}}^{2^{k-1}} u^{p'}(t) t^{\gamma p'} dt \right)^{\frac{q(p-1)}{p-q}} \left(\int_{2^k}^{2^{k+1}} \frac{v^q(x)}{x^q} dx \right)^{\frac{p}{p-q}} \right)^{\frac{p-q}{q}} \left(\sum_k \int_{2^{k-1}}^{2^{k+1}} f^p(t) dt \right)^{\frac{q}{p}}$$

$$\ll \left(\sum_k \int_{2^{k-2}}^{2^{k-1}} \left(\int_0^x u^{p'}(s) s^{\gamma p'} ds \right)^{\frac{p(q-1)}{p-q}} \left(\int_x^\infty \frac{v^q(t)}{t^q} dt \right)^{\frac{p}{p-q}} u^{p'}(x) x^{\gamma p'} dx \right)^{\frac{p-q}{q}}$$

$$\times \left(\sum_k \int_{2^{k-1}}^{2^{k+1}} f^p(t) dt \right)^{\frac{q}{p}}$$

$$\ll B^q \|f\|_p^q. \tag{33}$$

From (16), (31) and (33) we obtain the estimate

$$\|K_\gamma f\|_q \ll B \|f\|_p,$$

which together with (30) gives $\|K_\gamma\| \approx B$. The statement $i)$ is proved.

Proof of ii). *Necessity.* Let the operator K_γ be compact from L_p to L_q. Then the operator is bounded and therefore, by assertion $i)$, $B < \infty$.

Sufficiency. Let $A < \infty$. Here we have $K_\gamma f = P_d K_\gamma P_d f + P_d K_\gamma Q_d f + Q_d K_\gamma f$. Therefore

$$\|K_\gamma - P_d K_\gamma P_d\| \leqslant \|P_d K_\gamma Q_d\| + \|Q_d K_\gamma\|. \tag{34}$$

Since $d < \infty$, then from the Ando theorem and its generalizations (see e.g. [10]) the operator $P_d K_\gamma P_d$ is compact from $L_p(0,d)$ to $L_q(0,d)$, which is equivalent to the compactness of it from L_p to L_q. We show that the right-hand side (34) tends to zero as $d \to \infty$. Then the operator K_γ is compact from L_p to L_q as the uniform limit of compact operators. Similarly as in the proof of $ii)$ of Theorem 1 we find that

$$\|Q_d K_\gamma f\|_q = \left(\int_0^\infty v_d^q(x) \left| \int_0^x u(s) s^{\gamma-1} \ln \frac{x}{x-s} f(s) ds \right|^q dx \right)^{\frac{1}{q}}.$$

Then, in view of the statement $i)$,

$$\|Q_d K_\gamma\| \ll \left(\int_d^\infty \left(\int_z^\infty u^{p'}(s) s^{p'\gamma} ds \right)^{\frac{q(p-1)}{p-q}} \left(\int_d^z v^q(x) x^{-q} dx \right)^{\frac{q}{p-q}} v^q(z) z^{-q} dz \right)^{\frac{(p-q)}{pq}}.$$

From this estimate and the fact that $B < \infty$ it follows that

$$\lim_{d \to \infty} \|Q_d K_\gamma\| = 0. \tag{35}$$

Let $v_{dd} = P_d v$ and $u_d = Q_d u$. Then, using again statement $i)$, we obtain that

$$\begin{aligned}\|P_d K_\gamma Q_d f\|_q &= \left(\int_0^\infty v_{dd}^q(x) \left| \int_0^x u_d(s) s^{\gamma-1} \ln \frac{x}{x-s} f(s) ds \right|^q dx \right)^{\frac{1}{q}} \\ &\ll \left(\int_d^\infty u^{p'}(s) s^{p'\gamma} ds \right)^{\frac{1}{p'}} \left(\int_0^d v^q(x) x^{-q} dx \right)^{\frac{1}{q}} \|f\|_p \\ &= A(d)\|f\|_p. \end{aligned} \tag{36}$$

We also note that, by Remark 2, $B \approx \widetilde{B}$. Since

$$\begin{aligned} A(d) &\ll \widetilde{B}(d,\infty) \\ &= \left(\int_d^\infty \left(\int_x^\infty \frac{v^q(t)}{t^q} dt \right)^{\frac{q}{p-q}} \left(\int_0^x u^{p'}(s) s^{p'\gamma} ds \right)^{\frac{q(p-1)}{p-q}} \frac{v^q(x)}{x^q} dx \right)^{\frac{p-q}{pq}} \end{aligned}$$

then from (36) we have that $\lim\limits_{d\to\infty} \|P_d K_\gamma Q_d\| = 0$. From this and from (35) it follows that the right-hand side of (34) tends to zero at $d \to \infty$. Therefore also the sufficiency part of $ii)$ is proved. The proof is complete. □

Finally, we remark that as mentioned before the proofs of Theorem 3 and 4 follows by using Theorems 1 and 2, respectively, and a standard duality argument.

REMARK 3. The current status of the mentioned open question to characterize the Hardy type inequality (1) - (2) without restriction on the kernel $\mathscr{K}(x,s)$ was recently described in [13]. However, the cases considered in this paper are new and can not be found there.

Acknowledgement. The authors thank Professor Ryskul Oinarov for several generous advices, which have improved the final version of this paper.

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(Received September 7, 2016)

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