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SEPARATION BY STRONGLY h–CONVEX FUNCTIONS

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Abstract. The convex separation problem is studied intensively in many situation: It is answered for the cases of classical convexity, strong convexity, h-convexity and strong h-convexity with multiplicative h. In the case of h-convexity, multiplicativity turns out to be considerably relaxed. The aim of the present note is to give common generalization of these results, that is, to give sufficient and necessary conditions for the existence of strongly h-convex separators with no further assumption on multiplicativity.

1. Introduction

The notion of classical convexity has been extended and studied in several contexts. The problem of convex separation (or affine) has already been investigated in these situations. The aim of the present note is to give a common generalization of these recent results, placing them into the framework of strong h-convexity. To do this, the next notion plays the key role.

DEFINITION 1. Let D be a convex subset of a real vector space, let $h\colon [0,1]\to\mathbb{R}$ be nonnegative function and let $c\geqslant 0$. A function $f\colon D\to\mathbb{R}$ is said to be *strongly h_∞-convex* if for all $n\in\mathbb{N}$, for all $x_1,\dots,x_n\in D$, and for all $t_1,\dots,t_n\in[0,1]$ satisfying $t_1+\ldots+t_n=1$ the following inequality holds:

$$f\Big(\sum_{i=1}^{n} t_i x_i\Big)\leqslant \sum_{i=1}^{n} h(t_i)f(x_i)-c\sum_{i=1}^{n} t_i\Big\|x_i-\sum_{i=1}^{n} t_i x_i\Big\|^2.$$

Assume first that the inequality above is required only for $n=2$. Then, if $h=\mathrm{id}$ and $c=0$, this notion leads to the case of classical convexity. Similarly, for $h=\mathrm{id}$ and no further assumption on c, we get the case of strong convexity due to Polyak [13]. For further references, see [9], [15]. For $c=0$ and no further assumption on h, the definition gives the case of h-convexity introduced by Varošanec [14]. The definition also embraces the case of strong h-convex functions, a common generalization of these latter ones [1].

Let us emphasize that in many cases strong h_∞-convexity is equivalent to strong h-convexity, that is, to the case when our definition is restricted only to the $n=2$

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case. For example, if h is multiplicative then this phenomenon appears. These kind of cases can be interpreted that the underlying convexity notion satisfies a Jensen-type inequality.

Convex separation results are known in many important special cases. The classical case was obtained by Baron, Matkowski, and Nikodem [2]. The case of strong convexity was investigated by Merentes and Nikodem [10], while the case of h-convexity by Olbryś [12] under the assumption of multiplicativity. An attempt for a common generalization was presented by Lara, Merentes, and Nikodem [8] for strong h-convex functions. Besides some additional restrictions, the assumption on multiplicativity is still kept. For precise details, consult the Corollaries of the last section.

However, a recent paper of Bessenyei and Pénzes [5] clarifies that using an adequate form, multiplicativity can completely be omitted from the case of h-convexity. Our aim is to show that the same conclusion remains true in the above mentioned cases, in more general, for strong h_∞-convex functions. Namely, our main results give sufficient and necessary conditions for the existence of strong h_∞-convex separator without assuming multiplicativity on h.

2. Main results

Throughout in this note, $\mathbb{R}_+$ stands for the set of nonnegative reals and $\mathbb{N}$ denotes the set of positive integers. Our main results are presented int the next two theorems. The first one gives a necessary, while the second one a sufficient condition for the existence of strongly h_∞-convex separator. Let us emphasize, that these results reduce to a characterization theorem in some cases. This phenomenon will be enlightened among the applications of the last section.

THEOREM 1. *Let D be a convex subset of a real vector space, let $f,g\colon D\to\mathbb{R}$ and $h\colon[0,1]\to\mathbb{R}_+$ be given functions and let $c\geqslant 0$. If there exists a $\varphi\colon D\to\mathbb{R}$ strongly h_∞-convex function with modulus c such that $f\leqslant\varphi\leqslant g$ on D then we have*

$$f\Big(\sum_{i=1}^n t_i x_i\Big)\leqslant\sum_{i=1}^n h(t_i)g(x_i)-c\sum_{i=1}^n t_i\Big\|x_i-\sum_{i=1}^n t_i x_i\Big\|^2$$

for all $n\in\mathbb{N}$, $x_1,\dots,x_n\in D$ and $t_1,\dots,t_n\in[0,1]$ satisfying $t_1+\ldots+t_n=1$.

Proof. Using the definition of strong h_∞-convexity we get the desired inequality:

$$\begin{aligned} f\Big(\sum_{i=1}^n t_i x_i\Big)\leqslant\varphi\Big(\sum_{i=1}^n t_i x_i\Big)&\leqslant\sum_{i=1}^n h(t_i)\varphi(x_i)-c\sum_{i=1}^n t_i\Big\|x_i-\sum_{i=1}^n t_i x_i\Big\|^2\\ &\leqslant\sum_{i=1}^n h(t_i)g(x_i)-c\sum_{i=1}^n t_i\Big\|x_i-\sum_{i=1}^n t_i x_i\Big\|^2. \qquad\square\end{aligned}$$

The second main result reads as follows. Both in its formulation and in its proof, the basic ideas of [5] can be adopted to our situation.

THEOREM 2. *Let D be a convex subset of a real vector space, $f,g\colon D\to\mathbb{R}$ be given functions, $c\geqslant 0$ and let $h\colon[0,1]\to\mathbb{R}_+$ be such function that $h(t)\leqslant t$ holds for all $t\in[0,1]$ and $h(1)=1$. If for all $n,m_1,\dots,m_n\in\mathbb{N}$ $x_1,\dots,x_n\in D$ and $t_i,t_{ij}\in[0,1]$ satisfying $t_i=t_{i1}\cdot\ldots\cdot t_{im_i}$ and $t_1+\ldots+t_n=1$ the following inequality*

$$f\Big(\sum_{i=1}^{n}t_ix_i\Big)\leqslant\sum_{i=1}^{n}\prod_{j=1}^{m_i}h(t_{ij})g(x_i)-c\sum_{i=1}^{n}t_i\Big\|x_i-\sum_{i=1}^{n}t_ix_i\Big\|^2$$

holds then there exists a $\varphi\colon D\to\mathbb{R}$ strongly h_∞-convex function with modulus c such that

$$f\leqslant\varphi\leqslant g\qquad on\ D.$$

Proof. Fix $x\in D$ and define a function $\varphi\colon D\to\mathbb{R}$ by

$$\varphi(x)=\inf\Big\{\sum_{i=1}^{n}\prod_{j=1}^{m_i}h(t_{ij})g(x_i)-c\sum_{i=1}^{n}\ t_i\|x_i-x\|^2\ \Big|\ n\in\mathbb{N},x_1,\dots,x_n\in D,$$
$$t_1,\dots,t_n>0\ \text{ such that }\ \sum_{i=1}^{n}\ t_i=1,x=\sum_{i=1}^{n}t_ix_i,\prod_{j=1}^{m_i}t_{ij}=t_i\Big\}.$$

Our conditions grantee the correctness of this definition and $f(x)\leqslant\varphi(x)$ is also valid for all $x\in D$. On the other hand, taking $n=1$ and $m_1=1$ in the above definition and using the fact that $h(1)=1$, we get $\varphi(x)\leqslant g(x)$ for all $x\in D$.

Now we should prove that φ is strongly h_∞-convex with modulus c. Let us fix $x_1,\dots,x_n\in D$ and $t_1,\dots,t_n\in[0,1]$ such that $\sum_{i=1}^{n}t_i=1$. Let us choose arbitrary elements $x_{ik}\in D$ and $t_{ik}\in[0,1]$ for $i=1,\dots,n$; $k=1,\dots,n_i$ such that $x_i=\sum_{k=1}^{n_i}t_{ik}x_{ik}$ and $\sum_{k=1}^{n_i}t_{ik}=1$. Let us consider the coefficients

$$\lambda_{ik}:=t_i\prod_{j=1}^{m_{ik}}t_{ikj}=t_it_{ik}.$$

With this setting we get that

$$\sum_{i=1}^{n}\sum_{k=1}^{n_i}\lambda_{ik}x_{ik}=\sum_{i=1}^{n}t_i\sum_{k=1}^{n_i}t_{ik}x_{ik}=\sum_{i=1}^{n}t_ix_i=x,$$

and

$$\sum_{i=1}^{n}\sum_{k=1}^{n_i}\lambda_{ik}=\sum_{i=1}^{n}t_i\sum_{k=1}^{n_i}t_{ik}=\sum_{i=1}^{n}t_i=1.$$

That is, x is a convex combination of the elements x_{ik} with coefficients λ_{ik} where $i=1,\dots,n$ and $k=1,\dots,n_i$. Therefore, by the definition of φ we have

$$\varphi(x)\leqslant\sum_{i=1}^{n}\sum_{k=1}^{n_i}h(t_i)\prod_{j=1}^{m_{ik}}h(t_{ikj})g(x_{ik})-c\sum_{i=1}^{n}\sum_{k=1}^{n_i}t_it_{ik}\|x_{ik}-x\|^2$$

Using the property $h(t) \leqslant t$ let us notice that

$$\begin{aligned}\sum_{i=1}^{n}\sum_{k=1}^{n_i} t_i t_{ik}\|x_{ik}-x\|^2 &= \sum_{i=1}^{n} t_i\left(\sum_{k=1}^{n_i} t_{ik}\left(\|x_{ik}\|^2 - 2\langle x_{ik},x\rangle + \|x\|^2\right)\right)\\ &= \sum_{i=1}^{n} t_i\left(\sum_{k=1}^{n_i} t_{ik}\|x_{ik}\|^2 - 2\langle x_i,x\rangle + \|x\|^2\right)\\ &= \sum_{i=1}^{n} t_i\left(\sum_{k=1}^{n_i} t_{ik}\|x_{ik}-x_i\|^2 + \|x_i-x\|^2\right)\\ &\geqslant \sum_{i=1}^{n} h(t_i)\sum_{k=1}^{n_i} t_{ik}\|x_{ik}-x_i\|^2 + \sum_{i=1}^{n} t_i\|x_i-x\|^2.\end{aligned}$$

Applying this in the above inequality, we get

$$\varphi(x) \leqslant \sum_{i=1}^{n} h(t_i)\left(\sum_{k=1}^{n_i}\prod_{j=1}^{m_{ik}} h(t_{ikj})g(x_{ik}) - c\sum_{k=1}^{n_i} t_{ik}\|x_{ik}-x_i\|^2\right) - c\sum_{i=1}^{n} t_i\|x_i-x\|^2.$$

Finally, taking the infimum of the formula between parentheses on the right hand side and using the definition of φ, we have

$$\varphi\left(\sum_{i=1}^{n} t_i x_i\right) \leqslant \sum_{i=1}^{n} h(t_i)\varphi(x_i) - c\sum_{i=1}^{n} t_i\|x_i-x\|^2,$$

which shows the strongly h_∞-convexity of φ with modulus c. □

3. Applications

If $h\colon [0,1]\to\mathbb{R}$ is a multiplicative function with $h(1)=1$, then it is necessarily positive. Furthermore, multiplicativity also implies that the role of the product and the action of h can be interchanged in Theorem 2, that is,

$$\prod_{j=1}^{m_i} h(t_{ij}) = h(t_i) \qquad \text{whenever} \qquad t_i = t_{i1}\cdot\ldots\cdot t_{im_i}.$$

This means that the inequality of Theorem 2 reduces formally to the inequality of Theorem 1. Moreover, if h fulfills the additional property $h\leqslant \mathrm{id}$, the statements of Theorem 1 and Theorem 2 together give a *characterization* for the existence of an strongly h_∞-convex separator. Based on these simple observations, our main results lead to the theorems of Baron–Matkowski–Nikodem [2] and of Merentes–Nikodem [10] with $h=\mathrm{id}$. Similarly, for multiplicative h with $h\leqslant\mathrm{id}$, the Lara–Merentes–Nikodem Theorem [8] can be obtained from Theorem 2.

COROLLARY 1. *Real functions $f,g\colon D\to\mathbb{R}$ defined on a convex subset of a vector space can be separated by a convex function if and only if*

$$f\left(\sum_{i=1}^{n} t_i x_i\right) \leqslant \sum_{i=1}^{n} t_i g(x_i),$$

for all $n \in \mathbb{N}$, $x_1,\ldots,x_n \in D$ *and* $t_1,\ldots,t_n \in [0,1]$ *with* $t_1+\ldots+t_n = 1$.

COROLLARY 2. *Let* D *be a convex subset of a real vector space, let* $f,g\colon \to \mathbb{R}$ *be given and* $c \geqslant 0$. *There exists a strongly convex function* $\varphi\colon D \to \mathbb{R}$ *such that* $f \leqslant \varphi \leqslant g$ *on* D *if and only if*

$$f\Big(\sum_{i=1}^{n} t_i x_i\Big) \leqslant \sum_{i=1}^{n} t_i g(x_i) - c\sum_{i=1}^{n} t_i \|x_i - \bar{x}\|^2$$

for all $n \in \mathbb{N}$, $x_1\ldots,x_n \in D$ *and* $t_1,\ldots,t_n \in [0,1]$ *satisfying* $t_1+\ldots+t_n = 1$.

COROLLARY 3. *Let* D *be a convex subset of a real vector space, let* $f,g\colon \to \mathbb{R}$ *and* $h\colon [0,1] \to \mathbb{R}$ *be a multiplicative function such that* $h(t) \leqslant t$ *for all* $t \in [0,1]$ *and let* $c \geqslant 0$. *If for all* $n \in \mathbb{N}$, $x_1\ldots,x_n \in D$ *and* $t_1,\ldots,t_n \in [0,1]$ *satisfying* $t_1+\ldots+t_n = 1$ *the following inequality*

$$f\Big(\sum_{i=1}^{n} t_i x_i\Big) \leqslant \sum_{i=1}^{n} h(t_i) g(x_i) - c\sum_{i=1}^{n} t_i \|x_i - \bar{x}\|^2$$

holds where and $\bar{x} = t_1x_1+\ldots+t_nx_n$ *then there exists a* $\varphi\colon D \to \mathbb{R}$ *strongly* h*-convex function with modulus* c *such that* $f \leqslant \varphi \leqslant g$ *on* D.

Let us mention here two further consequences of Theorem 1 and Theorem 2 for the particular case $c = 0$ as Corollaries. Remarkable that they are partial generalizations of the theorems due to Olbryś [12] and Bessenyei–Pénzes [5]. Their results show, that the assumption $h \leqslant \mathrm{id}$ can be omitted.

COROLLARY 4. *Let* D *be a convex subset of a real vector space, let* $f,g\colon D \to \mathbb{R}$ *and* $h\colon [0,1] \to \mathbb{R}$ *be a non-zero multiplicative function with* $h \leqslant \mathrm{id}$. *There exists an* h*-convex function* $\varphi\colon D \to \mathbb{R}$ *such that* $f \leqslant \varphi \leqslant g$ *on* D *if and only if the following inequality*

$$f\Big(\sum_{i=1}^{n} t_i x_i\Big) \leqslant \sum_{i=1}^{n} h(t_i) g(x_i).$$

holds for all $n \in \mathbb{N}$, $x_1\ldots,x_n \in D$ *and* $t_1,\ldots,t_n \in [0,1]$ *satisfying* $t_1+\ldots+t_n = 1$.

COROLLARY 5. *Let* D *be a convex subset of a real vector space, let* $f,g\colon D \to \mathbb{R}$ *and* $h\colon [0,1] \to \mathbb{R}_+$ *be such that* $h(1) = 1$ *and* $h \leqslant \mathrm{id}$. *If for all* $n, m_1,\ldots,m_n \in \mathbb{N}$, $x_1,\ldots,x_n \in D$ *and for all* $t_i, t_{ij} \in [0,1]$ *satisfying* $t_i = t_{i1}\cdot\ldots\cdot t_{im_i}$ *and* $t_1+\ldots+t_n = 1$ *the following inequality*

$$f\Big(\sum_{i=1}^{n} t_i x_i\Big) \leqslant \sum_{i=1}^{n}\prod_{j=1}^{m_i} h(t_{ij}) g(x_i)$$

holds then there exists an h_∞*-convex function* $\varphi\colon D \to \mathbb{R}$ *such that* $f \leqslant \varphi \leqslant g$ *on* D.

Finally let us mention that the main motivating result, the Baron–Matkowski–Nikodem Theorem [2] has motivated many recent investigations. The papers [3] and [6] illustrate such situations and contain a detailed overview of this topic. Besides the problem of convex separation, its affine counterpart is also important. It was obtained by Nikodem and Wąsowicz [11]. It can be extended for Chebyshev systems [4] or, in more general, for Beckenbach families [7]. However, a similar extension of convex separation is still an open problem.

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