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INEQUALITIES AND BOUNDS FOR A CERTAIN BIVARIATE ELLIPTIC MEAN II

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Abstract. Further investigations of the bivariate elliptic mean introduced in [16] are presented. In particular new bounds for the mean under discussion are obtained. Also, the Wilker and Huygens-type inequalities as well as the Landen transformation are established. Results presented in this work are complimentary to those derived recently in [20] and [21].

1. Introduction

In recent years certain bivariate means have been investigated extensively by several researchers. A list of published papers which deal with those means is long and impressive. For obvious reasons that list is not included here.

The goal of this paper is to obtain bounds and inequalities for the particular mean introduced recently by this author in [16]. Its definition is included below (see (2)).

In what follows the letters a and b will always stand for positive and unequal numbers unless otherwise stated.

First we recall definition of the Schwab-Borchardt mean of a and b:

$$SB(a,b) \equiv SB = \begin{cases} \dfrac{\sqrt{b^2-a^2}}{\cos^{-1}(a/b)} & \text{if } a<b, \\ \dfrac{\sqrt{a^2-b^2}}{\cosh^{-1}(a/b)} & \text{if } a>b \end{cases} \tag{1}$$

(see, e.g., [1], [2]). This mean has been studied extensively in [22], [23] and in [8]. It is well known that the mean SB is strict, nonsymmetric and homogeneous of degree one in its variables.

Mean SB can also be expressed in terms of the degenerated completely symmetric elliptic integral of the first kind (see, e.g., [16]). It has been pointed out in [22] that some well known bivariate means such as logarithmic mean and two Seiffert means (see [27, 28]) can be represented by the Schwab-Borchardt mean of two simpler means such as geometric and arithmetic means or as the Schwab-Borchardt mean of arithmetic and the square - mean root mean. This idea was utilized lately by this author and other

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researchers as well. For more details the interested reader is referred to [4, 5, 6, 7, 8, 9, 11, 14, 15, 25, 26, 29, 30]

The mean studied in this paper is defined as follows:

$$N(a,b) \equiv N = \frac{1}{2}\left(a + \frac{b^2}{SB(a,b)}\right) \tag{2}$$

(see [16]). It's easy to see that mean N is also strict, nonsymmetric and homogeneous of degree one in its variables. Some authors call this mean, Neuman mean of the second kind (see, e.g., [5, 7, 25, 26, 29, 30]). Mean N can also be represented in terms of the degenerated completely symmetric elliptic integral of the second kind (see, e.g., [16]). By taking the $N-$ mean of two other means one can generate several new bivariate means. This idea was utilized in [16].

This paper is continuation of investigations reported in author's earlier works [20, 21, 19, 8, 16, 12, 11, 9, 14, 17, 15, 13, 18] and is organized as follows. Some preliminary results and formulas needed in this paper are given in Section 2. Bounds for the mean under discussion are obtained in Section 3. We close this paper with derivation of Landen's transformation for the mean under discussion. The Wilker and Huygens - type inequalities involving mean N are derived in Section 4. The Landen transformation for the mean under discussion is obtained in Section 5. Therein it is shown that this transformation is descending one.

2. Preliminary results and formulas needed in this paper

First of all let us record another formulas for means SB and N. Those will be utilized frequently in susequent sections of this paper.

One can easily verify that (1) implies

$$SB(a,b) \equiv SB = \begin{cases} b\dfrac{\sin r}{r} = a\dfrac{\tan r}{r} & \text{if } a < b, \\ b\dfrac{\sinh s}{s} = a\dfrac{\tanh s}{s} & \text{if } b < a, \end{cases} \tag{3}$$

where

$$\cos r = a/b \quad \text{if} \quad a < b \quad \text{and} \quad \cosh s = a/b \quad \text{if} \quad a > b. \tag{4}$$

Clearly

$$0 < r < \frac{\pi}{2} \tag{5}$$

and

$$s > 0. \tag{6}$$

Corresponding formulas for the mean N, obtained with the aid of (2) and (3), read as follows:

$$N(a,b) \equiv N = \frac{1}{2}b\left(\cos r + \frac{r}{\sin r}\right) = \frac{1}{2}a\left(1 + \frac{r}{\sin r \cos r}\right) \tag{7}$$

provided $a < b$. Similarly, if $a > b$, then

$$N(a,b) \equiv N = \frac{1}{2}b\left(\cosh s + \frac{s}{\sinh s}\right) = \frac{1}{2}a\left(1 + \frac{s}{\sinh s \cosh s}\right). \tag{8}$$

Here the domains for r and s are the same as these in (5) and (6).

Also, we will use the bivariate weighted power means of two positive numbers x_1 and x_2. The associated weights w_1 and w_2 are positive numbers which satisfy $w_1 + w_2 = 1$. With $X = (x_1, x_2)$ and $w = (w_1, w_2)$ the power mean of order p ($p \in \mathbb{R}$) is defined as follows

$$A_p(w;X) = \begin{cases} (w_1 x_1^p + w_2 x_2^p)^{1/p}, & p \neq 0 \\ x_1^{w_1} x_2^{w_2}, & p = 0. \end{cases} \tag{9}$$

It is well-known that the function $p \to A_p$ increases with increase in p.

For reader's convenience we recall definition of the celebrated Gauss hypergeometric function

$$F(\alpha,\beta;\gamma;z) = \sum_{n=0}^{\infty} \frac{(\alpha,n)(\beta,n)}{(\gamma,n)n!} z^n$$

($\alpha, \beta, \gamma \in \mathbb{R}$, $\gamma \neq 0, -1, \ldots$, $|z| < 1$) which is also denoted by ${}_2F_1(\alpha,\beta;\gamma;z)$. Here the symbol (α,n) stands for the shifted factorial also called the Appell symbol which is defined as $(\alpha,0) = 1$ for $a \neq 0$ and $(\alpha,n) = \alpha(\alpha-1)\cdot\ldots\cdot(\alpha-n+1)$ for $n = 1,2,\ldots$ (see, e.g., [3]).

We will need the following lower bound for the Gauss function. It has been established in [24] and reads as follows.

Suppose that

$$0 \leqslant \alpha \leqslant 1, \qquad \beta > 0, \qquad \text{and} \qquad \gamma \geqslant max(-\alpha,\beta). \tag{10}$$

If

$$\gamma \geqslant max(1-2\alpha, 2\beta) \qquad \text{and} \qquad p \leqslant \frac{\alpha+\gamma}{1+\gamma}, \tag{11}$$

then the inequality

$$A_p(\beta/\gamma, 1-\beta/\gamma, 1-x, 1) < [{}_2F_1(-\alpha,\beta;\gamma;x)]^{1/\alpha} \tag{12}$$

holds for all $x \in (0,1)$.

3. Bounds for mean N

In this section we provide new bounds for the mean $N(a,b)$. We begin with bounds expressed in terms of the power means of a and b. It has been proven in [16] that

$$A_1(w;X) < N(a,b) < A_2(w;X), \tag{13}$$

where

$$w = (1/3, 2/3) \qquad \text{and} \qquad X = (a,b). \tag{14}$$

An improvement of the left inequality in (13) reads as follows:

THEOREM 1. *Let w and X be the same as defined in (14). If $a < b$, then*

$$A_p(w;X) < N(a,b), \tag{15}$$

holds true for all $p \in [1,8/5]$.

Proof. First of all we will give a formula for mean N in terms of Gauss' function ${}_2F_1$

$$N(a,b) = b\,{}_2F_1(-1/2,1/2;3/2;1-(a/b)^2). \tag{16}$$

This result follows easily from [20, (3.4)], [3, (9.2-1)] and from [3, (5.9-1)]. We omit further details. Thus (16) yields $\alpha = \beta = 1/2$, $\gamma = 3/2$ and $x = 1-(a/b)^2$. Clearly conditions (10) and (11) are satisfied with $p \leqslant 4/5$. Making use of (16) and (12) we obtain, after a little algebra,

$$\left(\frac{1}{3}a^{2p} + \frac{2}{3}b^{2p}\right)^{1/(2p)} < N(a,b)$$

Since $p \leqslant 4/5$, $2p \leqslant 8/5$. The assertion now follows.

The next result reads as follows:

THEOREM 2. *Let A stand for the unweighted arithmetic mean of a and b and let*

$$\lambda = \frac{A}{b}. \tag{17}$$

Then the two - sided inequality

$$\frac{3}{\lambda + 2\sqrt{\lambda}} < \frac{2N(a,b)-a}{b} < \lambda^{-2/3} \tag{18}$$

is valid.

Proof. We shall utilize the invariance property of the Schwab-Borchardt mean

$$SB(a,b) = SB(A,\sqrt{Ab}) \tag{19}$$

(see, e.g., [1], [2]) together with the two-sided bounds

$$(ab^2)^{1/3} < SB(a,b) < \frac{a+2b}{3} \tag{20}$$

(see [22]). Using (20) with a replaced by A and with b replaced by $\sqrt{Ab}$ we obtain

$$\frac{3b}{A+2\sqrt{Ab}} < \frac{b}{SB(a,b)} < \left(\frac{b}{A}\right)^{-2/3} \tag{21}$$

Utilizing (2) we obtain

$$\frac{b}{SB(a,b)} = \frac{2N(a,b)-a}{b}. \tag{22}$$

This in conjunction with (21) and (17) yields the asserted inequality (18). The proof is complete.

4. Wilker and Huygens- type inequalities for the mean N

The main result of this section is an inequality involving a sum of powers of the quotients N/a and N/b and is easily derived using the following result [10]:

PROPOSITION 1. Let u, v, λ, μ be positive numbers. Assume that

$$1 < u^{\gamma} v^{\delta} \tag{23}$$

holds for some nonnegative numbers γ and δ whose sum equals to 1. If $u < 1 < v$, then the inequality

$$1 < \frac{\lambda}{\lambda+\mu} u^p + \frac{\mu}{\lambda+\mu} v^q \tag{24}$$

holds true if

$$q > 0 \qquad \text{and} \qquad p\lambda\delta \leqslant q\mu\gamma. \tag{25}$$

If $v < 1 < u$, then inequality (24) is valid if

$$p > 0 \qquad \text{and} \qquad q\mu\gamma \leqslant p\lambda\delta. \tag{26}$$

For brevity we will write below N instead of $N(a,b)$. We have the following:

THEOREM 3. *Let a and b be positive unequal numbers. Further let λ and μ be positive numbers. Then the inequality*

$$1 < \frac{\lambda}{\lambda+\mu}\left(\frac{N}{b}\right)^p + \frac{\mu}{\lambda+\mu}\left(\frac{N}{a}\right)^q \tag{27}$$

holds true if either

$$q > 0 \qquad \textit{and} \qquad p\lambda \leqslant 2q\mu \tag{28}$$

or if

$$p > 0 \qquad \textit{and} \qquad 2q\mu \leqslant p\lambda \tag{29}$$

Proof. Consider first the case when $a < b$. This implies that $a < N < b$. The last inequality can also be written as

$$u < 1 < v,$$

where

$$u = \frac{N}{b} \qquad \text{and} \qquad v = \frac{N}{a}. \tag{30}$$

Making use of inequality (15) together with the inequality of arithmetic and geometric means we obtain

$$A_0(w,X) < N.$$

Writing the last inequality in the form

$$(ab^2)^{1/3} < N$$

we obtain using (30)

$$u^{2/3}v^{1/3} < 1.$$

Thus

$$\gamma = 2/3 \qquad \text{and} \qquad \delta = 1/3. \tag{31}$$

Clearly conditions(25) imply (28). In the case when $b < a$ we have $b < N < a$. With u and v as defined in (30) we see that $v < 1 < u$. This in conjunction with (31) and (26) gives the asserted result. The proof is complete.

5. Landen's transformation for the mean N

Landen's transformation plays an important role in theory of special functions. This is well documented in B.C. Carlson's monograph [3].

In this section we shall derive Landen's transformation for the mean N. For the sake of presentation we introduce quantities c and d, where

$$c = A \qquad \text{and} \qquad d = \sqrt{Ab}. \tag{32}$$

Recall that the symbol A stands for the unweighted arithmetic mean of a and b.

The main result of this section reads as follows:

THEOREM 4. *The following formula*

$$N(c,d) = \frac{A}{b}[N(a,b) + \frac{1}{2}(b-a)] \tag{33}$$

is valid for all positive numbers a and b.

Proof. We use first formula (2) with a and b replaced, respectively, by c and d followed by application of the second part of (32) and (19) to obtain

$$N(c,d) = \frac{1}{2}\left(c + \frac{d^2}{SB(c,d)}\right) = \frac{1}{2}\left(A + \frac{Ab}{SB(a,b)}\right) = \frac{1}{2}A\left(1 + \frac{b}{SB(a,b)}\right).$$

Utilizing formula (22) we obtain

$$N(c,d) = \frac{1}{2}A\left(1 + \frac{2N(a,b)-a}{b}\right) = \frac{A}{b}\left[N(a,b) + \frac{1}{2}(b-a)\right].$$

This completes the proof.

We close this section by demonstrating that the Landen transformation for N is descending. Its easy to see that

$$a < c < d < b \tag{34}$$

provided $a < b$. Inequalities (34) are reversed if $a > b$.

We shall now prove the announced earlier monotonicity result:

THEOREM 5. *The following inequality*

$$N(c,d) < N(a,b) \tag{35}$$

is satisfied for all positive and unequal numbers a and b.

Proof. Using the invariance property (19) we get

$$SB(c,d) = SB(a,b) =: \lambda.$$

Utilizing formula (2) twice we obtain

$$N(c,d) = \frac{1}{2}\left(c + \frac{d^2}{\lambda}\right) \tag{36}$$

and

$$N(a,b) = \frac{1}{2}\left(a + \frac{b^2}{\lambda}\right). \tag{37}$$

In order to obtain the desired result it suffices to show that

$$c + \frac{d^2}{\lambda} < a + \frac{b^2}{\lambda}$$

or what is the same that

$$\lambda < \frac{b^2 - d^2}{c - a}.$$

Letting above $c = A$ and $d = Ab$ we see that the last inequality simplifies to $\lambda < b$ which is satisfied because $\lambda = SB(a,b) < b$. This completes the proof when $a < b$. If $a > b$, then the proof goes along lines already used above. We omit further details.

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