SOBOLEV'S INEQUALITIES FOR
HERZ–MORREY–ORLICZ SPACES ON THE HALF SPACE

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Dedicated to Professor Eiichi Nakai
on the occasion of his sixtieth birthday

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Abstract. We introduce Herz-Morrey-Orlicz spaces on the half space, and study the boundedness of the Hardy-Littlewood maximal operator. As an application, we establish Sobolev's inequality for Riesz potentials of functions in such spaces, which is one of mixed norm type inequalities.

1. Introduction

Let $\mathbf{R}^n$ be the Euclidean space. In harmonic analysis, the maximal operator is a classical tool when studying Sobolev functions and partial differential equations. This also plays a central role in the study of differentiation, singular integrals, smoothness of functions and so on (see [4, 9, 18], etc.).

It is well known that the maximal operator is bounded on the Lebesgue space $L^p(\mathbf{R}^n)$ if $p > 1$ (see [18]). The boundedness of the maximal operator was studied on Morrey spaces in [7, 14], on Orlicz-Morrey spaces in [16], and also on non-homogeneous Herz spaces in [10]. For Morrey spaces, which were introduced to estimate solutions of partial differential equations, we refer to [13, 17].

Recently, the boundedness of the maximal operator was studied for non-homogeneous central Herz-Morrey-Orlicz spaces on the whole space $\mathbf{R}^n$ (see [12]). Let $\mathbf{H}_+$ denote the half plane :

$$\mathbf{H}_+ = \{x = (x', x_n) \in \mathbf{R}^{n-1} \times \mathbf{R} : x_n > 0\}.$$

Our first aim in this paper is to study the boundedness of the Hardy-Littlewood maximal operator on $\mathscr{H}^{\Phi, \omega, q}(\mathbf{H}_+)$ defined by a general function Φ and a weight ω satisfying certain conditions and $0 < q \leqslant \infty$ (see Theorems 3.4 and 3.5 below). The space $\mathscr{H}^{\Phi, \omega, q}(\mathbf{H}_+)$ is referred to as a Herz-Morrey-Orlicz space on the half space. See Section 2 for the definitions of Φ, ω and $\mathscr{H}^{\Phi, \omega, q}(\mathbf{H}_+)$. To prove Theorem 3.4, we need

the boundedness of maximal operator on L^Φ (see Lemma 3.1), and treat only the case $1 < q < \infty$, because the remaining case is easily settled.

One of the important applications of the boundedness of the maximal operator is Sobolev's inequality; in classical Lebesgue spaces, we know Sobolev's inequality:

$$\|I_\alpha f\|_{L^{p^*}(\mathbf{R}^n)} \leqslant C \|f\|_{L^p(\mathbf{R}^n)}$$

for $f \in L^p(\mathbf{R}^n)$, $0 < \alpha < n$ and $1 < p < n/\alpha$, where I_α is the Riesz kernel of order α and $1/p^* = 1/p - \alpha/n$ (see, e.g. [2, Theorem 3.1.4]). Sobolev's inequality for Morrey spaces was given by D. R. Adams [1] (also [7, 14]), and then the result was extended to Orlicz-Morrey spaces in [15]. For local Morrey-type spaces, we refer the reader to [5, 6], for example. See also [10] for non-homogeneous Herz spaces and [8] for non-homogeneous central Morrey spaces.

In [12], the boundedness of Riesz potential operators was studied for non-homogeneous central Herz-Morrey-Orlicz spaces on $\mathbf{R}^n$. Our second aim in this paper is to give a general version of Sobolev's inequality for Riesz potentials of functions in $\mathcal{H}^{\Phi,\omega,q}(\mathbf{H}_+)$ as an application of the boundedness of the maximal operator on $\mathcal{H}^{\Phi,\omega,q}(\mathbf{H}_+)$ (see Theorem 4.7 below). This seems to be new even for the case $\Phi(r) = r^p$ and $\omega(r) = r^{-\nu}$ (see Corollary 4.8 and Remark 4.9 below). The key lemma for our main Theorem 4.7 is Lemma 4.5 below.

2. Preliminaries

Throughout this paper, let C denote various positive constants independent of the variables in question.

Let Φ be a convex function on $[0,\infty)$ such that

(Φ1) $\Phi(0) = 0$ and $\Phi(r) > 0$ for $r > 0$;

(Φ2) for some $p_2 > 1$, $r^{-p_2}\Phi(r)$ is almost decreasing, that is, there exists a constant $A_1 > 0$ such that

$$\Phi(rt) \leqslant A_1 r^{p_2} \Phi(t) \qquad \text{when } r > 1 \text{ and } t > 0;$$

(Φ3) for some $1 < p_1 \leqslant p_2$, $r^{-p_1}\Phi(r)$ is almost increasing, that is, there exists a constant $A_2 > 0$ such that

$$\Phi(rt) \geqslant A_2 r^{p_1} \Phi(t) \qquad \text{when } r > 1 \text{ and } t > 0.$$

Here note from (Φ2) that Φ is doubling, that is, there exists a constant $A > 0$ such that

$$\Phi(2r) \leqslant A\Phi(r) \qquad \text{for } r > 0.$$

Further consider a weight ω such that

(ω1) $\omega(r) > 0$ for $r > 0$;

(ω2) ω is almost decreasing in $(0,\infty)$;

(ω3) ω is doubling.

We see that $\omega(r) = r^{-\nu}(\log(e+r))^{\theta}$ is almost decreasing when $\nu > 0$ and $\theta \in \mathbf{R}$.
    Note here that (Φ2) holds if and only if

(Φ2′) $\Phi(rt) \geqslant A_1^{-1} r^{p_2} \Phi(t)$ when $0 < r < 1$ and $t > 0$;

and (Φ3) holds if and only if

(Φ3′) $\Phi(rt) \leqslant A_2^{-1} r^{p_1} \Phi(t)$ when $0 < r < 1$ and $t > 0$.

Moreover, if Φ is of the form $r^p (\log(e+r))^{\theta}$ for $p > 1$ and $\theta \in \mathbf{R}$, then (Φ2) and (Φ3) hold when $1 < p_1 < p < p_2$. In particular, $p_1 = p = p_2$ if and only if $\theta = 0$.
    For $r > 0$, set

$$H(r) = \{x = (x', x_n) \in \mathbf{R}^{n-1} \times \mathbf{R} : 0 < x_n < r\}$$

and

$$A(r) = H(2r) \setminus H(r).$$

For $0 < q \leqslant \infty$, we denote by $\mathscr{H}^{\Phi,\omega,q}(\mathbf{H}_+)$ the class of measurable functions f on the half space $\mathbf{H}_+$ satisfying

$$\|f\|_{\mathscr{H}^{\Phi,\omega,q}(\mathbf{H}_+)} = \inf\left\{\lambda > 0 : \int_{H(2)} \Phi\left(\frac{|f(y)|}{\lambda}\right) dy \right.$$
$$\left. + \left(\int_1^{\infty} \left(\omega(r) \int_{A(r)} \Phi\left(\frac{|f(y)|}{\lambda}\right) dy\right)^q \frac{dr}{r}\right)^{1/q} \leqslant 1\right\} < \infty$$

when $0 < q < \infty$, $\mathscr{H}^{\Phi,\omega,\infty}(\mathbf{H}_+)$ is the space of all measurable functions f on $\mathbf{H}_+$ such that

$$\|f\|_{\mathscr{H}^{\Phi,\omega,\infty}(\mathbf{H}_+)}$$
$$= \inf\left\{\lambda > 0 : \int_{H(2)} \Phi\left(\frac{|f(y)|}{\lambda}\right) dy + \sup_{r \geqslant 1}\left(\omega(r) \int_{A(r)} \Phi\left(\frac{|f(y)|}{\lambda}\right) dy\right) \leqslant 1\right\} < \infty.$$

The space $\mathscr{H}^{\Phi,\omega,q}(\mathbf{H}_+)$ is referred to as a non-homogeneous central Herz-Morrey-Orlicz space on the half space $\mathbf{H}_+$, which is one of spaces with mixed norm. In connection with $\mathscr{H}^{\Phi,\omega,q}(\mathbf{H}_+)$, we consider the space $\underline{\mathscr{H}}^{\Phi,\omega,q}(\mathbf{H}_+)$ of all measurable functions f on $\mathbf{H}_+$ satisfying

$$\|f\|_{\underline{\mathscr{H}}^{\Phi,\omega,q}(\mathbf{H}_+)} = \inf\left\{\lambda > 0 : \int_{H(2)} \Phi\left(\frac{|f(y)|}{\lambda}\right) dy \right.$$
$$\left. + \left(\int_1^{\infty} \left(\omega(r) \int_{H(r)} \Phi\left(\frac{|f(y)|}{\lambda}\right) dy\right)^q \frac{dr}{r}\right)^{1/q} \leqslant 1\right\} < \infty$$

when $0 < q < \infty$; if $q = \infty$, then we need necessary modifications. For these spaces, see e.g. [3, 5, 6, 13].

LEMMA 2.1. *Suppose*

(ω4) $r^{\delta_0}\omega(r)$ *is almost decreasing on* $[1,\infty)$ *for some* $\delta_0 > 0$.

Then $\mathscr{H}^{\Phi,\omega,q}(\mathbf{H}_+) = \underline{\mathscr{H}}^{\Phi,\omega,q}(\mathbf{H}_+)$.

Proof. Let $0 < \delta < \delta_0$. First, we show the case $0 < q < \infty$. Let f be a measurable function on $\mathbf{H}_+$ with

$$\int_{H(2)} \Phi(|f(y)|)\,dy + \left(\int_1^\infty \left(\omega(r) \int_{A(r)} \Phi(|f(y)|)\,dy \right)^q \frac{dr}{r} \right)^{1/q} \leqslant 1.$$

In case $0 < q < \infty$, by Hölder's inequality, we obtain

$$\int_{H(r)\setminus H(2)} \Phi(|f(y)|)\,dy$$
$$\leqslant C \int_1^r \left(\int_{A(t)} \Phi(|f(y)|)\,dy \right) \frac{dt}{t}$$
$$\leqslant C \left(\int_1^r \left(t^{-\delta}\omega(t)^{-1} \right)^{q'} \frac{dt}{t} \right)^{1/q'} \left(\int_1^r \left(t^{\delta}\omega(t) \int_{A(t)} \Phi(|f(y)|)\,dy \right)^q \frac{dt}{t} \right)^{1/q}$$
$$\leqslant C r^{-\delta}\omega(r)^{-1} \left(\int_1^r \left(t^{\delta}\omega(t) \int_{A(t)} \Phi(|f(y)|)\,dy \right)^q \frac{dt}{t} \right)^{1/q}$$

for $r \geqslant 1$ since

$$\int_1^r \left(t^{-\delta}\omega(t)^{-1} \right)^{q'} \frac{dt}{t} \leqslant C \left(r^{-\delta_0}\omega(r)^{-1} \right)^{q'} \int_1^r t^{(\delta_0 - \delta)q'} \frac{dt}{t}$$
$$= C \left(r^{-\delta_0}\omega(r)^{-1} \right)^{q'} r^{(\delta_0 - \delta)q'} = C r^{-\delta q'}\omega(r)^{-q'}$$

by (ω4). For the case $0 < q \leqslant 1$, we use the fact that $(a+b)^q \leqslant a^q + b^q$ for all $a,b \geqslant 0$ instead of Hölder's inequality and use

$$t^{-\delta}\omega(t)^{-1} \leqslant C r^{-\delta}\omega(r)^{-1}$$

for all $1 \leqslant t \leqslant r$ by (ω4). Hence we have by (ω4)

$$\int_1^\infty \left(\omega(r) \int_{H(r)} \Phi(|f(y)|)\,dy \right)^q \frac{dr}{r}$$
$$\leqslant C \left\{ \int_1^\infty \left(\omega(r) \int_{H(r)\setminus H(2)} \Phi(|f(y)|)\,dy \right)^q \frac{dr}{r} + \int_1^\infty \left(\omega(r) \int_{H(2)} \Phi(|f(y)|)\,dy \right)^q \frac{dr}{r} \right\}$$
$$\leqslant C \left\{ \int_1^\infty \left(t^{\delta}\omega(t) \int_{A(t)} \Phi(|f(y)|)\,dy \right)^q \left(\int_t^\infty r^{-\delta q} \frac{dr}{r} \right) \frac{dt}{t} + \left(\int_{H(2)} \Phi(|f(y)|)\,dy \right)^q \right\}$$
$$\leqslant C \left\{ \int_1^\infty \left(\omega(t) \int_{A(t)} \Phi(|f(y)|)\,dy \right)^q \frac{dr}{r} + \left(\int_{H(2)} \Phi(|f(y)|)\,dy \right)^q \right\},$$

so that

$$\int_{H(2)} \Phi(|f(y)|)\,dy + \left(\int_1^{\infty} \left(\omega(r) \int_{H(r)} \Phi(|f(y)|)\,dy \right)^q \frac{dr}{r} \right)^{1/q} \leqslant C.$$

Next, we deal with the case $q = \infty$. Let f be a measurable function on $\mathbf{H}_+$ with

$$\int_{H(2)} \Phi(|f(y)|)\,dy + \sup_{r \geqslant 1} \left(\omega(r) \int_{A(r)} \Phi(|f(y)|)\,dy \right) \leqslant 1.$$

By $(\omega 4)$, we obtain

$$\int_{H(r)\backslash H(2)} \Phi(|f(y)|)\,dy \leqslant C \int_1^r \left(\int_{A(t)} \Phi(|f(y)|)\,dy \right) \frac{dt}{t}$$

$$\leqslant C \left(\int_1^r t^{-\delta} \omega(t)^{-1} \frac{dt}{t} \right) \sup_{1 \leqslant t \leqslant r} \left(t^{\delta} \omega(t) \int_{A(t)} \Phi(|f(y)|)\,dy \right)$$

$$\leqslant C r^{-\delta} \omega(r)^{-1} \sup_{1 \leqslant t \leqslant r} \left(t^{\delta} \omega(t) \int_{A(t)} \Phi(|f(y)|)\,dy \right)$$

$$\leqslant C \omega(r)^{-1} \sup_{1 \leqslant t \leqslant r} \left(\omega(t) \int_{A(t)} \Phi(|f(y)|)\,dy \right)$$

for $r \geqslant 1$. Hence we have by $(\omega 2)$

$$\sup_{r \geqslant 1} \left(\omega(r) \int_{H(r)} \Phi(|f(y)|)\,dy \right)$$

$$\leqslant \sup_{r \geqslant 1} \left(\omega(r) \int_{H(r)\backslash H(2)} \Phi(|f(y)|)\,dy \right) + \sup_{r \geqslant 1} \left(\omega(r) \int_{H(2)} \Phi(|f(y)|)\,dy \right)$$

$$\leqslant C \left\{ \sup_{r \geqslant 1} \left(\sup_{1 \leqslant t \leqslant r} \left(\omega(t) \int_{A(t)} \Phi(|f(y)|)\,dy \right) \right) + \int_{H(2)} \Phi(|f(y)|)\,dy \right\}$$

$$\leqslant C \left\{ \sup_{t \geqslant 1} \left(\omega(t) \int_{A(t)} \Phi(|f(y)|)\,dy \right) + \int_{H(2)} \Phi(|f(y)|)\,dy \right\},$$

so that

$$\int_{H(2)} \Phi(|f(y)|)\,dy + \sup_{r \geqslant 1} \left(\omega(r) \int_{H(r)} \Phi(|f(y)|)\,dy \right) \leqslant C. \quad \square$$

3. The boundedness of the maximal operator

We define the maximal function of a locally integrable function f on $\mathbf{H}_+$ by

$$Mf(x) = \sup_{r > 0} \frac{1}{|B(x,r)|} \int_{B(x,r) \cap \mathbf{H}_+} |f(y)|\,dy,$$

where $B(x,r)$ denotes the open ball centered at $x \in \mathbf{R}^n$ with radius $r > 0$.

In view of $(\Phi 3)$, we have the boundedness of maximal operator in $L^{\Phi}(\mathbf{H}_+)$ $= \{ f \in L^1_{loc}(\mathbf{H}_+) : \int_{\mathbf{H}_+} \Phi(|f(y)|)\,dy < \infty \}$.

LEMMA 3.1. ([11, Corollary 4.4]) *There exists a constant $C > 0$ such that*

$$\int_{\mathbf{H}_+} \Phi(Mf(x))\,dx \leqslant C \int_{\mathbf{H}_+} \Phi(|f(y)|)\,dy.$$

Let us start with the following lemma which will be useful in the sequel.

LEMMA 3.2. *Let $a < 1$. Then there exists a constant $C > 0$ such that*

$$\int_{\mathbf{R}^{n-1}} |x-y|^{a-n}\,dx' \leqslant C|x_n - y_n|^{a-1},$$

where $x = (x', x_n)$ and $y = (y', y_n)$.

Proof. For $x = (x', x_n)$ and $y = (y', y_n)$, set $r = |x'-y'|$ and $b = |x_n - y_n|$. Then

$$\int_{\mathbf{R}^{n-1}} |x-y|^{a-n}\,dx' = C \int_0^\infty (r^2+b^2)^{(a-n)/2} r^{n-2}\,dr \leqslant Cb^{a-1}. \quad \square$$

For a real number κ, $x \in H(r)$ and $r > 1$, we define

$$J_{\kappa,r}(x) = \int_{\mathbf{H}_+ \setminus (H(2r) \cup B(x,r))} |x-y|^{\kappa-n} f(y)\,dy$$

and set

$$f_{1,r} = f_{1,r;x} = f\chi_{\mathbf{H}_+ \setminus (H(2r) \cup B(x,r))}.$$

Next result is instrumental for the boundedness of the maximal operator.

LEMMA 3.3. *Suppose*

$(\omega5)$ $r^{\kappa p_2 - 1 + \beta_0} \omega(r)^{-1}$ *is almost decreasing on $[1,\infty)$ for some $\beta_0 > 0$.*

Let $\varepsilon > 0$ and $\beta > 0$ such that

$$\kappa p_2 + \varepsilon(p_2 - 1) - 1 < 0 \qquad and \qquad \varepsilon(p_2 - 1) + \beta < \beta_0.$$

Then there exists a constant $C > 0$ such that

$$r^{-\varepsilon p_2} \int_{H(r)} \Phi(r^\varepsilon J_{\kappa,r}(x))\,dx \leqslant Cr^{\kappa p_2 + \beta} \omega(r)^{-1} \left(\int_r^\infty \left(t^{-\beta} \omega(t) \int_{A(t)} \Phi(f(y))\,dy \right)^q \frac{dt}{t} \right)^{1/q}$$

when $0 < q < \infty$ and

$$r^{-\varepsilon p_2} \int_{H(r)} \Phi(r^\varepsilon J_{\kappa,r}(x))\,dx \leqslant Cr^{\kappa p_2} \omega(r)^{-1} \sup_{t \geqslant r} \left(\omega(t) \int_{A(t)} \Phi(f(y))\,dy \right)$$

when $q = \infty$, for all nonnegative measurable functions f on $\mathbf{H}_+$.

Proof. We show only the case $1 < q < \infty$. Let ε be given as in the present lemma. For a nonnegative measurable function f on $\mathbf{H}_+$ and $x \in H(r)$, we have

$$
\begin{aligned}
J_{\kappa,r}(x) &= \int_{\mathbf{H}_+ \setminus (H(2r) \cup B(x,r))} |x-y|^{\kappa-n} f(y)\, dy \\
&\leqslant C \int_r^\infty \left(\frac{1}{|B(x,t)|} \int_{B(x,t)} f_{1,r}(y)\, dy \right) t^{\kappa-1}\, dt.
\end{aligned}
$$

We obtain by Jensen's inequality and ($\Phi 2$)

$$
\begin{aligned}
\Phi(r^\varepsilon J_{\kappa,r}(x)) &\leqslant C\Phi \left(r^\varepsilon \int_r^\infty \left(\frac{1}{|B(x,t)|} \int_{B(x,t)} t^{\kappa+\varepsilon} f_{1,r}(y)\, dy \right) t^{-\varepsilon-1}\, dt \right) \\
&\leqslant C r^\varepsilon \int_r^\infty \left(\frac{1}{|B(x,t)|} \int_{B(x,t)} \Phi(t^{\kappa+\varepsilon} f_{1,r}(y))\, dy \right) t^{-\varepsilon-1}\, dt \\
&\leqslant C r^\varepsilon \int_r^\infty t^{(\kappa+\varepsilon)p_2-n} \left(\int_{B(x,t)} \Phi(f_{1,r}(y))\, dy \right) t^{-\varepsilon-1}\, dt \\
&\leqslant C r^\varepsilon \int_{\mathbf{H}_+ \setminus (H(2r) \cup B(x,r))} |x-y|^{\kappa p_2 + \varepsilon(p_2-1)-n} \Phi(f(y))\, dy
\end{aligned}
$$

since $r > 1$ and

$$
\kappa p_2 + \varepsilon(p_2 - 1) - n < \kappa p_2 + \varepsilon(p_2 - 1) - 1 < 0.
$$

If $x \in H(r)$ and $y \in \mathbf{H}_+ \setminus B(x,r)$, then for $\bar{x} = (x', -x_n)$

$$
|\bar{x} - y| \leqslant 2x_n + |x-y| < 2r + |x-y| < 3|x-y|.
$$

Therefore, Lemma 3.2 gives

$$
\begin{aligned}
\int_{H(r)} |x-y|^{\kappa p_2 + \varepsilon(p_2-1)-n}\, dx &\leqslant \int_{H(r)} (|\bar{x}-y|/3)^{\kappa p_2 + \varepsilon(p_2-1)-n}\, dx \\
&\leqslant C \int_0^r (x_n + y_n)^{\kappa p_2 + \varepsilon(p_2-1)-1}\, dx_n \\
&\leqslant C r y_n^{\kappa p_2 + \varepsilon(p_2-1)-1}
\end{aligned}
$$

for $y \in \mathbf{H}_+ \setminus (H(2r) \cup B(x,r))$ since $\kappa p_2 + \varepsilon(p_2 - 1) < 1$. Hence we obtain by Hölder's inequality and ($\omega 5$)

$$
\begin{aligned}
\int_{H(r)} \Phi(r^\varepsilon J_{\kappa,r}(x))\, dx &\leqslant C r^\varepsilon \int_{\mathbf{H}_+ \setminus H(2r)} \left(\int_{H(r)} |x-y|^{\kappa p_2 + \varepsilon(p_2-1)-n}\, dx \right) \Phi(f(y))\, dy \\
&\leqslant C r^{\varepsilon+1} \int_{\mathbf{H}_+ \setminus H(2r)} y_n^{\kappa p_2 + \varepsilon(p_2-1)-1} \Phi(f(y))\, dy \\
&\leqslant C r^{\varepsilon+1} \sum_{j=1}^\infty (2^j r)^{\kappa p_2 + \varepsilon(p_2-1)-1} \int_{A(2^j r)} \Phi(f(y))\, dy
\end{aligned}
$$

$$\leqslant Cr^{\varepsilon+1}\left(\sum_{j=1}^{\infty}\left((2^j r)^{\kappa p_2+\varepsilon(p_2-1)-1+\beta}\omega(2^j r)^{-1}\right)^{q'}\right)^{1/q'}$$

$$\times\left(\sum_{j=1}^{\infty}\left((2^j r)^{-\beta}\omega(2^j r)\int_{A(2^j r)}\Phi(f(y))\,dy\right)^q\right)^{1/q}$$

$$\leqslant Cr^{(\kappa+\varepsilon)p_2+\beta}\omega(r)^{-1}$$

$$\times\left(\sum_{j=1}^{\infty}\left((2^j r)^{-\beta}\omega(2^j r)\int_{A(2^j r)}\Phi(f(y))\,dy\right)^q\right)^{1/q},$$

which gives

$$\int_{H(r)}\Phi(r^\varepsilon J_{\kappa,r}(x))\,dx$$

$$\leqslant Cr^{(\kappa+\varepsilon)p_2+\beta}\omega(r)^{-1}\left(\int_r^\infty\left(t^{-\beta}\omega(t)\int_{A(t)}\Phi(f(y))\,dy\right)^q\frac{dt}{t}\right)^{1/q},$$

which proves the assertion. $\square$

Now we are ready to prove the boundedness of maximal operator on $\underline{\mathscr{H}}^{\Phi,\omega,q}(\mathbf{H}_+)$.

THEOREM 3.4. *Suppose*

$(\omega5')$ $r^{-1+\beta_0}\omega(r)^{-1}$ *is almost decreasing on* $[1,\infty)$ *for some* $\beta_0>0$.

Let $\varepsilon>0$ *be given such that*

$$\varepsilon(p_2-1)-1<0\qquad and\qquad \varepsilon(p_2-1)<\beta_0.$$

Then there exists a constant $C>0$ *such that*

$$\int_{H(2)}\Phi(Mf(x))\,dx+\left(\int_1^\infty\left(r^{-\varepsilon p_2}\omega(r)\int_{H(r)}\Phi(r^\varepsilon Mf(x))\,dx\right)^q\frac{dr}{r}\right)^{1/q}\leqslant C$$

when $0<q<\infty$ *and*

$$\int_{H(2)}\Phi(Mf(x))\,dx+\sup_{r\geqslant1}\left(r^{-\varepsilon p_2}\omega(r)\int_{H(r)}\Phi(r^\varepsilon Mf(x))\,dx\right)\leqslant C$$

when $q=\infty$, *for all* $f\in\underline{\mathscr{H}}^{\Phi,\omega,q}(\mathbf{H}_+)$ *with* $\|f\|_{\underline{\mathscr{H}}^{\Phi,\omega,q}(\mathbf{H}_+)}\leqslant1$.

Proof. We show only the case $1<q<\infty$. Let f be a nonnegative measurable function on $\mathbf{H}_+$ such that

$$\int_{H(2)}\Phi(f(y))\,dy+\left(\int_1^\infty\left(\omega(r)\int_{H(r)}\Phi(f(y))\,dy\right)^q\frac{dr}{r}\right)^{1/q}\leqslant1.$$

For $r > 1$ write

$$f = f\chi_{\mathbf{H}_+ \setminus H(2r)} + f\chi_{H(2r)} = f_{2,r} + f_{3,r}.$$

First note by ($\Phi 2$) and Lemma 3.1 that

$$r^{-\varepsilon p_2} \int_{H(r)} \Phi(r^{\varepsilon} M f_{3,r}(x)) \, dx \leqslant C \int_{H(r)} \Phi(M f_{3,r}(y)) \, dy$$

$$\leqslant C \int_{\mathbf{H}_+} \Phi(f_{3,r}(y)) \, dy$$

$$= C \int_{H(2r)} \Phi(f(y)) \, dy,$$

so that

$$\int_{H(2)} \Phi(M f_{3,2}(x)) \, dx \leqslant C \int_{H(4)} \Phi(f(y)) \, dy \leqslant C$$

and

$$\int_1^{\infty} \left(r^{-\varepsilon p_2} \omega(r) \int_{H(r)} \Phi(r^{\varepsilon} M f_{3,r}(x)) \, dx \right)^q \frac{dr}{r} \leqslant C \int_1^{\infty} \left(\omega(r) \int_{H(2r)} \Phi(f(y)) \, dy \right)^q \frac{dr}{r}$$

$$\leqslant C \int_1^{\infty} \left(\omega(r) \int_{H(r)} \Phi(f(y)) \, dy \right)^q \frac{dr}{r}$$

$$\leqslant C.$$

For $x \in H(r)$, note that

$$M f_{2,r}(x) \leqslant C J_{0,r}(x).$$

Take $\beta > 0$ such that

$$\varepsilon(p_2 - 1) + \beta < \beta_0.$$

Then we obtain by Lemma 3.3 and ($\omega 5'$)

$$r^{-\varepsilon p_2} \int_{H(2)} \Phi(r^{\varepsilon} M f_{2,r}(x)) \, dx \leqslant C r^{\beta} \omega(r)^{-1} \left(\int_r^{\infty} \left(t^{-\beta} \omega(t) \int_{A(t)} \Phi(f(y)) \, dy \right)^q \frac{dt}{t} \right)^{1/q},$$

so that

$$\int_{H(2)} \Phi(M f_{2,2}(x)) \, dx \leqslant C \left(\int_2^{\infty} \left(t^{-\beta} \omega(t) \int_{A(t)} \Phi(f(y)) \, dy \right)^q \frac{dt}{t} \right)^{1/q}$$

$$\leqslant C \left(\int_2^{\infty} \left(\omega(t) \int_{H(t)} \Phi(f(y)) \, dy \right)^q \frac{dt}{t} \right)^{1/q}$$

$$\leqslant C$$

and again by Lemma 3.3

$$\int_1^\infty \left(r^{-\varepsilon p_2} \omega(r) \int_{H(r)} \Phi(r^\varepsilon M f_{2,r}(x)) \, dx \right)^q \frac{dr}{r}$$

$$\leqslant C \int_1^\infty r^{\beta q} \left(\int_r^\infty \left(t^{-\beta} \omega(t) \int_{A(t)} \Phi(f(y)) \, dy \right)^q \frac{dt}{t} \right) \frac{dr}{r}$$

$$\leqslant C \int_1^\infty \left(t^{-\beta} \omega(t) \int_{A(t)} \Phi(f(y)) \, dy \right)^q \left(\int_1^t r^{\beta q} \frac{dr}{r} \right) \frac{dt}{t}$$

$$\leqslant C \int_1^\infty \left(\omega(t) \int_{A(t)} \Phi(f(y)) \, dy \right)^q \frac{dt}{t}$$

$$\leqslant C \left\{ \int_{H(2)} \Phi(f(y)) \, dy + \left(\int_1^\infty \left(\omega(t) \int_{H(t)} \Phi(f(y)) \, dy \right)^q \frac{dt}{t} \right)^{1/q} \right\}$$

$$\leqslant C.$$

Consequently,

$$\int_{H(2)} \Phi(M f(x)) \, dx \leqslant C$$

and

$$\int_1^\infty \left(r^{-\varepsilon p_2} \omega(r) \int_{H(r)} \Phi(r^\varepsilon M f(x)) \, dx \right)^q \frac{dr}{r} \leqslant C,$$

and the proof of the theorem is completed. $\square$

In view of Theorem 3.4 and Lemma 2.1, we present the boundedness of maximal operator on $\mathscr{H}^{\Phi,\omega,q}(\mathbf{H}_+)$.

THEOREM 3.5. *Suppose* $(\omega 4)$ *and* $(\omega 5')$ *hold. Let* $\varepsilon > 0$ *be given such that*

$$\varepsilon(p_2 - 1) - 1 < 0 \qquad and \qquad \varepsilon(p_2 - 1) < \beta_0.$$

Then there exists a constant $C > 0$ *such that*

$$\int_{H(2)} \Phi(M f(x)) \, dx + \left(\int_1^\infty \left(r^{-\varepsilon p_2} \omega(r) \int_{A(r)} \Phi(r^\varepsilon M f(x)) \, dx \right)^q \frac{dr}{r} \right)^{1/q} \leqslant C$$

when $0 < q < \infty$ *and*

$$\int_{H(2)} \Phi(M f(x)) \, dx + \sup_{r \geqslant 1} \left(r^{-\varepsilon p_2} \omega(r) \int_{A(r)} \Phi(r^\varepsilon M f(x)) \, dx \right) \leqslant C$$

when $q = \infty$, *for all* $f \in \mathscr{H}^{\Phi,\omega,q}(\mathbf{H}_+)$ *with* $\|f\|_{\mathscr{H}^{\Phi,\omega,q}(\mathbf{H}_+)} \leqslant 1$.

COROLLARY 3.6. *Let* $0 < \nu < 1$. *If* $p > 1$, *then there exists a constant* $C > 0$ *such that*

$$\left(\int_{\mathbf{H}_+} M f(x)^p (1 + x_n)^{-\nu} \, dx \right)^{1/p} \leqslant C \left(\int_{\mathbf{H}_+} |f(y)|^p (1 + y_n)^{-\nu} \, dy \right)^{1/p}.$$

4. Sobolev inequality

The Riesz potential of order α $(0 < \alpha < n)$ of f is defined by

$$I_\alpha f(x) = \int_{\mathbf{H}_+} |x-y|^{\alpha-n} f(y)\,dy$$

with $f \in L^1_{loc}(\mathbf{H}_+)$.

For $x = (x', x_n) \in \mathbf{H}_+$ and $r > 0$, write

$$I_{1,r} f(x) = \int_{B(x,r)} |x-y|^{\alpha-n} f(y)\,dy,$$

$$I_{2,r} f(x) = \int_{B(x,1)\setminus B(x,r)} |x-y|^{\alpha-n} f(y)\,dy,$$

$$I_{3,r} f(x) = \int_{B(x,x_n/2)\setminus B(x,r)} |x-y|^{\alpha-n} f(y)\,dy,$$

$$I_{4,r} f(x) = \int_{\mathbf{H}_+ \setminus B(x,r)} |x-y|^{\alpha-n} f(y)\,dy.$$

In order to get our aims we shall need some lemmas.

LEMMA 4.1. *There exists a constant $C > 0$ such that*

$$I_{1,r} f(x) \leqslant Cr^\alpha M f(x)$$

for all nonnegative measurable function f on $\mathbf{H}_+$.

As in the proof of Lemma 3.3, we show the following result.

LEMMA 4.2. *Let $\varepsilon > 0$ be a number such that $\alpha p_1 + \varepsilon(p_1 - 1) - n < 0$. Then there exists a constant $C > 0$ such that*

$$\Phi(r^\varepsilon I_{2,r} f(x)) \leqslant Cr^{\alpha p_1 + \varepsilon p_1 - n} \int_{B(x,1)\setminus B(x,r)} \Phi(f(y))\,dy$$

for all nonnegative measurable functions f on $\mathbf{H}_+$, $x \in \mathbf{H}_+$ and $0 < r < 1/2$.

Proof. Let f be a nonnegative measurable function on $\mathbf{H}_+$ and $x \in \mathbf{H}_+$. For $0 < r < 1/2$, we write

$$\tilde{f}_{2,r} = f\chi_{B(x,1)\setminus B(x,r)}.$$

Then we have by Jensen's inequality and $(\Phi 3')$

$$\Phi(r^\varepsilon I_{2,r} f(x)) \leqslant C\Phi\left(r^\varepsilon \int_r^2 \left(\frac{1}{|B(x,t)|}\int_{B(x,t)} t^{\alpha+\varepsilon} \tilde{f}_{2,r}(y)\,dy\right) t^{-\varepsilon-1}\,dt\right)$$

$$\leqslant Cr^\varepsilon \int_r^2 \left(\frac{1}{|B(x,t)|}\int_{B(x,t)} \Phi(t^{\alpha+\varepsilon} \tilde{f}_{2,r}(y))\,dy\right) t^{-\varepsilon-1}\,dt$$

$$\leqslant Cr^{\varepsilon} \int_r^2 t^{\alpha p_1 + \varepsilon p_1 - n} \left(\int_{B(x,t)} \Phi(\tilde{f}_{2,r}(y)) \, dy \right) t^{-\varepsilon-1} \, dt$$

$$\leqslant Cr^{\varepsilon} \int_{B(x,1)\setminus B(x,r)} |x-y|^{\alpha p_1 + \varepsilon(p_1-1)-n} \Phi(f(y)) \, dy$$

$$\leqslant Cr^{\alpha p_1 + \varepsilon p_1 - n} \int_{B(x,1)\setminus B(x,r)} \Phi(f(y)) \, dy$$

since $\alpha p_1 + \varepsilon(p_1 - 1) - n < 0$. $\square$

LEMMA 4.3. *If $\varepsilon > 0$ and $\alpha p_2 + \varepsilon(p_2 - 1) - n < 0$, then there exists a constant $C > 0$ such that*

$$\Phi(r^{\varepsilon} I_{3,r} f(x)) \leqslant Cr^{\alpha p_2 + \varepsilon p_2 - n} \int_{B(x,x_n/2)} \Phi(f(y)) \, dy$$

for all nonnegative measurable functions f on $\mathbf{H}_+$, $x = (x', x_n) \in \mathbf{H}_+$ and $r \geqslant 1/2$.

Proof. Let f be a nonnegative measurable function on $\mathbf{H}_+$. It is sufficient to prove the result for $1/2 \leqslant r < x_n/2$. We write

$$\tilde{f}_{3,r} = f \chi_{B(x,x_n/2)\setminus B(x,r)}.$$

Then

$$I_{3,r} f(x) = \int_{B(x,x_n/2)\setminus B(x,r)} |x-y|^{\alpha-n} f(y) \, dy$$

$$\leqslant C \int_r^{x_n} \left(\frac{1}{|B(x,t)|} \int_{B(x,t)} \tilde{f}_{3,r}(y) \, dy \right) t^{\alpha-1} \, dt.$$

We have by Jensen's inequality and $(\Phi 2)$

$$\Phi(r^{\varepsilon} I_{3,r} f(x)) \leqslant C\Phi \left(r^{\varepsilon} \int_r^{x_n} \left(\frac{1}{|B(x,t)|} \int_{B(x,t)} t^{\alpha+\varepsilon} \tilde{f}_{3,r}(y) \, dy \right) t^{-\varepsilon-1} \, dt \right)$$

$$\leqslant Cr^{\varepsilon} \int_r^{x_n} \left(\frac{1}{|B(x,t)|} \int_{B(x,t)} \Phi(t^{\alpha+\varepsilon} \tilde{f}_{3,r}(y)) \, dy \right) t^{-\varepsilon-1} \, dt$$

$$\leqslant Cr^{\varepsilon} \int_r^{\infty} t^{\alpha p_2 + \varepsilon p_2 - n} \left(\int_{B(x,t)} \Phi(\tilde{f}_{3,r}(y)) \, dy \right) t^{-\varepsilon-1} \, dt$$

$$\leqslant Cr^{\varepsilon} \int_{B(x,x_n/2)\setminus B(x,r)} |x-y|^{\alpha p_2 + \varepsilon(p_2-1)-n} \Phi(f(y)) \, dy$$

$$\leqslant Cr^{\alpha p_2 + \varepsilon p_2 - n} \int_{B(x,x_n/2)} \Phi(f(y)) \, dy,$$

which gives the assertion. $\square$

LEMMA 4.4. *Suppose*

($\omega6$) $r^{\alpha p_2 + \varepsilon_0(p_2-1)-n}\omega(r)^{-1}$ *is almost decreasing on* $[1,\infty)$ *for some* $\varepsilon_0 > 0$.

If $0 < \varepsilon < \varepsilon_0$, *then there exists a constant* $C > 0$ *such that*

$$r^\varepsilon I_{4,r}f(x) \leqslant C\Phi^{-1}(r^{\alpha p_2 + \varepsilon p_2 - n}\omega(r)^{-1})$$

for all $r \geqslant 1/2$, $x = (x', x_n) \in \mathbf{H}_+$ *with* $r \geqslant x_n/2$ *and nonnegative measurable functions* f *on* $\mathbf{H}_+$ *with* $\|f\|_{\mathscr{H}^{\Phi,\omega,q}(\mathbf{H}_+)} \leqslant 1$.

Proof. We show only the case $1 < q < \infty$. Let f be a nonnegative measurable function on $\mathbf{H}_+$ such that

$$\int_{H(2)} \Phi(f(y))\,dy + \left(\int_1^\infty \left(\omega(r)\int_{H(r)} \Phi(f(y))\,dy\right)^q \frac{dr}{r}\right)^{1/q} \leqslant 1.$$

Let ε such that $0 < \varepsilon < \varepsilon_0$. For $x = (x', x_n) \in \mathbf{H}_+$ with $r \geqslant x_n/2$ and $r \geqslant 1/2$, we write

$$\tilde{f}_{4,r} = f\chi_{\mathbf{H}_+ \setminus B(x,r)}.$$

Then

$$I_{4,r}f(x) \leqslant C\int_r^\infty \left(\frac{1}{|B(x,t)|}\int_{B(x,t)} \tilde{f}_{4,r}(y)\,dy\right)t^{\alpha-1}\,dt.$$

Note from ($\omega6$) that

$$\alpha p_2 + \varepsilon_0(p_2-1) - n < 0$$

since $\omega(r)$ is almost decreasing on $[1,\infty)$, so that

$$\alpha p_2 + \varepsilon(p_2-1) - n = \alpha p_2 + \varepsilon_0(p_2-1) - n + (\varepsilon - \varepsilon_0)(p_2-1) < 0.$$

We have by Jensen's inequality and ($\Phi2$)

$$\Phi(r^\varepsilon I_{4,r}f(x)) \leqslant C\Phi\left(r^\varepsilon \int_r^\infty \left(\frac{1}{|B(x,t)|}\int_{B(x,t)} t^{\alpha+\varepsilon}\tilde{f}_{4,r}(y)\,dy\right)t^{-\varepsilon-1}\,dt\right)$$

$$\leqslant Cr^\varepsilon \int_r^\infty \left(\frac{1}{|B(x,t)|}\int_{B(x,t)} \Phi(t^{\alpha+\varepsilon}\tilde{f}_{4,r}(y))\,dy\right)t^{-\varepsilon-1}\,dt$$

$$\leqslant Cr^\varepsilon \int_r^\infty t^{\alpha p_2 + \varepsilon p_2 - n}\left(\int_{B(x,t)} \Phi(\tilde{f}_{4,r}(y))\,dy\right)t^{-\varepsilon-1}\,dt$$

$$\leqslant Cr^\varepsilon \int_{\mathbf{H}_+ \setminus B(x,r)} |x-y|^{\alpha p_2 + \varepsilon(p_2-1)-n}\Phi(f(y))\,dy.$$

Since $y_n \geqslant 4x_n/2 = 2x_n$ when $y \in \mathbf{H}_+ \setminus (H(r) \cup B(x,r))$ and $y_n \geqslant 4r$, $|x-y| \geqslant |x_n - y_n| \geqslant y_n/2$. When $y \in \mathbf{H}_+ \setminus (H(r) \cup B(x,r))$ and $r < y_n < 4r$, $|x-y| > r > y_n/4$. Therefore,

$$r^\varepsilon \int_{\mathbf{H}_+ \setminus (H(r) \cup B(x,r))} |x-y|^{\alpha p_2 + \varepsilon(p_2-1)-n}\Phi(f(y))\,dy$$

$$\leqslant Cr^\varepsilon \int_{\mathbf{H}_+ \setminus H(r)} y_n^{\alpha p_2 + \varepsilon(p_2-1)-n}\Phi(f(y))\,dy$$

since $\alpha p_2 + \varepsilon(p_2 - 1) - n < 0$. Hence

$$\Phi(r^\varepsilon I_{4,r} f(x))$$
$$\leqslant C\left\{ r^\varepsilon \int_{\mathbf{H}_+ \setminus H(r)} y_n^{\alpha p_2 + \varepsilon(p_2-1) - n} \Phi(f(y))\,dy + r^{\alpha p_2 + \varepsilon p_2 - n} \int_{H(r)} \Phi(f(y))\,dy \right\}.$$

Here we find by Hölder's inequality and $(\omega 6)$

$$\int_{\mathbf{H}_+ \setminus H(r)} y_n^{\alpha p_2 + \varepsilon(p_2-1) - n} \Phi(f(y))\,dy \leqslant C \sum_{j=0}^{\infty} \int_{A(2^j r)} (2^j r)^{\alpha p_2 + \varepsilon(p_2-1) - n} \Phi(f(y))\,dy$$

$$\leqslant C \left(\sum_{j=0}^{\infty} \left((2^j r)^{\alpha p_2 + \varepsilon(p_2-1) - n} \omega(2^j r)^{-1} \right)^{q'} \right)^{1/q'} \left(\sum_{j=0}^{\infty} \left(\omega(2^j r) \int_{A(2^j r)} \Phi(f(y))\,dy \right)^q \right)^{1/q}$$

$$\leqslant C r^{\alpha p_2 + \varepsilon(p_2-1) - n} \omega(r)^{-1} \left(\sum_{j=0}^{\infty} \left(\omega(2^j r) \int_{A(2^j r)} \Phi(f(y))\,dy \right)^q \right)^{1/q}.$$

Further,

$$r^{\alpha p_2 + \varepsilon p_2 - n} \int_{H(r)} \Phi(f(y))\,dy \leqslant C r^{\alpha p_2 + \varepsilon p_2 - n} \omega(r)^{-1}.$$

Hence

$$\Phi(r^\varepsilon I_{4,r} f(x)) \leqslant C r^{\alpha p_2 + \varepsilon p_2 - n} \omega(r)^{-1},$$

which gives the assertion. $\square$

LEMMA 4.5. *Suppose $(\omega 6)$ holds. Let $0 < \varepsilon < \varepsilon_0$. Let f be a nonnegative measurable function on $\mathbf{H}_+$ such that $\|f\|_{\mathcal{H}^{\Phi,\omega,q}(\mathbf{H}_+)} \leqslant 1$.*

(1) *Let $N_{1,\varepsilon} = n + (\alpha + \varepsilon)(p_2 - p_1)$. If $\alpha p_2 + \varepsilon_0 p_2 - n < 0$, then there exists a constant $C > 0$ such that*

$$I_\alpha f(x) \leqslant C M f(x) \left(\omega(x_n) \Phi(M f(x)) \right)^{-\alpha/N_{1,\varepsilon}}$$

for all $x = (x', x_n) \in \mathbf{H}_+$ with $x_n \geqslant 1$ and $\left(\omega(x_n) \Phi(M f(x)) \right)^{-1/N_{1,\varepsilon}} < 1$.

(2) *Let $N_{2,\varepsilon} = n - (\alpha + \varepsilon)(p_2 - p_1)$. If $\alpha p_2 + \varepsilon_0 p_2 - n < 0$, then there exists a constant $C > 0$ such that*

$$I_\alpha f(x) \leqslant C M f(x) \left(\omega(x_n) \Phi(M f(x)) \right)^{-\alpha/N_{2,\varepsilon}}$$

for all $x = (x', x_n) \in \mathbf{H}_+$ with $x_n \geqslant 2$ and $1 \leqslant \left(\omega(x_n) \Phi(M f(x)) \right)^{-1/N_{2,\varepsilon}} \leqslant x_n/2$.

(3) *Suppose*

$(\omega 7)$ *there exist a real number $N_{3,\varepsilon}$ and a constant $A > 0$ such that $0 < N_{3,\varepsilon} \leqslant N_{2,\varepsilon}$ and*

$$t^{N_{3,\varepsilon}} \leqslant A t^{N_{2,\varepsilon}} \omega(t)$$

for all $t \geqslant 1$.

Then there exists a constant $C > 0$ such that

$$I_\alpha f(x) \leqslant C M f(x) \Phi(M f(x))^{-\alpha/N_{3,\varepsilon}}$$

for all $x = (x', x_n) \in \mathbf{H}_+$ with $x_n \geqslant 1$ and $(\omega(x_n)\Phi(M f(x)))^{-1/N_{2,\varepsilon}} \geqslant x_n/2$.

Proof. First we show the case (1). By Lemmas 4.1–4.4, we have for $0 < t < 1$ and $x = (x', x_n) \in \mathbf{H}_+$ with $x_n \geqslant 1$

$$I_\alpha f(x) = \int_{B(x,t)} |x-y|^{\alpha-n} f(y)\,dy + \int_{B(x,1)\setminus B(x,t)} |x-y|^{\alpha-n} f(y)\,dy$$

$$+ \int_{B(x,x_n/2)\setminus B(x,1)} |x-y|^{\alpha-n} f(y)\,dy + \int_{\mathbf{H}_+ \setminus B(x,x_n/2)} |x-y|^{\alpha-n} f(y)\,dy$$

$$\leqslant C\Big\{ t^\alpha M f(x) + t^{-\varepsilon}\Phi^{-1}\Big(t^{\alpha p_1 + \varepsilon p_1 - n} \int_{H(2x_n)} \Phi(f(y))\,dy \Big)$$

$$+ \Phi^{-1}\Big(\int_{H(3x_n/2)} \Phi(f(y))\,dy \Big) + x_n^{-\varepsilon}\Phi^{-1}(x_n^{\alpha p_2 + \varepsilon p_2 - n}\omega(x_n)^{-1}) \Big\}$$

$$\leqslant C\{ t^\alpha M f(x) + t^{-\varepsilon}\Phi^{-1}(t^{\alpha p_1 + \varepsilon p_1 - n}\omega(x_n)^{-1})$$

$$+ \Phi^{-1}(\omega(x_n)^{-1}) + x_n^{-\varepsilon}\Phi^{-1}(x_n^{\alpha p_2 + \varepsilon p_2 - n}\omega(x_n)^{-1}) \}$$

$$\leqslant C\{ t^\alpha M f(x) + t^{-\varepsilon}\Phi^{-1}(t^{\alpha p_1 + \varepsilon p_1 - n}\omega(x_n)^{-1}) \}$$

since

$$\alpha p_1 + \varepsilon p_1 - n \leqslant \alpha p_2 + \varepsilon p_2 - n < \alpha p_2 + \varepsilon_0 p_2 - n < 0$$

and

$$\alpha p_1 + \varepsilon(p_1 - 1) - n \leqslant \alpha p_2 + \varepsilon(p_2 - 1) - n < 0.$$

If $t = (\omega(x_n)\Phi(M f(x)))^{-1/N_{1,\varepsilon}} < 1$, then

$$\Phi(t^{-\alpha-\varepsilon}\Phi^{-1}(t^{\alpha p_1 + \varepsilon p_1 - n}\omega(x_n)^{-1})) \leqslant A_1 t^{(-\alpha-\varepsilon)p_2} t^{\alpha p_1 + \varepsilon p_1 - n}\omega(x_n)^{-1}$$

$$= A_1 t^{-N_{1,\varepsilon}}\omega(x_n)^{-1}$$

by $(\Phi 2)$, so that

$$I_\alpha f(x) \leqslant C\{ t^\alpha M f(x) + t^\alpha \Phi^{-1}(A_1 t^{-N_{1,\varepsilon}}\omega(x_n)^{-1}) \}$$

$$\leqslant C t^\alpha M f(x)$$

$$= C M f(x)\,(\omega(x_n)\Phi(M f(x)))^{-\alpha/N_{1,\varepsilon}},$$

which gives Assertion (1).

Next we show the case (2). We have for $x = (x', x_n) \in \mathbf{H}_+$ and $1 \leqslant t < x_n/2$

$$I_\alpha f(x) = \int_{B(x,t)} |x-y|^{\alpha-n} f(y)\,dy + \int_{B(x,x_n/2)\setminus B(x,t)} |x-y|^{\alpha-n} f(y)\,dy$$

$$+ \int_{\mathbf{H}_+ \setminus B(x,x_n/2)} |x-y|^{\alpha-n} f(y)\,dy$$

$$\leqslant C\left\{t^{\alpha}Mf(x)+t^{-\varepsilon}\Phi^{-1}(t^{\alpha p_2+\varepsilon p_2-n}\omega(x_n)^{-1})+x_n^{-\varepsilon}\Phi^{-1}(x_n^{\alpha p_2+\varepsilon p_2-n}\omega(x_n)^{-1})\right\}$$

$$\leqslant C\left\{t^{\alpha}Mf(x)+t^{-\varepsilon}\Phi^{-1}(t^{\alpha p_2+\varepsilon p_2-n}\omega(x_n)^{-1})\right\}$$

by Lemmas 4.1, 4.3 and 4.4. If $1<t=(\omega(x_n)\Phi(Mf(x)))^{-1/N_{2,\varepsilon}}<x_n/2$, then

$$\Phi(t^{-\alpha-\varepsilon}\Phi^{-1}(t^{\alpha p_2+\varepsilon p_2-n}\omega(x_n)^{-1}))\leqslant A_2^{-1}t^{(-\alpha-\varepsilon)p_1}t^{\alpha p_2+\varepsilon p_2-n}\omega(x_n)^{-1}$$

$$=A_2^{-1}t^{-N_{2,\varepsilon}}\omega(x_n)^{-1}$$

by $(\Phi 3')$, so that

$$I_{\alpha}f(x)\leqslant C\left\{t^{\alpha}Mf(x)+t^{\alpha}\Phi^{-1}(A_2^{-1}t^{-N_{2,\varepsilon}}\omega(x_n)^{-1})\right\}$$

$$\leqslant Ct^{\alpha}Mf(x)$$

$$=CMf(x)\left(\omega(x_n)\Phi(Mf(x))\right)^{-\alpha/N_{2,\varepsilon}},$$

which gives Assertion (2).

Finally we show the case (3). Let $x=(x',x_n)\in\mathbf{H}_+$ with $x_n\geqslant 1$.

If $(\omega(x_n)\Phi(Mf(x)))^{-1/N_{2,\varepsilon}}\geqslant x_n/2\geqslant 1/2$, then we take $t=\Phi(Mf(x))^{-1/N_{3,\varepsilon}}$. Since $x_n\geqslant 1$, we see that

$$x_n^{N_{3,\varepsilon}}\leqslant Ax_n^{N_{2,\varepsilon}}\omega(x_n)\leqslant A2^{N_{2,\varepsilon}}\Phi(Mf(x))^{-1}=A2^{N_{2,\varepsilon}}t^{N_{3,\varepsilon}},$$

so that $t\geqslant A^{-1/N_{3,\varepsilon}}2^{-N_{2,\varepsilon}/N_{3,\varepsilon}}x_n\geqslant C$. We find

$$I_{\alpha}f(x)=\int_{B(x,t)\cap\mathbf{H}_+}|x-y|^{\alpha-n}f(y)\,dy+\int_{\mathbf{H}_+\setminus B(x,t)}|x-y|^{\alpha-n}f(y)\,dy$$

$$\leqslant C\left\{t^{\alpha}Mf(x)+t^{-\varepsilon}\Phi^{-1}(t^{\alpha p_2+\varepsilon p_2-n}\omega(t)^{-1})\right\}$$

by Lemmas 4.1 and 4.4. Since

$$\Phi(t^{-\alpha-\varepsilon}\Phi^{-1}(t^{\alpha p_2+\varepsilon p_2-n}\omega(t)^{-1}))\leqslant Ct^{(-\alpha-\varepsilon)p_1}t^{\alpha p_2+\varepsilon p_2-n}\omega(t)^{-1}$$

$$=Ct^{-N_{2,\varepsilon}}\omega(t)^{-1}\leqslant Ct^{-N_{3,\varepsilon}}$$

by $(\Phi 3')$, we have

$$I_{\alpha}f(x)\leqslant C\left\{t^{\alpha}Mf(x)+t^{-\varepsilon}\Phi^{-1}(t^{\alpha p_2+\varepsilon p_2-n}\omega(t)^{-1})\right\}$$

$$\leqslant C\left\{t^{\alpha}Mf(x)+t^{\alpha}\Phi^{-1}(t^{-N_{3,\varepsilon}})\right\}$$

$$\leqslant Ct^{\alpha}Mf(x)$$

$$=CMf(x)\Phi(Mf(x))^{-\alpha/N_{3,\varepsilon}},$$

which gives Assertion (3). $\quad\square$

Suppose there exist convex functions $\Psi_{i,\varepsilon}$ $(i=1,2,3)$ on $[0,\infty)$ such that

$(\Psi_{i,\varepsilon})$ there exists a constant $A_3>0$ such that

$$\Psi_{i,\varepsilon}\left(t\Phi(t)^{-\alpha/N_{i,\varepsilon}}\right)\leqslant A_3\Phi(t)$$

for $t>0$.

In view of previous results we have the following.

LEMMA 4.6. *Suppose* $(\omega6)$ *and* $(\omega7)$ *hold for* $\alpha p_2 + \varepsilon_0 p_2 - n < 0$. *Let* $0 < \varepsilon < \varepsilon_0$. *Then there exists a constant* $C > 0$ *such that*

$$
\int_{A(r)} \left(\Psi_{1,\varepsilon} \left(\left(x_n^{-\varepsilon p_2} \omega(x_n) \right)^{\alpha/N_{1,\varepsilon}} x_n^{\varepsilon} I_\alpha f(x) \right) \chi_{\{x=(x',x_n)\in\mathbf{H}_+ : (\omega(x_n)\Phi(Mf(x)))^{-1/N_{1,\varepsilon}}<1\}} \right.
$$
$$
+ \Psi_{2,\varepsilon} \left(\left(x_n^{-\varepsilon p_2} \omega(x_n) \right)^{\alpha/N_{2,\varepsilon}} x_n^{\varepsilon} I_\alpha f(x) \right) \chi_{\{x=(x',x_n)\in\mathbf{H}_+ : 1\leqslant(\omega(x_n)\Phi(Mf(x)))^{-1/N_{2,\varepsilon}}\leqslant x_n/2\}}
$$
$$
\left. + \Psi_{3,\varepsilon} \left(\left(x_n^{-\varepsilon p_2} \right)^{\alpha/N_{3,\varepsilon}} x_n^{\varepsilon} I_\alpha f(x) \right) \chi_{\{x=(x',x_n)\in\mathbf{H}_+ : (\omega(x_n)\Phi(Mf(x)))^{-1/N_{2,\varepsilon}}\geqslant x_n/2\}} \right) dx
$$
$$
\leqslant C \int_{A(r)} \Phi(r^{\varepsilon} Mf(x)) \, dx
$$

for all $r \geqslant 1$ *and nonnegative measurable functions* f *on* $\mathbf{H}_+$ *such that* $\|f\|_{\mathscr{H}^{\Phi,\omega,q}(\mathbf{H}_+)} \leqslant 1$.

Proof. Let $x = (x', x_n) \in A(r)$ with $r \geqslant 1$.

If $(\omega(x_n)\Phi(Mf(x)))^{-1/N_{1,\varepsilon}} < 1$ with $N_{1,\varepsilon} = n + (\alpha + \varepsilon)(p_2 - p_1)$, then we have by Lemma 4.5 (1) and $(\Phi2)$

$$
I_\alpha f(x) \leqslant C Mf(x) \left(\omega(x_n)\Phi(Mf(x)) \right)^{-\alpha/N_{1,\varepsilon}}
$$
$$
\leqslant C Mf(x) \left(\omega(x_n) x_n^{-\varepsilon p_2} \Phi(x_n^{\varepsilon} Mf(x)) \right)^{-\alpha/N_{1,\varepsilon}},
$$

so that in view of $(\Psi_{1,\varepsilon})$

$$
\Psi_{1,\varepsilon} \left(\left(x_n^{-\varepsilon p_2} \omega(x_n) \right)^{\alpha/N_{1,\varepsilon}} x_n^{\varepsilon} I_\alpha f(x) \right) \leqslant C \Psi_{1,\varepsilon} \left(x_n^{\varepsilon} Mf(x) \Phi(x_n^{\varepsilon} Mf(x))^{-\alpha/N_{1,\varepsilon}} \right)
$$
$$
\leqslant C \Phi(x_n^{\varepsilon} Mf(x)).
$$

If $1 \leqslant (\omega(x_n)\Phi(Mf(x)))^{-1/N_{2,\varepsilon}} \leqslant x_n/2$ with $N_{2,\varepsilon} = n - (\alpha + \varepsilon)(p_2 - p_1)$, then we have by Lemma 4.5 (2) and $(\Phi2)$

$$
I_\alpha f(x) \leqslant C Mf(x) \left(\omega(x_n)\Phi(Mf(x)) \right)^{-\alpha/N_{2,\varepsilon}}
$$
$$
\leqslant C Mf(x) \left(\omega(x_n) x_n^{-\varepsilon p_2} \Phi(x_n^{\varepsilon} Mf(x)) \right)^{-\alpha/N_{2,\varepsilon}}
$$

and hence, in view of $(\Psi_{2,\varepsilon})$,

$$
\Psi_{2,\varepsilon} \left(\left(x_n^{-\varepsilon p_2} \omega(x_n) \right)^{\alpha/N_{2,\varepsilon}} x_n^{\varepsilon} I_\alpha f(x) \right) \leqslant C \Phi(x_n^{\varepsilon} Mf(x)).
$$

Finally, if $(\omega(x_n)\Phi(Mf(x)))^{-1/N_{2,\varepsilon}} \geqslant x_n/2$, then we obtain by Lemma 4.5 (3) and $(\Phi2)$

$$
I_\alpha f(x) \leqslant C Mf(x) \Phi(Mf(x))^{-\alpha/N_{3,\varepsilon}} \leqslant C Mf(x) \left(x_n^{-\varepsilon p_2} \Phi(x_n^{\varepsilon} Mf(x)) \right)^{-\alpha/N_{3,\varepsilon}},
$$

so that in view of $(\Psi_{3,\varepsilon})$

$$\Psi_{3,\varepsilon}\left(\left(x_n^{-\varepsilon p_2}\right)^{\alpha/N_{3,\varepsilon}} x_n^\varepsilon I_\alpha f(x)\right) \leqslant C\Phi(x_n^\varepsilon Mf(x)),$$

as required.   □

Now we establish our main result.

THEOREM 4.7. *Suppose* $(\omega 4)$, $(\omega 5')$, $(\omega 6)$ *and* $(\omega 7)$ *hold for* $\alpha p_2 + \varepsilon_0 p_2 - n < 0$. *If* $0 < \varepsilon < \varepsilon_0$, $\varepsilon(p_2 - 1) - 1 < 0$ *and* $\varepsilon(p_2 - 1) < \beta_0$, *then there exists a constant* $C > 0$ *such that*

$$\int_1^\infty \left(r^{-\varepsilon p_2}\omega(r) \right.$$

$$\times \int_{A(r)} \left(\Psi_{1,\varepsilon}\left(\left(x_n^{-\varepsilon p_2}\omega(x_n)\right)^{\alpha/N_{1,\varepsilon}} x_n^\varepsilon |I_\alpha f(x)|\right) \chi_{\{x=(x',x_n)\in \mathbf{H}_+:(\omega(x_n)\Phi(Mf(x)))^{-1/N_{1,\varepsilon}}<1\}}$$

$$+ \Psi_{2,\varepsilon}\left(\left(x_n^{-\varepsilon p_2}\omega(x_n)\right)^{\alpha/N_{2,\varepsilon}} x_n^\varepsilon |I_\alpha f(x)|\right) \chi_{\{x=(x',x_n)\in \mathbf{H}_+:1\leqslant(\omega(x_n)\Phi(Mf(x)))^{-1/N_{2,\varepsilon}}\leqslant x_n/2\}}$$

$$\left. + \Psi_{3,\varepsilon}\left(\left(x_n^{-\varepsilon p_2}\right)^{\alpha/N_{3,\varepsilon}} x_n^\varepsilon |I_\alpha f(x)|\right) \chi_{\{x=(x',x_n)\in \mathbf{H}_+:(\omega(x_n)\Phi(Mf(x)))^{-1/N_{2,\varepsilon}}\geqslant x_n/2\}} \right) dx \right)^q \frac{dr}{r}$$

$$\leqslant C$$

when $0 < q < \infty$ *and*

$$\sup_{r\geqslant 1}\left(r^{-\varepsilon p_2}\omega(r) \right.$$

$$\times \int_{A(r)} \left(\Psi_{1,\varepsilon}\left(\left(x_n^{-\varepsilon p_2}\omega(x_n)\right)^{\alpha/N_{1,\varepsilon}} x_n^\varepsilon |I_\alpha f(x)|\right) \chi_{\{x=(x',x_n)\in \mathbf{H}_+:(\omega(x_n)\Phi(Mf(x)))^{-1/N_{1,\varepsilon}}<1\}}$$

$$+ \Psi_{2,\varepsilon}\left(\left(x_n^{-\varepsilon p_2}\omega(x_n)\right)^{\alpha/N_{2,\varepsilon}} x_n^\varepsilon |I_\alpha f(x)|\right) \chi_{\{x=(x',x_n)\in \mathbf{H}_+:1\leqslant(\omega(x_n)\Phi(Mf(x)))^{-1/N_{2,\varepsilon}}\leqslant x_n/2\}}$$

$$\left. + \Psi_{3,\varepsilon}\left(\left(x_n^{-\varepsilon p_2}\right)^{\alpha/N_{3,\varepsilon}} x_n^\varepsilon |I_\alpha f(x)|\right) \chi_{\{x=(x',x_n)\in \mathbf{H}_+:(\omega(x_n)\Phi(Mf(x)))^{-1/N_{2,\varepsilon}}\geqslant x_n/2\}} \right) dx \right)$$

$$\leqslant C$$

when $q = \infty$, *for all measurable functions* f *on* $\mathbf{H}_+$ *such that* $\|f\|_{\mathscr{H}^{\Phi,\omega,q}(\mathbf{H}_+)} \leqslant 1$.

Proof. We show only the case $1 < q < \infty$. In view of Lemma 4.6 and Theorem 3.5, we obtain

$$\int_1^\infty \left(r^{-\varepsilon p_2}\omega(r) \right.$$

$$\times \int_{A(r)} \left(\Psi_{1,\varepsilon}\left(\left(x_n^{-\varepsilon p_2}\omega(x_n)\right)^{\alpha/N_{1,\varepsilon}} x_n^\varepsilon |I_\alpha f(x)|\right) \chi_{\{x=(x',x_n)\in \mathbf{H}_+:(\omega(x_n)\Phi(Mf(x)))^{-1/N_{1,\varepsilon}}<1\}}$$

$$+ \Psi_{2,\varepsilon}\left(\left(x_n^{-\varepsilon p_2}\omega(x_n)\right)^{\alpha/N_{2,\varepsilon}} x_n^\varepsilon |I_\alpha f(x)|\right) \chi_{\{x=(x',x_n)\in \mathbf{H}_+:1\leqslant(\omega(x_n)\Phi(Mf(x)))^{-1/N_{2,\varepsilon}}\leqslant x_n/2\}}$$

$$+ \Psi_{3,\varepsilon}\left((x_n^{-\varepsilon p_2})^{\alpha/N_{3,\varepsilon}} x_n^{\varepsilon} |I_\alpha f(x)| \right) \chi_{\{x=(x',x_n)\in \mathbf{H}_+ : (\omega(x_n)\Phi(Mf(x)))^{-1/N_{2,\varepsilon}} \geqslant x_n/2\}} \right)^q \frac{dr}{r}$$

$$\leqslant C \int_1^\infty \left(r^{-\varepsilon p_2} \omega(r) \int_{A(r)} \Phi(r^\varepsilon Mf(x))\, dx \right)^q \frac{dr}{r}$$

$$\leqslant C,$$

as required. $\square$

When $\Phi(r) = r^p$ and $\omega(r) = r^{-\nu}$, we obtain the following corollary.

COROLLARY 4.8. *Let*

$$E_1 = \{x = (x',x_n) \in \mathbf{H}_+ : Mf(x) \geqslant (2^n x_n^{\nu-n})^{1/p}\}$$

and

$$E_2 = \{x = (x',x_n) \in \mathbf{H}_+ : Mf(x) < (2^n x_n^{\nu-n})^{1/p}\}.$$

Let $1/p^* = 1/p - \alpha/n > 0$ *and* $1/p_\nu = 1/p - \alpha/(n-\nu) > 0$. *If* $p > 1$ *and* $0 < \nu < 1$, *then there exists a constant* $C > 0$ *such that*

$$\int_{E_1 \setminus H(1)} \left(|I_\alpha f(x)|(1+x_n)^{-\nu/p} \right)^{p^*} dx + \int_{E_2 \setminus H(1)} |I_\alpha f(x)|^{p_\nu} x_n^{-\nu}\, dx \leqslant C$$

for all measurable functions f *on* $\mathbf{H}_+$ *such that*

$$\int_{\mathbf{H}_+} |f(y)|^p (1+y_n)^{-\nu}\, dy \leqslant 1.$$

In order to prove Corollary 4.8, it is enough to notice that for $\Phi(r) = r^p$ and $\omega(r) = r^{-\nu}$ we have $N_{1,\varepsilon} = N_{2,\varepsilon} = n$ and $N_{3,\varepsilon} = n - \nu$ and then we may take $\Psi_{1,\varepsilon}(r) = \Psi_{2,\varepsilon}(r) = r^{p^*}$ and $\Psi_{3,\varepsilon}(r) = r^{p_\nu}$ in view of ($\Psi_{i,\varepsilon}$) ($i = 1,2,3$).

REMARK 4.9. We shall show that the exponent p_ν in Corollary 4.8 is needed. To show this, consider the function

$$f(y) = |y|^{-(n-\nu)/p-\varepsilon} \chi_{\mathbf{H}_+ \setminus H(1)}(y)$$

for $0 < \nu < 1$ and $0 < \varepsilon < n - (n - \nu)/p$. Then, in view of Lemma 3.2, we have

$$\int_{\mathbf{H}_+} f(y)^p (1+y_n)^{-\nu}\, dy \leqslant \int_1^\infty y_n^{-\nu} \left(\int_{\mathbf{R}^{n-1}} |y|^{\nu - \varepsilon p - n}\, dy' \right) dy_n$$

$$\leqslant C \int_1^\infty y_n^{-\varepsilon p - 1}\, dy_n < \infty.$$

We find

$$Mf(x) \leqslant C|x|^{-(n-\nu)/p-\varepsilon} \leqslant C x_n^{-(n-\nu)/p-\varepsilon} < (2^n x_n^{\nu-n})^{1/p}$$

for $x \in \mathbf{H}_+ \setminus H(c)$ with some $c \geqslant 1$. Note that

$$
\begin{aligned}
I_\alpha f(x) &\geqslant \int_{\mathbf{H}_+ \cap B(x,|x|/2)} |x-y|^{\alpha-n} f(y)\,dy \\
&\geqslant C|x|^{-(n-\nu)/p-\varepsilon} \int_{\mathbf{H}_+ \cap B(x,|x|/2)} |x-y|^{\alpha-n}\,dy \\
&\geqslant C|x|^{-(n-\nu)/p-\varepsilon+\alpha}
\end{aligned}
$$

for $x \in \mathbf{H}_+ \setminus H(1)$. Let $s > 0$ such that

$$
1/s < ((n-\nu)/p - \alpha + \varepsilon)/(n-1) = (1/p_\nu + \varepsilon/(n-\nu))(n-\nu)/(n-1).
$$

Hence it follows from Lemma 3.2 that

$$
\begin{aligned}
\int_{E_2 \setminus H(1)} I_\alpha f(x)^s (1+x_n)^{-\nu}\,dx &\geqslant \int_{\mathbf{H}_+ \setminus H(c)} I_\alpha f(x)^s (1+x_n)^{-\nu}\,dx \\
&\geqslant C \int_c^\infty x_n^{-\nu} \left(\int_{\mathbf{R}^{n-1}} |x|^{(-(n-\nu)/p-\varepsilon+\alpha)s}\,dx' \right) dx_n \\
&\geqslant C \int_c^\infty x_n^{-\nu+(-(n-\nu)/p-\varepsilon+\alpha)s+n-1}\,dx_n
\end{aligned}
$$

since $(-(n-\nu)/p - \varepsilon + \alpha)s + n < 1$.

　　If

$$
1/s \geqslant 1/p_\nu + \varepsilon/(n-\nu),
$$

then

$$
\int_{E_2 \setminus H(1)} I_\alpha f(x)^s (1+x_n)^{-\nu}\,dx = \infty.
$$

This implies that $s \geqslant p_\nu$ by letting $\varepsilon \to 0$.

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