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A COUNTEREXAMPLE TO A QUESTION OF BAPAT & SUNDER

STEPHEN DRURY

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Abstract. The objective of this article is to provide a counterexample to a question of Bapat and Sunder concerning the relative magnitudes of the permanent of a positive semidefinite matrix and the largest eigenvalue of a related matrix. We also discuss the significance of this result in connection with the eigenvalues of the Schur matrix.

1. Introduction

The permanent of a $n \times n$ matrix $A = (a_{jk})$ is defined as the quantity

$$\operatorname{per}(A) = \sum_{\sigma \in S_n} \prod_{j=1}^{n} a_{j,\sigma(j)}.$$

It is an important concept useful in combinatorial applications. For a recent survey of permanent inequalities and open questions the reader is referred to [11] and the references therein.

Let A be a $n \times n$ positive semidefinite matrix. Denote by $A(i,j)$ the $(n-1)\times(n-1)$ submatrix of A obtained by deleting the i^{th} row and j^{th} column of A. Now define a $n \times n$ matrix B by

$$b_{ij} = a_{ij} \operatorname{per}(A(i,j)). \tag{1}$$

Then it is clear that B is again a positive semidefinite matrix and it follows from the Laplace expansion of the permanent that all its row and column sums are equal to $\operatorname{per}(A)$. Thus, $\operatorname{per}(A)$ is an eigenvalue of B and $\mathbb{1}$ is the corresponding eigenvector.

In [2, Conjecture 3], Bapat and Sunder raise the question of whether $\operatorname{per}(A)$ is necessarily the largest eigenvalue of B. We provide a counterexample to this question.

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2. The counterexample

With $n = 8$, we take

$$X = \begin{pmatrix} -7+4i & 9-3i & -6+2i & 3+4i & 7+6i & 4-4i & i & 5-8i \\ 4-5i & 1+4i & -8-2i & -7+4i & 1-4i & 1-8i & 8-6i & 1-3i \end{pmatrix}$$

and set $A = X^*X$. Then we obtain

$$A = \begin{pmatrix} 106 & -91+6i & 28-38i & -53-59i & -1-81i & -15i & 66+9i & -48+29i \\ -91-6i & 107 & -76+30i & 24+77i & 30+67i & 17-36i & -19-29i & 58-64i \\ 28+38i & -76-30i & 108 & 38-76i & -30-16i & -24+82i & -50+58i & -48+64i \\ -53+59i & 24-77i & 38+76i & 90 & 22+14i & -43+24i & -76+13i & -36-27i \\ -1+81i & 30-67i & -30+16i & 22-14i & 102 & 37-56i & 38+33i & -85i \\ 15i & 17+36i & -24-82i & -43-24i & 37+56i & 97 & 52+62i & 77-7i \\ 66-9i & -19+29i & -50-58i & -76-13i & 38-33i & 52-62i & 101 & 18-23i \\ -48-29i & 58+64i & -48-64i & -36+27i & 85i & 77+7i & 18+23i & 99 \end{pmatrix}$$

a rank two positive semidefinite matrix. Calculations show that

$$\mathrm{per}(A) = 2977257622144118400$$

and that the largest eigenvalue of B exceeds 3028080150918724811.

This example was found using a hill-climbing computer search and the example found was rounded for easy presentation. The ratio $\lambda_1(B)/\mathrm{per}(A)$ found in the search was approximately 1.01956. No example was found for $n = 7$.

3. The Schur matrix

For a positive semidefinite $n \times n$ matrix A, define the convolution operator $\Pi(A)$ on the symmetric group S_n by its matrix

$$\Pi(A)_{\sigma,\rho} = \prod_{j=1}^{n} a_{\sigma(j),\rho(j)}.$$

This (usually huge) matrix is known as the Schur matrix. As is well-known, $\Pi(A)$ is unitarily similar to a block diagonal matrix indexed by the set of irreducible representations of S_n. The diagonal block corresponding to the irreducible representation π is a matrix multiplication operator by a hermitian matrix $\widehat{\Pi(A)}(\pi)$. Thus, the eigenvalues of $\Pi(A)$ coming from the diagonal block corresponding to π are the eigenvalues of $\widehat{\Pi(A)}(\pi)$ each repeated d_π times where d_π is the dimension of π. The reader may consult [5] for the Fourier analysis of compact (and hence finite) groups.

The permanent on top conjecture was originally formulated by G. Soules in his Ph.D. dissertation 1966 [10] and published in [7]. It asks if the largest eigenvalue of $\Pi(A)$ is $\mathrm{per}(A)$, namely the eigenvalue arising from the trivial representation. In 2016, Shchesnovich [9] presented an example of a 5×5 positive semidefinite matrix A and a unit column vector X indexed by S_5 such that $X^*\Pi(A)X > \mathrm{per}(A)$ thereby demolishing Soule's conjecture.

The irreducible representations of S_n are well-known to be in one-to-one correspondence with the Ferrers diagrams with n entries. Details can be found in [6, 8]. Shchesnovich does not identify in his paper the representation that is responsible for his counterexample, but calculations reveal that it is the Ferrers diagram $(3,2)$ that has 3 entries in the first row and 2 in the second.

It is well-known that the representation $\sigma \mapsto P(\sigma)$, the representation that takes each permutation to its permutation matrix decomposes as the direct sum of the trivial representation (Ferrers diagram (n)) and the representation π_1 with Ferrers diagram $(n-1,1)$. We have, denoting ε the identity permutation,

$$\sum_{\rho \in S_n} \Pi(A)_{\rho,\varepsilon} P_{j,k}(\rho) = \sum_{\rho \in S_n} \left(\prod_{i=1}^{n} a_{\rho i,i} \right) \delta_{j,\rho k} = a_{jk} \operatorname{per}(A(j,k)) = b_{jk}.$$

It follows that the eigenvalues of B are $\operatorname{per}(A)$ together with the eigenvalues of $\widehat{\Pi(A)}(\pi_1)$. Thus for $n=8$ we have yet another counterexample to the permanent on top conjecture [9, 3, 4].

4. A new question

So, the relevant question is now:

QUESTION 1. *For a given n, which irreducible representations π of S_n have the property that the largest eigenvalue of $\widehat{\Pi(A)}(\pi)$ is bounded above by $\operatorname{per}(A)$ for every positive definite $n \times n$ hermitian matrix A?*

The branching rule is a rule that determines how a given representation of S_n decomposes when it is restricted to S_m, the subgroup of S_n of permutations of $\{1,2,\dots,m\}$ for $m<n$. It is a consequence of the branching rule [8, §2.8] that if the Ferrers diagram of π contains either of the Ferrers diagrams $(3,2)$ or $(7,1)$ then the representation does not have the property of Question 1. Here, we are using the word 'contains' in a very loose sense. A Ferrers diagram α contains another β if each row count of β is dominated by the corresponding row count of α. This means that β 'fits inside' α.

CONJECTURE 1. *A representation satisfies the property asked in the question if it contains neither of the Ferrers diagrams $(3,2)$ or $(7,1)$.*

Some other conjectures that we believe might be true and that do not appear in [11] are:

CONJECTURE 2. *If A is a real rank two correlation matrix (i.e. $a_{jk} = \cos(\theta_j - \theta_k)$ with the θ_j real) then $\operatorname{per}(A \circ A) \leqslant \operatorname{per}(A)$. Here $\circ$ denotes the Hadamard (entrywise) product.*

This is a special case of a question raised in [1].

CONJECTURE 3. *If A is a real positive semidefinite matrix then $\lambda_1(B) = \operatorname{per}(A)$. Here B is defined by (1) and $\lambda_1(B)$ is its largest eigenvalue.*

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Stephen Drury
Department of Mathematics and Statistics
McGill University
Montreal, Canada H3A 0B9
e-mail: stephen.drury@mcgill.ca