LANDEN INEQUALITIES FOR A CLASS OF
HYPERGEOMETRIC FUNCTIONS WITH APPLICATIONS

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Abstract. In this paper, we study a class of Gaussian hypergeometric function $_2F_1(a,b;(a+b+1)/2;x)(a,b>0)$, and find the maximal regions of ab plane in the first quadrant where the well-known Landen identities for the complete elliptic integrals of the first kind turn on respective inequalities valid for each $x \in (0,1)$. Besides, the generalized Grötzsch ring function with two parameters $\mu_{a,b}(r)$ is introduced, and the analogs of duplication formula satisfied by Grötzsch ring function $\mu(r)$ for $\mu_{a,b}(r)$, in the form of inequalities, will be derived.

1. Introduction

For real numbers a,b, and c with $c \neq 0, -1, -2, \cdots$, the Gaussian hypergeometric function is defined by [1, 8, 30, 40, 48, 49, 55, 72]

$$F(a,b;c;x) =_2 F_1(a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a,n)(b,n)}{(c,n)} \frac{x^n}{n!}$$

for $x \in (-1,1)$, where

$$(a,n) = a(a+1)(a+2)\cdots(a+n-1)$$

for $n = 1, 2, \cdots$, and $(a,0) = 1$ for $a \neq 0$. The function $F(a,b;c;x)$ is called zero-balanced when $c = a+b$. The limiting values for $F(a,b;c;x)$ at 1 are as follows (see [8, Theorem 1.19 and 1.48])

$$\begin{cases} F(a,b;c;1) = \Gamma(c)\Gamma(c-a-b)/[\Gamma(c-a)\Gamma(c-b)], & a+b < c, \\ B(a,b)F(a,b;c;z) + \log(1-z) = R(a,b) + O((1-z)\log(1-z)), & a+b = c, \quad (1.1) \\ F(a,b;c;z) = (1-z)^{c-a-b}F(c-a,c-b;c;z), & a+b > c. \end{cases}$$

Here $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$ $(\mathrm{Re}(x) > 0)$ [60, 61, 68, 69, 70, 74, 75, 76] and $B(p,q) = [\Gamma(p)\Gamma(q)]/[\Gamma(p+q)]$ $(\mathrm{Re}(p) > 0,\ \mathrm{Re}(q) > 0)$ are the classical gamma function and beta functions, respectively, and

$$R(a,b) = -\psi(a) - \psi(b) - 2\gamma, \quad R\left(\frac{1}{2},\frac{1}{2}\right) = \log 16,$$

$\psi(z) = \Gamma'(z)/\Gamma(z)$ $(\mathrm{Re}(z) > 0)$ [66, 67, 77] and γ is the Euler-Mascheroni constant defined by

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^n \frac{1}{k} - \log n\right) = 0.577\cdots.$$

As special cases of hypergeometric functions, the complete elliptic integral of the first kind $\mathcal{K}(r)$ and the second kind $\mathcal{E}(r)$ can be expressed as

$$\mathcal{K}(r) = \frac{\pi}{2}F\left(\frac{1}{2},\frac{1}{2};1;r^2\right) = \int_0^{\pi/2}(1 - r^2\sin^2 t)^{-1/2}dt, \quad r \in [0,1), \quad \mathcal{K}(1^-) = +\infty,$$

$$\mathcal{E}(r) = \frac{\pi}{2}F\left(-\frac{1}{2},\frac{1}{2};1;r^2\right) = \int_0^{\pi/2}(1 - r^2\sin^2 t)^{1/2}dt, \quad r \in [0,1],$$

respectively. Elliptic integrals have been studied over three centuries. Many important applications were found in physics, engineering, theory of mean values, number theory and other related fields [1, 2, 13, 14, 16, 25, 26, 27, 28, 32, 37, 38, 41, 43, 47, 58]. Recently, they also occur frequently in geometric function theory, particularly, these integrals enable us to investigate a lot of special functions that are indispensable for the study of quasiconformal mappings [3, 4, 5, 6, 11, 33]. For more properties and inequalities for $\mathcal{K}(r)$ and $\mathcal{E}(r)$ see e.g. [15, 17, 18, 19, 20, 21, 22, 23, 24, 31, 45, 50, 51, 52, 53, 57, 62, 64, 65, 73]. Recall that some of the most important properties of the elliptic integral $\mathcal{K}(r)$ are Landen identities proved in 1771 [59, p. 507]

$$\mathcal{K}\left(\frac{2\sqrt{r}}{1+r}\right) = (1+r)\mathcal{K}(r), \quad \mathcal{K}\left(\frac{1-r}{1+r}\right) = \frac{1+r}{2}\mathcal{K}(r'). \tag{1.2}$$

Here and in what follows $r' \equiv \sqrt{1-r^2}$. Equation (1.2) is equivalent to

$$F\left(\frac{1}{2},\frac{1}{2};1;\frac{4r}{(1+r)^2}\right) = (1+r)F\left(\frac{1}{2},\frac{1}{2};1;r^2\right), \tag{1.3}$$

$$F\left(\frac{1}{2},\frac{1}{2};1;\left(\frac{1-r}{1+r}\right)^2\right) = \frac{1+r}{2},F\left(\frac{1}{2},\frac{1}{2};1;1-r^2\right).$$

In 1992, a natural problem related to Landen identities was proposed by Anderson et al. in [7, p. 79]:

PROBLEM 1.1. Find an analog of Landen's transformation formulas in (1.2) for the zero-balanced hypergeometric function $F(a,b;a+b;x)(a,b > 0)$.

Later, Qiu and Vuorinen [35, Theorem 1.2] claimed that the function

$$s(r) = (1 + \sqrt{r})F(a,b;a+b;r) - F\left(a,b;a+b;\frac{4\sqrt{r}}{(1+\sqrt{r})^2}\right)$$

is increasing in $r \in (0,1)$ for $(a,b) \in \{a,b > 0 | a+b \leqslant 1\}$, and derived the Landen inequality

$$F\left(a,b;a+b;\frac{4r}{(1+r)^2}\right) \leqslant (1+r)F\left(a,b;a+b;r^2\right) \tag{1.4}$$

for each $r \in (0,1)$ and $(a,b) \in \{a,b > 0 | a+b \leqslant 1\}$. Unfortunately, there is an error in the proof of the monotonicity of $s(r)$. Simić and Vuorinen [39, Lemma 3.2] gave a correct proof, and showed that $s(r)$ is also increasing on $(0,1)$ for $(a,b) \in \{a,b > 0 | a+b > 1, ab \leqslant 1/4\}$, and decreasing for $(a,b) \in \{a,b > 0 | 1/a + 1/b \leqslant 4\}$. Besides, for each $r \in (0,1)$, they found the maximal regions of ab plane in the first quadrant where equalities (1.2) become (1.4) or its reversed inequality.

Let $\mathbf{B}^2$ be the unit disk in the plane and $\mu(r)$ the modulus of the plane Grötzsch ring $\mathbf{B}^2 \setminus [0,r]$ for $r \in (0,1)$. Then $\mu(r)$, called the Grötzsch ring function, has the following explicit expression in terms of complete elliptic integrals (see [29, p. 60])

$$\mu(r) = \frac{\pi}{2} \frac{\mathcal{K}(r')}{\mathcal{K}(r)}, \quad r \in (0,1). \tag{1.5}$$

By (1.2) and (1.5), it is easy to see that

$$\mu(r) = 2\mu\left(\frac{2\sqrt{r}}{1+r}\right), \quad \mu(r) = \frac{1}{2}\mu\left(\frac{1-r'}{1+r'}\right). \tag{1.6}$$

The Grötzsch ring function plays a very important role in quasi-conformal theory and quasi-regular theory (see [8, 42]). Moreover, using $\mu(r)$, the classical modular equation of signature 2 and degree p ($p > 1$) in number theory also can be rewritten as (see [43])

$$\mu(s) = p\mu(r), \quad 0 < r < 1.$$

Because of its importance, numerous properties and inequalities of $\mu(r)$ have been obtained by many authors in the recent years (see [8, 10, 34]).

On the other hand, in the 1990's, Anderson, Vuorinen, Vamanamurthy and Qiu et al. began to study a natural generalization of $\mu(r)$ defined by

$$\mu_a(r) = \frac{\pi}{2\sin(\pi a)} \frac{F(a, 1-a; 1; 1-r^2)}{F(a, 1-a; 1; r^2)}, \quad r \in (0,1)$$

in the theory of Ramanujan generalized modular equation, and a lot of beautiful properties and inequalities of $\mu(r)$ have been extended to the function $\mu_a(r)$ [12, 34, 36, 44, 46, 71]. For instance, Qiu and Vuorinen [34, 36] derived the analogs of the duplication formula and logarithmic inequalities satisfied by $\mu(r)$ for $\mu_a(r)$. The infinite-product representations for $\mu_{1/3}(r)$ and $\mu_{1/4}(r)$ which involves only r are established in [54, 56].

The objective of this paper is twofold. The first one is to study a class of hypergeometric function $F(a,b;(a+b+1)/2;x)$ with $a,b > 0$, and prove a generalization of Landen identities for it in the form of inequalities analogously. The other one is to introduce the generalized Grötzsch ring function with two parameters defined by

$$\mu_{a,b}(r) = \frac{B(a,b)}{2}\frac{F\left(a,b;(a+b+1)/2;;1-r^2\right)}{F\left(a,b;(a+b+1)/2;r^2\right)}, \quad r \in (0,1), \tag{1.7}$$

and find the analogs of identities (1.6) for $\mu_{a,b}(r)$.

According to the definition of $\mu_{a,b}(r)$ in (1.7), it follows that $\mu_a(r) = \mu_{a,1-a}(r)$, $\mu(r) = \mu_{1/2}(r) = \mu_{1/2,1/2}(r)$, and the limiting values of $\mu_{a,b}(r)$ at 0 and 1 are

$$
\begin{aligned}
\mu_{a,b}(0^+) &= \lim_{r\to 0^+}\frac{B(a,b)}{2}F\left(a,b;\frac{a+b+1}{2};1-r^2\right)\\
&= \begin{cases} \frac{B(a,b)}{2}H(a,b), & a+b < 1;\\ +\infty, & a+b \geqslant 1, \end{cases}
\end{aligned}
\tag{1.8}
$$

$$
\mu_{a,b}(1^-) = \lim_{r\to 1^-}\frac{B(a,b)}{2F\left(a,b;\frac{a+b+1}{2};r^2\right)} = \begin{cases} \frac{B(a,b)}{2H(a,b)}, & a+b < 1;\\ 0, & a+b \geqslant 1, \end{cases}
\tag{1.9}
$$

respectively. Here and in what follows

$$H(a,b) = \frac{B(\frac{a+b+1}{2},\frac{1-a-b}{2})}{B(\frac{1+b-a}{2},\frac{1+a-b}{2})}.$$

Moreover, due to [9, Lemmas 4.5, 4.6 and 4.8], $\mu_{a,b}(r)$ satisfies the following derivative formula

$$\frac{d\mu_{a,b}(r)}{dr} = -\frac{\Gamma(\frac{a+b+1}{2})^2}{\Gamma(a+b)}\frac{1}{r^{a+b}r'^{a+b+1}F\left(a,b;(a+b+1)/2;r^2\right)^2}. \tag{1.10}$$

2. Main results

For convenience, we first list some regions in $\{(a,b) \in \mathbb{R}^2 | a > 0, b > 0\}$ (see Figure 1):

$$D_1 = \{(a,b)|a,b > 0, a+b \leqslant 1\},$$

$$D_2 = \left\{(a,b)|a,b > 0, a+b > 1, ab + \frac{a+b}{2} - \frac{3}{4} < 0\right\},$$

$$D_3 = \left\{(a,b)|a,b > 0, ab + \frac{a+b}{2} - \frac{3}{4} \geqslant 0, ab - \frac{a+b+1}{8} < 0\right\},$$
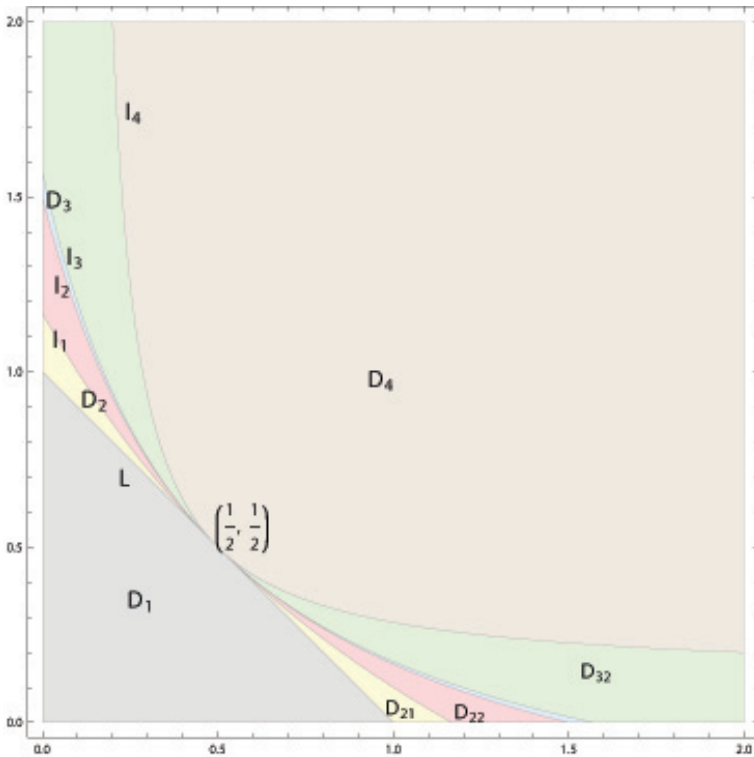
$$D_4 = \left\{(a,b)|a,b > 0, ab - \frac{a+b+1}{8} \geqslant 0\right\},$$

Figure 1: *The regions D_1, D_2, D_3, D_4, D_{21}, D_{22}, D_{31}, and D_{32}, where $L : a+b = 1$, $l_1 : ab + 3(a+b)/2 - 7/4 = 0$, $l_2 : ab + (a+b)/2 - 3/4 = 0$, $l_3 : ab + 7(a+b)/16 - 11/16 = 0$, $l_4 : ab - (a+b+1)/8 = 0$.*

$$D_{21} = \left\{ (a,b) | a,b > 0, a+b > 1, ab + \frac{3}{2}(a+b) - \frac{7}{4} < 0 \right\},$$

$$D_{22} = \left\{ (a,b) | a,b > 0, ab + \frac{3}{2}(a+b) - \frac{7}{4} \geqslant 0, ab + \frac{a+b}{2} - \frac{3}{4} < 0 \right\},$$

$$D_{31} = \left\{ (a,b) | a,b > 0, ab + \frac{a+b}{2} - \frac{3}{4} \geqslant 0, ab + \frac{7}{16}(a+b) - \frac{11}{16} < 0 \right\},$$

$$D_{32} = \left\{ (a,b) | a,b > 0, ab + \frac{7}{16}(a+b) - \frac{11}{16} \geqslant 0, ab - \frac{a+b+1}{8} < 0 \right\}.$$

Clearly, $D_1 \cup D_2 \cup D_3 \cup D_4 = \{(a,b) \in \mathbb{R}^2 | a > 0, b > 0\}$, $D_{21} \subset D_2$, $D_{22} \subset D_2$, $D_{31} \subset D_3$ and $D_{32} \subset D_3$.

THEOREM 2.1. *Let $a,b \in (0,\infty)$. Then the inequality*

$$F\left(a,b;\frac{a+b+1}{2};\frac{4r}{(1+r)^2}\right) \leqslant (1+r)F\left(a,b;\frac{a+b+1}{2};r^2\right) \tag{2.1}$$

holds for all $r \in (0,1)$ *if and only if* $(a,b) \in D_1$, *and the reversed inequality*

$$F\left(a,b;\frac{a+b+1}{2};\frac{4r}{(1+r)^2}\right) \geqslant (1+r)F\left(a,b;\frac{a+b+1}{2};r^2\right) \qquad (2.2)$$

takes place for each $r \in (0,1)$ *if and only if* $(a,b) \in D_4$, *with equality in each instance if and only if* $(a,b) = (1/2,1/2)$.

In the remaining region $(a,b) \in D_2 \cup D_3$, neither of the above inequalities holds for each $r \in (0,1)$.

THEOREM 2.2. *For* $a,b > 0$ *with* $(a,b) \neq (1/2,1/2)$, *define the function* f *on* $(0,1)$ *by*

$$f(r) = (1+\sqrt{r})F\left(a,b;\frac{a+b+1}{2};r\right) - F\left(a,b;\frac{a+b+1}{2};\frac{4\sqrt{r}}{(1+\sqrt{r})^2}\right).$$

Then the following statements are true:

(1) $f(r)$ *is strictly increasing from* $(0,1)$ *onto* $(0,[R(a,b)-\log 16]/B(a,b))$ *when* $(a,b) \in L = \{(a,b)|a,b > 0, a+b = 1\}$;

(2) $f(r)$ *is strictly increasing from* $(0,1)$ *onto* $(0,H(a,b))$ *when* $(a,b) \in D_1 \setminus L$;

(3) $f(r)$ *is strictly decreasing from* $(0,1)$ *onto* $(-\infty,0)$ *when* $(a,b) \in D_4$.

Moreover, for $(a,b) \in D_1 \setminus L$, *the inequality*

$$F\left(a,b;\frac{a+b+1}{2};\frac{4r}{(1+r)^2}\right) \leqslant (1+r)F\left(a,b;\frac{a+b+1}{2};r^2\right)$$

$$\leqslant F\left(a,b;\frac{a+b+1}{2};\frac{4r}{(1+r)^2}\right) + H(a,b) \qquad (2.3)$$

holds for all $r \in (0,1)$, *and for* $(a,b) \in L$, *the following inequality is valid for each* $r \in (0,1)$,

$$F\left(a,b;\frac{a+b+1}{2};\frac{4r}{(1+r)^2}\right) \leqslant (1+r)F\left(a,b;\frac{a+b+1}{2};r^2\right)$$

$$\leqslant F\left(a,b;\frac{a+b+1}{2};\frac{4r}{(1+r)^2}\right) + \frac{R(a,b)-\log 16}{B(a,b)}. \qquad (2.4)$$

THEOREM 2.3. *For* $(a,b) \in \{(a,b)|a,b > 0, ab \geqslant a+b-10/9, a+b \geqslant 2\}$, *define the function g on* $(0,1)$ *by*

$$g(r) = 2\mu_{a,b}\left(\frac{2\sqrt{r}}{1+r}\right) - \mu_{a,b}(r).$$

Then g is strictly increasing from $(0,1)$ *onto* $(-\infty,0)$. *In particular, the inequality*

$$2\mu_{a,b}\left(\frac{2\sqrt{r}}{1+r}\right) < \mu_{a,b}(r) \qquad (2.5)$$

holds for each $r \in (0,1)$ *with* $(a,b) \in \{(a,b)|a,b > 0, ab \geqslant a+b-10/9, a+b \geqslant 2\}$.

REMARK 2.4. If $a+b=1$, then our Theorems 2.1 and 2.2 lead to Theorem 2.1 and Lemma 3.2 in [39], respectively.

3. Proofs of Theorems 2.1 and 2.2

Thought out this section, we let

$$F(x) = F\left(a,b;\frac{a+b+1}{2};x\right), \quad G(x) = F\left(a+1,b+1;\frac{a+b+3}{2};x\right) \qquad (3.1)$$

with $a,b > 0$ and $(a,b) \neq (1/2,1/2)$, and

$$F^*(x) = F\left(\frac{1}{2},\frac{1}{2};1;x\right), \quad G^*(x) = F\left(\frac{3}{2},\frac{3}{2};2;x\right). \qquad (3.2)$$

LEMMA 3.1. ([63, Theorem 2.1]) *Suppose that the power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$ have the radius of convergence $r > 0$ with $b_n > 0$ for all $n \in \{0,1,2,\cdots\}$. Let $h(x) = f(x)/g(x)$ and $H_{f,g} = (f'/g')g - f$, then the following statements are true:*

(1) If the non-constant sequence $\{a_n/b_n\}_{n=0}^{\infty}$ is increasing (decreasing), then $h(x)$ is strictly increasing (decreasing) on $(0,r)$;

(2) If the non-constant sequence $\{a_n/b_n\}$ is increasing (decreasing) for $0 < n \leqslant n_0$ and decreasing (increasing) for $n > n_0$, then the function h is strictly increasing (decreasing) on $(0,r)$ if and only if $H_{f,g}(r^-) \geqslant (\leqslant)0$. While if $H_{f,g}(r^-) < (>)0$, then there exists $x_0 \in (0,r)$ such that $h(x)$ is strictly increasing (decreasing) on $(0,x_0)$ and strictly decreasing (increasing) on (x_0,r).

LEMMA 3.2. (1) *The function $\phi(x) = F(x)/F^*(x)$ is strictly decreasing on $(0,1)$ if $(a,b) \in D_1$ and strictly increasing on $(0,1)$ if $(a,b) \in D_4$. Moreover, if $(a,b) \in D_2 \cup D_3$, then there exists $x_0 \in (0,1)$ such that $\phi(x)$ is strictly decreasing on $(0,x_0)$ and strictly increasing on $(x_0,1)$;*

(2) The function $\varphi(x) = G(x)/G^*(x)$ is strictly decreasing on $(0,1)$ if $(a,b) \in D_1$ and strictly increasing on $(0,1)$ if $(a,b) \in D_4 \cup D_{32}$. In the remaining region, namely $x \in D_2 \cup D_{31}$, $\varphi(x)$ is piecewise monotone on $(0,1)$.

Proof. For part (1), let

$$A_n = \frac{(a,n)(b,n)}{\left(\frac{a+b+1}{2},n\right)n!}, \quad A_n^* = \frac{(1/2,n)(1/2,n)}{(1,n)(n)!},$$

then

$$\phi(x) = \frac{F(x)}{F^*(x)} = \frac{\sum\limits_{n=0}^{\infty} A_n x^n}{\sum\limits_{n=0}^{\infty} A_n^* x^n}. \qquad (3.3)$$

Simple computations show that the monotonicity of $\{A_n/A_n^*\}$ depends on the sign of

$$H_n = \left(\frac{a+b-1}{2}\right)n^2 + \left(ab + \frac{a+b}{2} - \frac{3}{4}\right)n + ab - \frac{a+b+1}{8}. \tag{3.4}$$

We divide the proof into three cases.

Case 1 $(a,b) \in D_1$. Then $H_n < 0$ for $n = 0, 1, 2, \cdots$ follows from (3.4). Therefore, (3.3) and Lemma 3.1(1) lead to that $\phi(x)$ is strictly decreasing on $(0,1)$.

Case 2 $(a,b) \in D_4$. Then (3.4) implies $H_n > 0$ for $n = 0, 1, 2, \cdots$, and $\phi(x)$ is strictly increasing on $(0,1)$ by (3.3) and Lemma 3.1(1).

Case 3 $(a,b) \in D_2 \cup D_3$. Then from (3.4) we conclude that the sequence $\{A_n/A_n^*\}$ is decreasing for $n \in \{1, 2, \cdots, n_0\}$ and increasing for $n \geqslant n_0$, for some integer n_0. Moreover, using (1.1) and the derivative formula of hypergeometric function

$$\frac{dF(a,b;c;x)}{dx} = \frac{ab}{c}F(a+1,b+1;c+1;x),$$

we get

$$\begin{aligned}
H_{F,F^*}(x) =& \frac{8ab}{a+b+1}(1-x)\frac{F(a+1,b+1;(a+b+3)/2;x)}{F(1/2,1/2;2;x)}F\left(\frac{1}{2},\frac{1}{2};1;x\right) \\
& - F\left(a,b;\frac{a+b+1}{2};x\right) \\
=& \frac{8ab}{a+b+1}(1-x)\frac{(1-x)^{-(a+b+1)/2}F\left(\frac{b-a+1}{2},\frac{a-b+1}{2};\frac{a+b+3}{2};x\right)}{F(1/2,1/2;2;x)}F\left(\frac{1}{2},\frac{1}{2};1;x\right) \\
& - (1-x)^{\frac{1-a-b}{2}}F\left(\frac{b-a+1}{2},\frac{a-b+1}{2};\frac{a+b+1}{2};x\right) \\
=& (1-x)^{\frac{1-a-b}{2}}\left[\frac{8ab}{a+b+1}\frac{F\left(\frac{b-a+1}{2},\frac{a-b+1}{2};\frac{a+b+3}{2};x\right)}{F(1/2,1/2;2;x)}F\left(\frac{1}{2},\frac{1}{2};1;x\right)\right. \\
& \left. - F\left(\frac{b-a+1}{2},\frac{a-b+1}{2};\frac{a+b+1}{2};x\right)\right],
\end{aligned}$$

$$\lim_{x \to 1^-} H_{F,F^*}(x) = +\infty. \tag{3.5}$$

Combining (3.5), (3.3) and Lemma 3.1(2) together with (1.1), we have that there exists $x_0 \in (0,1)$ such that $\phi(x)$ is strictly decreasing on $(0,x_0)$ and strictly increasing on $(x_0,1)$.

For part (2), let

$$B_n = \frac{(a+1,n)(b+1,n)}{\left(\frac{a+b+3}{2},n\right)n!}, \quad B_n^* = \frac{(3/2,n)(3/2,n)}{(2,n)(n)!},$$

then

$$\varphi(x) = \frac{G(x)}{G^*(x)} = \frac{\sum\limits_{n=0}^{\infty} B_n x^n}{\sum\limits_{n=0}^{\infty} B_n^* x^n}. \tag{3.6}$$

Note that the monotonicity of $\{B_n/B_n^*\}$ depends on the sign of

$$H_n^* = \left(\frac{a+b-1}{2}\right)n^2 + \left[ab + \frac{3(a+b)}{2} - \frac{7}{4}\right]n + 2ab + \frac{7(a+b)}{8} - \frac{11}{8}. \tag{3.7}$$

Moreover, for $(a,b) \in D_2 \cup D_{31}^*$, we have

$$
\begin{aligned}
H_{G,G^*}(x) &= \frac{2(a+1)(b+1)}{a+b+3} \frac{F\left(a+2,b+2;\frac{a+b+5}{2};x\right)}{9F\left(\frac{5}{2},\frac{5}{2};3;x\right)/8} F\left(\frac{3}{2},\frac{3}{2};2;x\right) \\
&\quad - F\left(a+1,b+1;\frac{a+b+3}{2};x\right) \\
&= \frac{16(a+1)(b+1)}{9(a+b+3)}(1-x)^{-\frac{a+b+1}{2}} \frac{F\left(\frac{b-a+1}{2},\frac{a-b+1}{2};\frac{a+b+5}{2};x\right)}{F(\frac{1}{2},\frac{1}{2};3;x)} F\left(\frac{1}{2},\frac{1}{2};2;x\right) \\
&\quad - (1-x)^{-\frac{a+b+1}{2}} F\left(\frac{b-a+1}{2},\frac{a-b+1}{2};\frac{a+b+3}{2};x\right) \\
&= (1-x)^{-\frac{a+b+1}{2}} \left[\frac{16(a+1)(b+1)}{9(a+b+3)} \frac{F\left(\frac{b-a+1}{2},\frac{a-b+1}{2};\frac{a+b+5}{2};x\right)}{F(\frac{1}{2},\frac{1}{2};3;x)} F\left(\frac{1}{2},\frac{1}{2};2;x\right) \right. \\
&\quad \left. - F\left(\frac{b-a+1}{2},\frac{a-b+1}{2};\frac{a+b+3}{2};x\right) \right],
\end{aligned}
$$

$$
\begin{aligned}
&\lim_{x\to 1^-}(1-x)^{\frac{a+b+1}{2}} H_{G,G^*}(x) \\
&= \frac{16(a+1)(b+1)}{9(a+b+3)} \frac{\Gamma\left(\frac{a+b+5}{2}\right)\Gamma\left(\frac{a+b+3}{2}\right)}{\Gamma(a+2)\Gamma(b+2)} \frac{\Gamma(2)\Gamma(1)}{\Gamma\left(\frac{3}{2}\right)^2} \frac{\Gamma\left(\frac{5}{2}\right)^2}{\Gamma(3)\Gamma(2)} - \frac{\Gamma\left(\frac{a+b+3}{2}\right)\Gamma\left(\frac{a+b+1}{2}\right)}{\Gamma(a+1)\Gamma(b+1)} \\
&= \left(\frac{a+b-1}{2}\right) \frac{\Gamma\left(\frac{a+b+3}{2}\right)\Gamma\left(\frac{a+b+1}{2}\right)}{\Gamma(a+1)\Gamma(b+1)} > 0,
\end{aligned}
$$

and thereby,

$$\lim_{x\to 1^-} H_{G,G^*}(x) = +\infty. \tag{3.8}$$

It follows from (3.6)-(3.8) together with the similar arguments in part (1) that the assertions of part (2) hold true. Here we omit the details. □

Proof of Theorem 2.1. Since $y(r) = 4r/(1+r)^2 > r^2$ for $0 < r < 1$, Lemma 3.2(1) shows that $\phi(r^2) > \phi(y(r))$ for $(a,b) \in D_1$ and $\phi(r^2) < \phi(y(r))$ for $(a,b) \in D_4$. Therefore, it follows from (1.2) that

$$\frac{F(r^2)}{F^*(r^2)} > \frac{F(y(r))}{F^*(y(r))}, \quad F(y(r)) < \frac{F^*(y(r))}{F^*(r^2)}F(r^2) = (1+r)F(r^2)$$

for $(a,b) \in D_1$, which implies (2.1).

Inequality (2.2) can be derived analogously. The remaining conclusions easily follow from Lemma 3.2(1). □

Proof of Theorem 2.2. Clearly $f(0^+) = 0$. From (1.1) we get

$$f(1^-) = \begin{cases} H(a,b), & a+b < 1, \\ \frac{R(a,b)-\log 16}{B(a,b)}, & a+b = 1, \\ -\infty, & a+b > 1. \end{cases} \tag{3.9}$$

Let $x = x(r) = 4\sqrt{r}/(1+\sqrt{r})^2$. Then

$$1-x = \left(\frac{1-\sqrt{r}}{1+\sqrt{r}}\right)^2, \quad \frac{dx}{dr} = \frac{2(1-\sqrt{r})}{\sqrt{r}(1+\sqrt{r})^3}. \tag{3.10}$$

Differentiating f yields

$$2\sqrt{r}(1-\sqrt{r})f'(r) = (1-\sqrt{r})F(r) + 2\sqrt{r}(1-r)\frac{2ab}{a+b+1}G(r)$$
$$- \frac{2ab}{a+b+1}\frac{4(1-x)}{1+\sqrt{r}}G(x). \tag{3.11}$$

On the other hand, differentiating Landen identity (1.2) with replacing r by $\sqrt{r}$, we get

$$\frac{1}{4}\frac{2(1-\sqrt{r})}{\sqrt{r}(1+\sqrt{r})^3}G^*(x) = \frac{1}{2\sqrt{r}}F^*(r) + \frac{1}{4}(1+\sqrt{r})G^*(r),$$

namely,

$$\frac{1-x}{1+\sqrt{r}}\frac{G^*(x)}{G^*(r)} = \frac{(1-\sqrt{r})F^*(r)}{G^*(r)} + \frac{\sqrt{r}}{2}(1-r). \tag{3.12}$$

Since φ defined as in Lemma 3.2(2) is strictly decreasing in $(0,1)$ on D_1, we conclude that $\varphi(r) > \varphi(x)$ for all $r \in (0,1)$, that is

$$G(x) < \frac{G*(x)}{G^*(r)}G(r). \tag{3.13}$$

Equations (3.11) and (3.12) together with inequality (3.13) lead to

$$2\sqrt{r}(1-\sqrt{r})f'(r)$$
$$= (1-\sqrt{r})F(r) + 2\sqrt{r}(1-r)\frac{2ab}{a+b+1}G(r) - \frac{2ab}{a+b+1}\frac{4(1-x)}{1+\sqrt{r}}G(x)$$
$$> (1-\sqrt{r})F(r) + 2\sqrt{r}(1-r)\frac{2ab}{a+b+1}G(r) - \frac{2ab}{a+b+1}\frac{4(1-x)}{1+\sqrt{r}}\frac{G*(x)}{G^*(r)}G(r)$$
$$= (1-\sqrt{r})F(r) + 2\sqrt{r}(1-r)\frac{2ab}{a+b+1}G(r) - \frac{8ab}{a+b+1}\left[\frac{(1-\sqrt{r})F^*(r)}{G^*(r)} + \frac{\sqrt{r}}{2}(1-r)\right]G(r)$$
$$= (1-\sqrt{r})\left[F(r) - \frac{8ab}{a+b+1}\frac{F^*(r)}{G^*(r)}G(r)\right]$$
$$= (1-\sqrt{r})\frac{4F(r)^2}{G^*(r)}\left(\frac{F^*(r)}{F(r)}\right)'. \tag{3.14}$$

Lemma 3.2(1) shows that $r \to F^*(r)/F(r)$ is strictly increasing in $(0,1)$ on $(a,b) \in D_1$. Thus by (3.14) one has $f'(r) > 0$ for all $r \in (0,1)$ when $(a,b) \in D_1$, and f is strictly increasing in $(0,1)$.

On the other hand, since φ defined as in Lemma 3.2(2) is strictly increasing in $(0,1)$ on D_4, we have $\varphi(x) > \varphi(r)$, namely

$$G(x) > \frac{G*(x)}{G^*(r)} G(r).$$

With the similar argument, one has

$$\sqrt{r} f'(r) < \frac{2F(r)^2}{G^*(r)} \left(\frac{F^*(r)}{F(r)} \right)' < 0,$$

because $F(r)/F^*(r)$ is strictly increasing in $(0,1)$ on D_4 by Lemma 3.2(1). Hence $f(r)$ is strictly decreasing in $(0,1)$ on D_4. It is now obvious that the remaining conclusions hold. □

4. Proof of Theorem 2.3

LEMMA 4.1. *Let $x' = \sqrt{1-x^2}$ for $0 < x < 1$. Then the function*

$$f(x) = \frac{(xx')^{(a+b-1)/2} F\left(a,b; \frac{a+b+1}{2}; x^2\right)}{F\left(\frac{1}{2}, \frac{1}{2}; 1; x^2\right)} \tag{4.1}$$

is strictly increasing on $(0,1)$ when $(a,b) \in \{(a,b) | a,b > 0, ab \geqslant a+b-10/9, a+b \geqslant 2\}$.

Proof. Differentiating f gives

$$f'(x) = \frac{1}{F\left(\frac{1}{2}, \frac{1}{2}; 1; x^2\right)^2} \left\{ \left[\frac{a+b-1}{2} (xx')^{\frac{a+b-3}{2}} \left(\frac{1-2x^2}{x'} \right) F\left(a,b; \frac{a+b+1}{2}; x^2\right) \right. \right.$$

$$+ (xx')^{\frac{a+b-1}{2}} \frac{2ab}{a+b+1} F\left(a+1, b+1; \frac{a+b+3}{2}; x^2\right) \cdot 2x \left] F\left(\frac{1}{2}, \frac{1}{2}; 1; x^2\right) \right.$$

$$\left. - (xx')^{\frac{a+b-1}{2}} F\left(a,b; \frac{a+b+1}{2}; x^2\right) \cdot \frac{x}{2} F\left(\frac{3}{2}, \frac{3}{2}; 2; x^2\right) \right\},$$

and thereby

$$f_1(x) \equiv x'(xx')^{-\frac{a+b-3}{2}} F\left(\frac{1}{2}, \frac{1}{2}; 1; x^2\right)^2 f'(x)$$

$$= \frac{a+b-1}{2} (1-2x^2) F\left(a,b; \frac{a+b+1}{2}; x^2\right) F\left(\frac{1}{2}, \frac{1}{2}; 1; x^2\right)$$

$$+ \frac{4ab}{a+b+1} x^2 x'^2 F\left(a+1, b+1; \frac{a+b+3}{2}; x^2\right) F\left(\frac{1}{2}, \frac{1}{2}; 1; x^2\right)$$

$$- \frac{x^2 x'^2}{2} F\left(a, b; \frac{a+b+1}{2}; x^2\right) F\left(\frac{3}{2}, \frac{3}{2}; 2; x^2\right). \tag{4.2}$$

Since $x'^2 F\left(3/2, 3/2; 2; x^2\right) = F\left(1/2, 1/2; 2; x^2\right) \leqslant F\left(1/2, 1/2; 1; x^2\right)$ for $x \in (0,1)$, it follows from (4.2) that

$$f_1(x) \geqslant \frac{a+b-1}{2}\left(1-2x^2\right) F\left(a, b; \frac{a+b+1}{2}; x^2\right) F\left(\frac{1}{2}, \frac{1}{2}; 1; x^2\right)$$

$$+ \frac{4ab}{a+b+1} x^2 x'^2 F\left(a+1, b+1; \frac{a+b+3}{2}; x^2\right) F\left(\frac{1}{2}, \frac{1}{2}; 1; x^2\right)$$

$$- \frac{x^2}{2} F\left(a, b; \frac{a+b+1}{2}; x^2\right) F\left(\frac{1}{2}, \frac{1}{2}; 1; x^2\right)$$

$$= F\left(\frac{1}{2}, \frac{1}{2}; 1; x^2\right)\left[\frac{1}{2}\left(a+b-1-(2a+2b-1)x^2\right) F\left(a, b; \frac{a+b+1}{2}; x^2\right)\right.$$

$$\left. + \frac{4ab}{a+b+1} x^2(1-x^2) F\left(a+1, b+1; \frac{a+b+3}{2}; x^2\right)\right]. \tag{4.3}$$

Let $f_2(x) = f_1(x)/F\left(1/2, 1/2; 1; x^2\right)$, then from (4.3) we get

$$f_2(x) \geqslant \frac{a+b-1}{2} \sum_{n=0}^{\infty} \frac{(a,n)(b,n)}{\left(\frac{a+b+1}{2}, n\right)} \frac{x^{2n}}{n!} - \left(a+b-\frac{1}{2}\right) \sum_{n=0}^{\infty} \frac{(a,n)(b,n)}{\left(\frac{a+b+1}{2}, n\right)} \frac{x^{2n+2}}{n!}$$

$$+ \frac{4ab}{a+b+1} \sum_{n=0}^{\infty} \frac{(a+1,n)(b+1,n)}{\left(\frac{a+b+3}{2}, n\right)} \frac{x^{2n+2}}{n!}$$

$$- \frac{4ab}{a+b+1} \sum_{n=0}^{\infty} \frac{(a+1,n)(b+1,n)}{\left(\frac{a+b+3}{2}, n\right)} \frac{x^{2n+4}}{n!}$$

$$= \frac{a+b-1}{2} + \left[\frac{ab(a+b-1)}{a+b+1} - \left(a+b-\frac{1}{2}\right) + \frac{4ab}{a+b+1}\right] x^2$$

$$+ \frac{a+b-1}{2} \sum_{n=0}^{\infty} \frac{(a,n+2)(b,n+2)}{\left(\frac{a+b+1}{2}, n+2\right)} \frac{x^{2n+4}}{(n+2)!}$$

$$- \left(a+b-\frac{1}{2}\right) \sum_{n=0}^{\infty} \frac{(a,n+1)(b,n+1)}{\left(\frac{a+b+1}{2}, n+1\right)} \frac{x^{2n+4}}{(n+1)!}$$

$$+ 2 \sum_{n=0}^{\infty} \frac{(a,n+2)(b,n+2)}{\left(\frac{a+b+1}{2}, n+2\right)} \frac{x^{2n+4}}{(n+1)!} - 2 \sum_{n=0}^{\infty} \frac{(a,n+1)(b,n+1)}{\left(\frac{a+b+1}{2}, n+1\right)} \frac{x^{2n+4}}{n!}$$

$$= \frac{a+b-1}{2}\left(1 - \frac{x^2}{a+b+1}\right) + \frac{ab(a+b-1)-(a-b)^2}{a+b+1} x^2$$

$$+ \sum_{n=0}^{\infty} \frac{(a,n+1)(b,n+1)}{\left(\frac{a+b+1}{2}, n+2\right)} \frac{C_n}{(n+2)!} x^{2n+4}, \tag{4.4}$$

where

$$C_n = \frac{a+b-1}{2}(n+a+1)(n+b+1) - (n+2)\left(a+b-\frac{1}{2}\right)\left(n+1+\frac{a+b+1}{2}\right)$$

$$+2(n+a+1)(n+b+1)(n+2) - 2(n+1)(n+2)\left(n+1+\frac{a+b+1}{2}\right)$$

$$=\left(\frac{a+b-2}{2}\right)n^2 + \frac{1}{4}(a+b+8ab-9)n$$

$$+\frac{1}{2}\left[ab(a+b+3) - (a-b)^2 - (a+b+2)\right]. \tag{4.5}$$

If $(a,b) \in \{(a,b)|a,b>0, ab \geqslant a+b-10/9, a+b \geqslant 2\}$, then by elementary computations one has

$$ab(a+b-1) - (a-b)^2 \geqslant \left(a+b-\frac{10}{9}\right)(a+b-1) - (a-b)^2$$

$$= 4ab - \frac{19(a+b)}{9} + \frac{10}{9} \geqslant \frac{17(a+b)}{9} - \frac{30}{9} \geqslant \frac{4}{9},$$

$$a+b+8ab-9 \geqslant a+b+8\left(a+b-\frac{10}{9}\right) - 9 = 9(a+b) - \frac{161}{9} \geqslant \frac{1}{9},$$

$$ab(a+b+3) - (a-b)^2 - (a+b+2)$$

$$\geqslant \left(a+b-\frac{10}{9}\right)(a+b+3) - (a-b)^2 - (a+b+2)$$

$$= 4ab + \frac{8(a+b)}{9} - \frac{16}{3} \geqslant 4\left(a+b-\frac{10}{9}\right) + \frac{8(a+b)}{9} - \frac{16}{3}$$

$$= \frac{44(a+b)}{9} - \frac{88}{9} \geqslant 0.$$

Thus it follows from (4.4) and (4.5) that $f_2(x) > 0$ for all $x \in (0,1)$. Therefore by (4.2) and (4.3), f is strictly increasing on $(0,1)$ when $(a,b) \in \{(a,b)|a,b>0, ab \geqslant a+b-10/9, a+b \geqslant 2\}$. $\square$

REMARK 4.2. The main purpose of Lemma 4.1 is to find some $a,b > 0$ such that the function f defined as in (4.1) is strictly monotone on $(0,1)$. In fact, for $a+b < 1$, we have that $f(0^+) = f(1^-) = +\infty$, so that f is not monotone on $(0,1)$; While for $a+b = 1$, Lemma 3.2(1) shows that f is strictly decreasing on $(0,1)$. In the remaining region $(a,b) \in \{a+b > 1\}$, note that $f(0^+) = 0$ and $f(1^-) = +\infty$, we find its subregion such that f is strictly increasing on $(0,1)$ in Lemma 4.1.

COROLLARY 4.3. *For* $(a,b) \in \{(a,b)|a,b>0, ab \geqslant a+b-10/9, a+b \geqslant 2\}$, *let* $x = x(r) = 2\sqrt{r}/(1+r)$, *then the Landen-type inequality*

$$(xx')^{(a+b-1)/2}F\left(a,b; \frac{a+b+1}{2}; x^2\right) > (1+r)(rr')^{(a+b-1)/2}F\left(a,b; \frac{a+b+1}{2}; r^2\right) \tag{4.6}$$

holds for all $r \in (0,1)$.

Proof. Since $x = x(r) = 2\sqrt{r}/(1+r) > r$ for all $r \in (0,1)$, Lemma 4.1 implies that $f(x) > f(r)$ for $0 < r < 1$, and consequently inequality (4.6) follows. □

Proof of Theorem 2.3. Note that $g(1^-) = 0$ by (1.9), and

$$
\lim_{r \to 0^+} g(r)
$$

$$
= \lim_{r \to 0^+} \frac{B(a,b)}{2} \left[2F\left(a,b; \frac{a+b+1}{2}; \left(\frac{1-r}{1+r}\right)^2\right) - F\left(a,b; \frac{a+b+1}{2}; 1-r^2\right) \right]
$$

$$
= \lim_{r \to 0^+} \left\{ B(a,b) \left[1 - \left(\frac{1-r}{1+r}\right)^2\right]^{(1-a-b)/2} F\left(\frac{-a+b+1}{2}, \frac{a-b+1}{2}; \frac{a+b+1}{2}; \left(\frac{1-r}{1+r}\right)^2\right) \right.
$$

$$
\left. - \frac{B(a,b)}{2} r^{1-a-b} F\left(\frac{-a+b+1}{2}, \frac{a-b+1}{2}; \frac{a+b+1}{2}; 1-r^2\right) \right\}
$$

$$
= \frac{B(a,b)}{2} \frac{\Gamma(\frac{a+b+1}{2})\Gamma(\frac{a+b-1}{2})}{\Gamma(a)\Gamma(b)} \lim_{r \to 0^+} \left[2 \cdot \left(\frac{2\sqrt{r}}{1+r}\right)^{1-a-b} - r^{1-a-b} \right] = -\infty.
$$

Let $x = 2\sqrt{r}/(1+r)$, then

$$
x' = \sqrt{1-x^2} = \frac{1-r}{1+r}, \quad \frac{dx}{dr} = \frac{x'(1+x')^2}{2x}.
$$

Differentiating g yields

$$
g'(r) = -2 \frac{\Gamma\left(\frac{a+b+1}{2}\right)^2}{\Gamma(a+b)} \frac{1}{x^{a+b} x'^{a+b+1} F\left(a,b; \frac{a+b+1}{2}; x^2\right)^2} \cdot \frac{x'(1+x')^2}{2x}
$$

$$
+ \frac{\Gamma\left(\frac{a+b+1}{2}\right)^2}{\Gamma(a+b)} \cdot \frac{1}{r^{a+b} r'^{a+b+1} F\left(a,b; \frac{a+b+1}{2}; r^2\right)^2}
$$

$$
= \frac{\Gamma\left(\frac{a+b+1}{2}\right)^2}{\Gamma(a+b)} \frac{(1+x')^2}{(1+r)^2 x^{a+b+1} x'^{a+b} F\left(a,b; \frac{a+b+1}{2}; x^2\right)^2}
$$

$$
\times \left[\frac{(xx')^{a+b-1} F\left(a,b; \frac{a+b+1}{2}; x^2\right)^2}{(rr')^{a+b-1} F\left(a,b; \frac{a+b+1}{2}; r^2\right)^2} - (1+r)^2 \right]. \tag{4.7}
$$

Therefore, the monotonicity of g directly follows from (4.6) and (4.7). Inequality (2.5) is clear. □

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