Mathematical
Inequalities
& Applications
Volume 21, Number 2 (2018), 539–552 doi:10.7153/mia-2018-21-39

POMPEIU–LIKE THEOREMS FOR THE MEDIANS OF A SIMPLEX

GHADA AL-AFIFI, MOWAFFAQ HAJJA, AHMAD HAMDAN
AND PANAGIOTIS T. KRASOPOULOS

(*Communicated by H. Martini*)

Abstract. A theorem of Pompeiu states that the distances from the vertices of an equilateral triangle to an arbitrary point in its plane can serve as the side lengths of a triangle. A similar theorem holds for the medians of any triangle. Generalizations to higher dimensions of Pompeiu's theorem are established by M. Fiedler in 1977 and by Gh. Al-Afifi et al in a forthcoming paper. This paper is concerned with similar possible higher dimensional theorems for the medians.

1. Introduction

A beautiful theorem of Pompeiu states that the distances from the vertices of a regular triangle to an arbitrary point in its plane form a triangle (in the sense that they can serve as the side lengths of a triangle). Another beautiful Pompeiu-like theorem states that the medians of an arbitrary triangle form a triangle. Visual elegant proofs of these two theorems are depicted in Figures 1 and 2 below. Figure 1, taken from [15, pp. 89–90] and [10, pp. 5–6], describes a proof of Pompeiu's theorem when the point P is inside the triangle. In the figure, $A'B''$, $B'C''$, and $C'A''$ are parallel to AB, BC, and CA, respectively, resulting in isosceles trapezoids $C''AA''P$, $A''BB''P$, $B''CC''P$, and hence in $PA = C''A''$, $PB = A''B''$, and $PC = B''C''$. Figure 2, taken from [12, §473, p. 282], describes a proof of the Pompeiu-like medians theorem. In the figure, CX and $B'X$ are parallel to BA and BC respectively, resulting in parallelograms $BB'XA'$ and $C'AXC$, and hence in $BB' = A'X$ and $CC' = AX$.

As shown in [5] and [6], the same proof applies to what are referred to as *generalized* or s*-medians*; see Figure 3 in Section 3.

This paper is concerned with higher dimensional analogues of these results. In Section 2, we consider the n-dimensional Pompeiu's configuration consisting of a regular n-simplex S of edge length a and an arbitrary point P in its affine hull, and we describe, in Theorem 1, the two known generalizations of Pompeiu's theorem pertaining to the distances $b_1,\dots,b_{n+1}$ from P to the vertices of S. We also show, in Theorem 2, that these two generalizations are equivalent to two inequalities involving the $n+1$ positive numbers $b_1,\dots,b_{n+1}$. In Section 3, we explore the possibility of obtaining similar generalizations for the medians (and the generalized medians) $c_1,\dots,c_{n+1}$

Mathematics subject classification (2010): 51M04, 51M16, 51M15, 51M25, 51M20.
Keywords and phrases: Content, facet, generalized medians, simplex, isodynamic simplex, medians, Pompeiu's theorem.

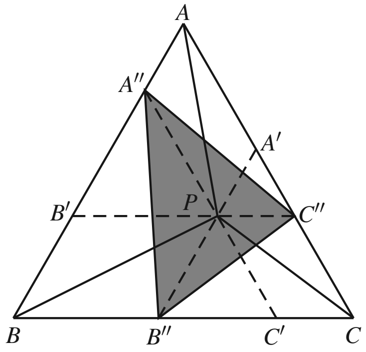

Figure 1: *The Pompeiu triangle* (A',B',C'). *Here* $A'B'' \parallel AB$, $B'C'' \parallel BC$, $C'A'' \parallel CA$, *Here* $A'B''$, $B'C''$, $C'A''$ *are parallel to* AB, BC, CA, *resulting in isosceles trapezoids* $C''AA''P$, $A''BB''P$, $B''CC''P$, *and hence in* $(PA,PB,PC) = (C''A'',A''B'',B''C'')$.

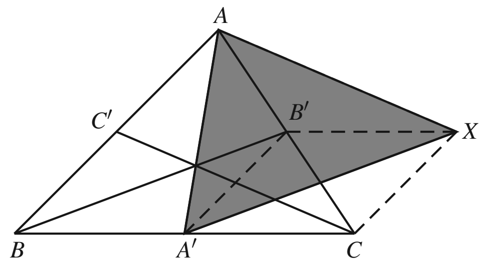

Figure 2: *The median triangle* (X,A,A'). *Here* $CX \parallel BA$, $B'X \parallel BA$, *resulting in parallelograms* $BB'XA'$, $C'AXC$, *and hence in* $BB' = A'X$, $CC' = AX$.

of an arbitrary n-simplex, and we show that one of these generalizations holds, while the other does not. In Section 4, we investigate the relation between the two generalizations in Theorem 2, and we prove that one is strictly stronger than the other. We do this by proving that one of the two inequalities mentioned earlier is strictly stronger than the other one. We then consider other closely related inequalities, and we establish relations among them that we expect to be useful in similar contexts. In Section 6, we consider the question whether other cevians, such as altitudes and angle bisectors, of an arbitrary triangle form a triangle, and we give references that imply negative answers.

2. Pompeiu's theorem

As mentioned above, the distances b_1,b_2,b_3 from the vertices of a regular triangle $T = A_1A_2A_3$ to a point B in its affine hull form a possibly degenerate triangle T_B, and that T_B is degenerate if and only if B lies on the circumcircle of T. To generalize this to higher dimensions, one would take a regular n-simplex, and consider the distances $b_1,\ldots,b_{n+1}$ from its vertices to an arbitrary point in its affine hull. Unfortunately, when $n > 2$, these distances, being only $n+1$ in number, can in no way serve as the $n(n+1)/2$ edge lengths of an n-simplex. To circumvent this dead end, M. Fiedler came up with a brilliant idea based on the observation that

$$\begin{aligned} &\text{the three numbers } a_1,\ a_2,\ a_3 \text{ form a triangle} \\ \iff &\text{the three numbers } \frac{1}{a_2a_3},\ \frac{1}{a_3a_1},\ \frac{1}{a_1a_2} \text{ do.} \end{aligned} \tag{1}$$

Thus letting $b_1,\dots,b_{n+1}$ be as above, Fiedler considered the possibility of finding an n-simplex $[B_1,\dots,B_{n+1}]$ having the property that

$$\|B_i - B_j\| = \frac{1}{b_i b_j} \text{ for } 1 \leqslant i < j \leqslant n+1, \tag{2}$$

and amazingly, he was able to prove that this is the case indeed, i.e., that there exists an n-simplex $[B_1,\dots,B_{n+1}]$ whose edge lengths satisfy (2).

We mention here that an n-simplex $[B_1,\dots,B_{n+1}]$ whose edge lengths satisfy (2) for some positive numbers $b_1,\dots,b_{n+1}$ is said to be *isodynamic*. These simplices are studied in [3], and are descibed in [8].

Pompeiu's theorem was also generalized in a different way in [1]. It is proved there that the distances $b_1,\dots,b_{n+1}$ can serve as the contents (or $(n-1)$-dimensional Lebesgue measures) of the facets of an n-simplex, This is motivated by the feeling that both edge lengths and facet contents (or "volumes") of an n-simplex, $n \geqslant 3$, correspond naturally to the side lengths of a triangle: Like a side of a triangle, an edge of an n-simplex is the convex hull of two vertices, and a facet of an n-simplex is the convex hull of any n of its vertices. For ease of reference, we record the results described above in the following theorem.

THEOREM 1. *Let $n \geqslant 2$, and let $S = [A_1,\dots,A_{n+1}]$ be a regular n-simplex. Let $a_1,\dots,a_{n+1}$ be the distances from the vertices of S to an arbitrary point P in its affine hull, and suppose that P is not a vertex of S.*

(i) There is a possibly degenerate n-simplex T whose facet contents are $a_1,\dots,a_{n+1}$. Also, T is degenerate if and only if $n = 2$ and P lies on the circumcircle of S.

(ii) There is an n-simplex $[B_1,\dots,B_{n+1}]$ whose edge lengths are given by

$$\|B_i - B_j\| = \frac{1}{a_i a_j} \textit{ for } 1 \leqslant i < j \leqslant n+1.$$

Proof. Part (i) is proved in [1], and Part (ii) is proved by M. Fiedler in [3]. □

Tests for deciding whether given positive numbers $b_1,\dots,b_{n+1}$ satisfy conditions (i) and (ii) in Theorem 1 above are given in the next theorem.

THEOREM 2. *Let $a_1,\dots,a_{n+1}$, $n \geqslant 2$, be given positive numbers, and let*

$$\mathscr{L} = a_1 + \cdots + a_{n+1},\ \mathscr{M} = a_1^2 + \cdots + a_{n+1}^2,\ \mathscr{N} = a_1^4 + \cdots + a_{n+1}^4. \tag{3}$$

(i) There is an n-simplex whose facet contents are $a_1,\dots,a_{n+1}$ if and only if

$$\prod_{i=1}^{n+1} (\mathscr{L} - 2a_i) > 0, \textit{ or equivalently } \mathscr{L} > 2a_i \textit{ for } 1 \leqslant i \leqslant n+1. \tag{4}$$

(ii) *There is an* n*-simplex* $[B_1,\dots,B_{n+1}]$ *whose edge lengths are given by*

$$\|B_i - B_j\| = \frac{1}{a_i a_j} \text{ for } 1 \leqslant i < j \leqslant n+1$$

if and only if

$$\mathscr{M}^2 - n\mathscr{N} > 0. \tag{5}$$

Proof. The first statement (i) was proved by L. Gerber in [4], and was later reproved by S. Izumi in [11]. The second statement (ii) was proved by M. Fiedler in [3]. It also appears as the middle statement of [7, Theorem 7.1 (b)] in a slightly altered form, namely, *there exists a* d*-simplex* $[B_1,B_2,\dots,B_{d+1}]$ *such that* $\|B_i - B_j\|^2 = \beta_i\beta_j$ *if and only if*

$$\left(\frac{1}{\beta_1} + \cdots + \frac{1}{\beta_{d+1}}\right)^2 - d\left(\frac{1}{\beta_1^2} + \cdots + \frac{1}{\beta_{d+1}^2}\right) > 0. \quad \square$$

Finally we find it convenient to introduce concise terms to describe properties (i) and (ii) of Theorem 2.

DEFINITION 1. We say that the positive numbers $b_1,\dots,b_{n+1}$ have the *Fiedler property* if there exists an n-simplex $[B_1,\dots,B_{n+1}]$ for which (2) holds. We say that they have the *Gerber property* if there exists an n-simplex whose facet contents are $b_1,\dots,b_{n+1}$.

Thus the results in [3] and in [1] say that the distances from the vertices of a regular n-simplex, $n \geqslant 2$, to an arbitrary point in its affine hull have both the Fiedler and Gerber properties. Notice that when $n = 2$, then the two properties are equivalent by (1).

3. Pompeiu-like theorems for the medians of an n-simplex

As mentioned earlier, and referring to Definition 1, the lengths of the medians of an arbitrary triangle have the Gerber property, i.e., can serve as the side lengths of a triangle. This is true of what have been referred to as the s-medians (or generalized medians). These are defined for triangle ABC and $s \in \mathbb{R}$ to be the cevians AA_s, BB_s, and CC_s, where A_s, B_s, and C_s are the points on the sidelines BC, CA, and AB that divide the sides BC, CA, and AB in the ratio $s : 1-s$. A proof of the fact that the lengths of the s-medians form a triangle is depicted in Figure 3 below. Notice that when $s = 1/2$, the s-medians are nothing but the medians.

In view of the above and in view of Theorem 1, it is natural to ask whether the medians of an n-simplex, $n \geqslant 2$, have the Fiedler and/or Gerber properties. It is also natural to define higher dimensional analogues of the notion of s-medians, and to decide whether these analogues have the Fiedler and/or the Gerber properties. All of these questions are answered below. Analogues to triangles' s-medians are the $\mathbf{u}$-medians

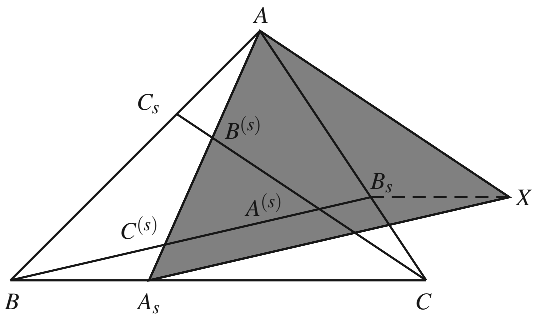

Figure 3: *The s-median triangle* (X,A,A_s).

defined for n-simplices for all $n \geqslant 2$ in definition 2, and Theorem 3 proves that the $\mathbf{u}$-medians, and hence the medians, of any n-simplex, $n \geqslant 2$, have the Gerber property. Example 1 shows that the medians of an n-simplex, $n \geqslant 2$, do not necessarily have the Fiedler property.

DEFINITION 2. Let $\mathbf{u} = (u_1, \ldots, u_n)$, where $u_1, \ldots, u_n > 0$ and $u_1 + \cdots + u_n = 1$. For an n-simplex $S = [A_1, \ldots, A_{n+1}]$, and any $1 \leqslant j \leqslant n+1$, we let

$$A'_j = \sum_{k=1}^{n} u_k A_{j+k}, \tag{6}$$

where indices are significant up to their values mod $n+1$. The cevians $A_jA'_j$, $1 \leqslant j \leqslant n+1$, are called the $\mathbf{u}$-medians of S. When $n = 2$, and when $\mathbf{u} = (s,t) = (s, 1-s)$, the $\mathbf{u}$-medians of S were studied in [5] and [6], where they were called s-medians, and where they were proved to qualify as the side lengths of a triangle.

THEOREM 3. *The lengths of the $\mathbf{u}$-medians of any n-simplex, $n \geqslant 2$, can serve as the facet contents of an n-simplex.*

Proof. It follows from (6) that

$$U_j = A'_j - A_j = \sum_{k=1}^{n} u_k A_{j+k} - \sum_{k=1}^{n} u_k A_j = \sum_{k=1}^{n} u_k \left(A_{j+k} - A_j\right).$$

Adding over j, we obtain

$$\sum_{j=1}^{n+1} U_j = \sum_{j=1}^{n+1} \left(\sum_{k=1}^{n} u_k \left(A_{j+k} - A_j\right) \right) = \sum_{k=1}^{n} \left(\sum_{j=1}^{n+1} A_{j+k} - \sum_{j=1}^{n+1} A_j \right) u_k = \sum_{k=1}^{n} (0) u_k = 0.$$

Therefore

$$\sum_{j=1}^{n+1} U_j = 0,$$

and hence

$$\|U_{n+1}\| = \| \sum_{j=1}^{n} U_j\| \ < \ \sum_{j=1}^{n} \|U_j\|,$$

where the inequality here is strict because $U_1, \dots, U_n$ are not collinear. Thus

$$2\|U_{n+1}\| < \sum_{j=1}^{n+1} \|U_j\|.$$

Similarly

$$2\|U_k\| < \sum_{j=1}^{n+1} \|U_j\|$$

for every $k = 1, \dots, n+1$. Therefore the lengths $\|A_k - A'_k\|$, $1 \leqslant k \leqslant n+1$, can serve as the facet contents of an n-simplex, as desired. □

COROLLARY 1. *The lengths of the medians of any n-simplex, $n \geqslant 2$, can serve as the facet contents of an n-simplex.*

Thus the medians (and the generalized medians) of an n-simplex satisfy the Gerber property. The next example shows that they do not necessarily satisfy the Fiedler property.

EXAMPLE 1. We now give an example of an n-simplex having medians of lengths $a_1, \dots, a_{n+1}$ such that there does not exist an n-simplex $[B_1, \dots, B_{n+1}]$ whose edge lengths are given by

$$\|B_i - B_j\| = \frac{1}{a_i a_j} \text{ for } 1 \leqslant i < j \leqslant n+1. \tag{7}$$

Let $e_1, \dots, e_n$ be the standard basis of the Euclidean space $\mathbb{R}^n$, i.e., e_i is the vector whose i-th coordinate is 1 and whose other coordinates are 0. Let x be a positive number, and let the n-simplex $T = [A_1, \dots, A_{n+1}]$ be defined by

$$A_i = xe_i \text{ for } 1 \leqslant i \leqslant n, \ \ A_{n+1} = -A_1 - \cdots - A_n.$$

Thus the centroid of T is the origin, and the centroid M_i, $1 \leqslant i \leqslant n+1$, of the i-th facet is given by

$$M_i = \frac{1}{n}\left(\sum_{j=1, j\neq i}^{n+1} A_j\right) \ = \ \frac{-A_i}{n}. \tag{8}$$

Also $\|A_i\| = x$ if $1 \leqslant i \leqslant n$, and $\|A_{n+1}\| = \sqrt{nx^2}$. Therefore the lengths a_i, $1 \leqslant i \leqslant n+1$, of the medians A_iM_i are given by

$$a_i = \|A_i - M_i\| = \frac{n+1}{n}\|A_i\| = \begin{cases} \frac{x(n+1)}{n} & \text{if } 1 \leqslant i \leqslant n, \\ \frac{x(n+1)\sqrt{n}}{n} & \text{if } i = n+1. \end{cases} \tag{9}$$

Hence we have

$$\begin{aligned}
\mathscr{M} &:= a_1^2 + \cdots + a_{n+1}^2 = \left(\frac{x^2(n+1)^2}{n^2}\right)(n+n) \ = \ \frac{2x^2(n+1)^2}{n}. \\
\mathscr{N} &:= a_1^4 + \cdots + a_{n+1}^4 = \left(\frac{x^4(n+1)^4}{n^4}\right)\left(n+n^2\right) \ = \ \frac{x^4(n+1)^5}{n^3}. \\
\mathscr{M}^2 - n\mathscr{N} &= \frac{4x^4(n+1)^4}{n^2} - \frac{x^4(n+1)^5}{n^2} \ = \ \frac{x^4(n+1)^4(3-n)}{n^2}.
\end{aligned}$$

Thus $\mathscr{M}^2 - n\mathscr{N} \leqslant 0$ for $n \geqslant 3$, and therefore there does not exist an n-simplex $[B_1, \ldots, B_{n+1}]$ whose edge lengths satisfy (7).

4. Relation between the Fiedler and Gerber properties

It is natural to ask whether any of the Fiedler and Gerber properties implies the other. We know from Corollary 1 and Example 1 that the Gerber property does not imply the Fiedler property, and therefore we ask whether the Fiedler property implies the Gerber property. In view of Theorem 2, this is equivalent to asking whether the inequality (5) implies inequality (4). An affirmative answer is given in Theorem 4 below.

THEOREM 4. *Let $a_1, \ldots, a_{n+1}$, $n \geqslant 2$, be positive numbers, and let $\mathscr{L}, \mathscr{M}, \mathscr{N}$ be defined as in* (3)*, i.e., by*

$$\mathscr{L} = a_1 + \cdots + a_{n+1}, \ \mathscr{M} = a_1^2 + \cdots + a_{n+1}^2, \ \mathscr{N} = a_1^4 + \cdots + a_{n+1}^4. \tag{10}$$

Consider the inequalities (4) *and* (5)*, i.e., the inequalities*

$$\prod_{i=1}^{n+1}(\mathscr{L} - 2a_i) > 0, \text{ or equivalently } \mathscr{L} > 2a_i \text{ for } 1 \leqslant i \leqslant n+1, \tag{11}$$

$$\mathscr{M}^2 - n\mathscr{N} > 0. \tag{12}$$

If $n = 2$, then (12) *and* (11) *are equivalent. If $n \geqslant 3$, then* (12) *implies* (11)*, but* (11) *does not imply* (12)*.*

Proof. We find it convenient to introduce the auxiliary inequalities

$$\mathscr{L}^2 - n\mathscr{M} > 0, \tag{13}$$

$$\prod_{i=1}^{n+1}(\mathscr{M} - 2a_i^2) > 0, \text{ or equivalently } \mathscr{M} > 2a_i^2 \text{ for } 1 \leqslant i \leqslant n+1, \tag{14}$$

and prove a little more than what is required. Precisely, we prove the implications in the table

	For	
[A]	$n=2$,	(14) $\Longrightarrow$ (11) $\Longleftrightarrow$ (12) $\Longrightarrow$ (13)
[B]	$n \geqslant 3$,	(12) $\Longrightarrow$ (13) $\Longrightarrow$ (11)
[C]	$n \geqslant 3$,	(11) does not imply (12)

Table 1.

We divide the proof into steps for convenience.

Step (A). The implications in [A] follow from the facts that

$$\begin{aligned} (11) &\Longleftrightarrow a_1, a_2, a_3 \text{ form a triangle} \\ (12) &\Longleftrightarrow a_1, a_2, a_3 \text{ form a triangle} \\ (13) &\Longleftrightarrow \sqrt{a_1}, \sqrt{a_2}, \sqrt{a_3} \text{ form a triangle} \\ (14) &\Longleftrightarrow a_1^2, a_2^2, a_3^2 \text{ form a triangle,} \end{aligned}$$

and the equivalences

$$a_1, a_2, a_3 \text{ form a triangle} \Longleftrightarrow \sqrt{a_1}, \sqrt{a_2}, \sqrt{a_3} \text{ form an acute triangle,} \tag{15}$$

$$a_1^2, a_2^2, a_3^2 \text{ form a triangle} \Longleftrightarrow a_1, a_2, a_3 \text{ form an acute triangle.} \tag{16}$$

The implication $\Longrightarrow$ in (16) follows from the simple implications

$$\begin{aligned} &x^2+y^2>z^2 \Longrightarrow x+y>z \text{ (by squaring),} \\ &x^2+y^2>z^2 \Longrightarrow \cos Z, \text{ being nothing but } \frac{x^2+y^2-z^2}{2xy}, \text{ is positive.} \end{aligned}$$

The implication $\Longleftarrow$ follows similarly from the law of cosines. The equivalence $\Longleftrightarrow$ in (15) follows from that in (16).

Step $(B\text{-}1)$. Here, we prove the first implication in [B]. We shall show that if $n \geqslant 3$ (and in fact if $n \geqslant 2$), and if (12) holds, then (13) holds. For this, let

$$S_p = \sum_{i=1}^{n+1} a_i^p.$$

Then we are given that

$$S_2^2 > nS_4, \text{ i.e., } n < \frac{S_2^2}{S_4},$$

and we are to prove that

$$S_1^2 > nS_2, \text{ i.e., } n < \frac{S_1^2}{S_2}.$$

It is clearly sufficient to show that

$$\frac{S_2^2}{S_4} \leqslant \frac{S_1^2}{S_2}, \text{ i.e., } S_2^3 \leqslant S_1^2 S_4.$$

But the last inequality follows from the log-convexity of S_p, since $(2)(3) = (1)(2) + (4)(1)$; see [2, (2), p. 213].

Step $(B\text{-}2)$. For the second implication in [B], we suppose that $n \geqslant 3$, and that (13) holds. We are to prove (11). Notice that (11) is equivalent to saying that

$$\mathscr{L} - 2a_k > 0 \text{ for } 1 \leqslant k \leqslant n+1.$$

We start with the case $n \geqslant 4$. It follows from (13) that $\mathscr{L} > \sqrt{n\mathscr{M}}$, and therefore

$$\mathscr{L} - 2a_k > \sqrt{n\mathscr{M}} - 2a_k$$

for all k. Thus to show that $\mathscr{L} > 2a_k > 0$, it suffices to show that $\sqrt{n\mathscr{M}} > 2a_k$. But

$$\sqrt{n\mathscr{M}} > 2a_k \iff n\sum_{i=1}^{n+1} a_i^2 > 4a_k^2 \iff n \sum_{i=1, i\neq k}^{n+1} a_i^2 > (4-n)a_k^2,$$

which clearly holds since $n \geqslant 4$. Thus $\mathscr{L} - 2a_k > 0$ for all k, as desired.

Next consider the case $n = 3$, and suppose that (13) holds. Rename $a_1, \ldots, a_4$ as x, y, z, w. Then by symmetry it is sufficient to show that $y + z + w > x$. This follows from the following simplifications:

$$\begin{aligned}
(x+y+z+w)^2 &> 3(x^2+y^2+z^2+w^2), \text{ by (13)}\\
2x(y+z+w) + 2(yz+zw+wy) &> 2(x^2+y^2+z^2+w^2),\\
2x(y+z+w) - 2x^2 &> 2(y^2+z^2+w^2) - 2(yz+zw+wy)\\
&= (y-z)^2 + (z-w)^2 + (w-y)^2\\
&\geqslant 0.
\end{aligned}$$

Therefore $2x(y+z+w) - 2x^2 > 0$, and hence $y+z+w > x$, as desired. Hence (13) implies (11).

Step (C). For [C], we shall construct an example where (11) holds, while (12) does not. For this, we consider the family of examples $(a_1, \ldots, a_{n+1})$ given by

$$a_1 = a_2 = \cdots, a_n = 1,\ a_{n+1} = x,$$

where $x > 0$. For such examples, it is easy to see that

$$\text{(11) is equivalent to } x < n \text{ and (12) is equivalent to } x < \sqrt{\tfrac{2n}{n-1}}.$$

Thus to show that (11) does not imply (12), it is sufficient to find $x > 0$ such that $x < n$ but $x \geqslant \sqrt{2n/(n-1)}$. This is possible if and only if $n > \sqrt{2n/(n-1)}$ which in turn holds if and only if $n \geqslant 3$. In fact,

$$\begin{aligned}
n > \sqrt{\frac{2n}{n-1}} &\iff n(n-1) > 2 \iff n^2 - n - 2 > 0\\
&\iff (n-2)(n+1) > 0 \iff n > 2.
\end{aligned}$$

This completes the proof of the theorem. □

REMARK 1. It follows from Theorems 4 and 2 that the existence of an n-simplex $U = [U_1, \dots, U_{n+1}]$ having the property that

$$\|U_i - U_j\| = \frac{1}{a_i a_j} \text{ for } 1 \leqslant i < j \leqslant n+1$$

implies the existence of an n-simplex $W = [W_1, \dots, W_{n+1}]$ having the property that

$$\text{content of the } i\text{-th facet of } W = a_i \text{ for } 1 \leqslant i \leqslant n+1,$$

but not conversely (unless $n = 2$). In particular, this shows that the condition (ii) in Theorem 1 is strictly stronger than (i) (for all $n > 2$). This also shows that Fiedler's result in [3] is strictly stronger than the result of Al-Afifi et al in [1] (for all $n > 2$).

REMARK 2. One can also show that the assumption, made in Theorem 1, regarding P lying in the affine hull of S is redundant. To see this, assume that $S = [A_1, \dots, A_{n+1}]$ is a regular n-simplex that lies in $\mathbb{R}^m$ for some $m > n \geqslant 2$, and that $P \in \mathbb{R}^m$. Without loss of generality, we may assume that P is not in the affine hull of S. Let H be the affine hull of $\{P, A_1, \dots, A_{n+1}\}$, and let $V = [A_0, A_1, \dots, A_{n+1}]$ be a regular $(n+1)$-simplex in H. That such a V exists follows by taking a regular $(n+1)$-simplex $W = [T_0, T_1, \dots, T_{n+1}]$ in H, identifying A_i, $1 \leqslant i \leqslant 1$, with T_i, and then setting $A_0 = T_0$. By Theorem 1(ii), there is an $(n+1)$-simplex $[B_0, B_1, \dots, B_{n+1}]$ with

$$\|B_i - B_j\| = \frac{1}{a_i a_j} \text{ for } 0 \leqslant i < j \leqslant n+1,$$

where $a_0 = \|P - A_0\|$. Restricting attention to the n-simplex $[B_1, \dots, B_{n+1}]$, it is obvious that

$$\|B_i - B_j\| = \frac{1}{a_i a_j} \text{ for } 1 \leqslant i < j \leqslant n+1.$$

Thus, referring to Theorem 1, (ii) holds, and hence (i) holds.

5. More related inequalities

Our main goal in the previous theorem was to establish the implication (12) $\Longrightarrow$ (11). Along the way, we established several other implications among the inequalities (11), (12), (13), and (14). In the next theorem, we establish all the possible remaining implications.

THEOREM 5. *Let $a_1, \dots, a_{n+1}$, $n \geqslant 2$, be positive numbers, and let $\mathscr{L}, \mathscr{M}, \mathscr{N}$ be as given in (10). Consider the inequalities (11), (12), (13), and (14) introduced in Theorem 4 and its proof. Then the implications in the following table hold.*

For	
$n = 2$,	(14) $\Longrightarrow$ (11) $\Longleftrightarrow$ (12) $\Longrightarrow$ (13).
$n \geqslant 3$,	(12) $\Longrightarrow$ (13) $\Longrightarrow$ (11).
$n \geqslant 3$,	(12) $\Longrightarrow$ (14) $\Longrightarrow$ (11).
$n \geqslant 6$,	(13) $\Longrightarrow$ (14).

Table 2.

None of the remaining implications hold. More precisely,

	For	
(1)	$n \geqslant 3,$	(11) *does not imply* (13).
(2)	$n = 2,$	(13) *does not imply* (11).
(3)	$n = 2,$	(12) *does not imply* (14).
(4)	$n \geqslant 2,$	(11) *does not imply* (14).
(5)	$n \geqslant 2,$	(13) *does not imply* (12).
(6)	$n \geqslant 3,$	(11) *does not imply* (12).
(7)	$n \geqslant 3,$	(14) *does not imply* (13).
(8)	$n \geqslant 3,$	(14) *does not imply* (12).
(9)	$n \leqslant 5,$	(13) *does not imply* (14).

Table 3.

Proof. The first two rows of Table 2 are included in Theorem 4. The implication (14) $\Longrightarrow$ (11) in the third row follows from the simple fact that if $a_{n+1}^2 < a_1^2 + \cdots + a_n^2$, $n \geqslant 2$, then $a_{n+1} < a_1 + \cdots + a_n$. The implication (12) $\Longrightarrow$ (14) in the third row follows from the implication (13) $\Longrightarrow$ (11) that was proved in Theorem 4. Thus it remains to prove the implication in the last row, i. e., (13) $\Longrightarrow$ (14) for $n \geqslant 6$.

So suppose that $n \geqslant 6$ and that (13) holds. We are to show that (14) holds. Let us assume (without loss of generality) that $a_1, \dots, a_{n+1}$ are in ascending order, and let us put

$$a_i = y_i \text{ for } i = 1, \dots, n, \text{ and } a_{n+1} = x.$$

Thus

$$y_1 \leqslant \cdots \leqslant y_n \leqslant x.$$

We also set

$$A = y_1 + \cdots + y_n, \ \ B \ = \ y_1^2 + \cdots + y_n^2,$$

and we recall the arithmetic-quadratic means inequality

$$\frac{A}{n} \leqslant \sqrt{\frac{B}{n}}, \text{ i.e., } A^2 \ \leqslant \ nB. \tag{17}$$

We also note that (14) is equivalent to saying that

$$B > x^2. \tag{18}$$

We proceed by contradiction. Thus we suppose that (14) does not hold, i.e., that

$$B \leqslant x^2, \tag{19}$$

and we show that $\mathscr{L}^2 - n\mathscr{M} \leqslant 0$. This is done as follows.

$$\begin{aligned}
\mathscr{L}^2 - n\mathscr{M} &= (x+A)^2 - n(x^2+B) \\
&= (1-n)x^2 + 2Ax + (A^2 - nB) \\
&\leqslant (1-n)x^2 + 2x\sqrt{nB} \text{ (by (17))} \\
&\leqslant (1-n)x^2 + 2x\sqrt{nx^2} \text{ (by (19))} \\
&\leqslant (1-n+2\sqrt{n})x^2 \\
&\leqslant 0, \text{ (because } n \geqslant 6 \text{ and hence } 1-n+2\sqrt{n} < 0).
\end{aligned}$$

The last step follows from the simple implications

$$1-n+2\sqrt{n}<0 \iff 4n<(n-1)^2 \iff n^2-6n+1>0,$$
$$n \geqslant 6 \implies n^2-6n+1=(n-3)^2-8 \geqslant 9-8>0.$$

It remains to give counterexamples that support the last set of statements. The following one-parameter family of counterexamples $(a_1,\dots,a_{n+1})$, where

$$a_1=a_2=\cdots,a_n=1,\ a_{n+1}=x,$$

and where $x>0$ will suffice for almost all our claims. For such examples, it is easy to see that

$$\begin{aligned}
&(11) \text{ is equivalent to } x<n,\\
&(12) \text{ is equivalent to } x<\sqrt{\frac{2n}{n-1}},\\
&(13) \text{ is equivalent to } x<\frac{2n}{n-1},\\
&(14) \text{ is equivalent to } x<\sqrt{n}.
\end{aligned}$$

We now consider each row of Table 2 separately.

(1) If $n \geqslant 4$, take an x such that $2n/(n-1) \leqslant x<n$. This is possible because $2n/(n-1)<n$, i.e., $3<n$. If $n=3$, take the example $(1,1,5,5)$.

(2) For $n=2$, (13) and (11) are equivalent to $x<4$ and $x<2$, respectively, and $(x<4)$ does not imply $(x<2)$.

(3) For $n=2$, (12) and (14) are equivalent to $x<2$ and $x<\sqrt{2}$, respectively, and $(x<2)$ does not imply $(x<\sqrt{2})$.

(4) For $n \geqslant 2$, (11) does not imply (14) because $(x<n)$ does not imply $(x<\sqrt{n})$.

(5) For $n \geqslant 2$, (13) does not imply (12), because $(x<2n/(n-1))$ does not imply $(x<\sqrt{2n/(n-1)})$. This in turn follows from $2n/(n-1)=2+2/(n-1)>2$.

(6) If $n \geqslant 3$, then (11) does not imply (12), because $(x<n)$ does not imply $(x<\sqrt{2n/(n-1)})$. This in turn follows from the fact that

$$\begin{aligned}
n \leqslant \sqrt{\frac{2n}{n-1}} &\iff n(n-1) \leqslant 2 \iff n^2-n-2 \leqslant 0\\
&\iff (n-2)(n+1) \leqslant 0 \iff n \leqslant 2.
\end{aligned}$$

(7) For $n \geqslant 6$, (14) does not imply (13), because $(x<\sqrt{n})$ does not imply $(x<2n/(n-1))$. This in turn follows from $2n/(n-1)<\sqrt{n}$ (for $n \geqslant 6$). In fact,

$$\frac{2n}{n-1}<\sqrt{n} \iff 4n<(n-1)^2 \iff n^2-6n+1>0 \iff n \geqslant 6. \tag{20}$$

For $n=5$, take $(a_1,\dots,a_6)=(1,2,3,4,5,6)$.
For $n=4$, take $(a_1,\dots,a_5)=(1,1,3,4,4)$.

For $n = 3$, take $(a_1, \ldots, a_4) = (1, 1, 4, 4)$.

(8) (14) and (12) are equivalent to $x < \sqrt{n}$ and $x < \sqrt{2n/(n-1)}$, respectively. If $n > 3$, then $n > 2n/(n-1)$, and therefore (14) does not imply (12). For $n = 3$, take the example (1,1,5,5).

(9) For $n < 6$, (13) does not imply (14), because $(x < 2n/(n-1))$ does not imply $(x < \sqrt{n})$. This in turn follows from $2n/(n-1) > \sqrt{n}$ (for $n < 6$), which can be proved in a way similar to (20). □

6. Other cevians

The Pompeiu-like theorem for medians and generalized medians for a triangle may give the false impression that Pompeiu-like theorems hold for the other classical cevians, namely, the altitudes and the angle bisectors of a triangle. This is obviously false for altitudes, since the altitudes h_i of a triangle with side lengths a_i and area K are given by $h_i = 2K/a_i$, and since the inequalities $a_1 + a_2 > a_3$, etc., do not imply the inequalities $h_1 + h_2 > h_3$, etc., as seen by the example $(a_1, a_2, a_3) = (1, 1, c)$, with $c < 2$. It may turn out to be interesting to find geometric characterizations of triangles whose altitudes form a triangle.

The question whether the (lengths of the internal) angle bisectors of a triangle form a triangle is much more interesting and has an unexpected answer. It turns out that given *any* three positive numbers b_1, b_2, b_3, there is a triangle whose angle bisectors have lengths b_1, b_2, b_3; see [13], [14], and [9]. This answers in the strongest terms the question whether the angle bisectors of an arbitrary triangle form a triangle. It also gives rise to the question whether triangles whose angle bisectors form a triangle have a geometric characterization.

REFERENCES

[1] GH. AL-AFIFI, M. HAJJA, AND A. HAMDAN, *Another n-dimensional generalization of Pompeiu's theorem*, Amer. Math. Monthly, to appear.

[2] P. S. BULLEN, *A Dictionary of Inequalities*, second edition, Taylor & Francis Group, LLC, Boca Raton, Florida, 2015.

[3] M. FIEDLER, *Isodynamic systems in Euclidean spaces and an n-dimensional analogue of a theorem by Pompeiu*, Časopis pro pěstování matematiky **102** (1977), 370–381.

[4] L. GERBER, *The orthocentric simplex as an extreme simplex*, Pacific J. Math. **56** (1975), 97–111.

[5] M. HAJJA, *On nested sequences of triangles*, Results Math. **54** (2009), 289–299.

[6] M. HAJJA, *The sequence of generalized median triangles and a new shape function*, J. Geom. **96** (2009), 71–79.

[7] M. HAJJA AND M. HAYAJNEH, *Impurity of the corner angles in certain special families of simplices*, J. Geom. **105** (2014), 539–560.

[8] M. HAJJA, M. HAYAJNEH, AND H. MARTINI, *More characterizations of certain special families of simplices*, Results Math. **69** (2016), 23–47.

[9] G. HEINDL, *How to compute a triangle with prescribed lengths of its internal angle bisectors*, Forum Geom. **16** (2016), 407–414.

[10] R. HONSBERGER, *Mathematical Delights*, The Dolciani Mathematical Expositions, no. 28, MAA, Washington, D. C., 2004.

[11] S. IZUMI, *Sufficiency of simplex inequalities*, Proc. Am. Math. Soc. **144** 2016, 1299–1307.

[12] R. A. JOHNSON, *Advanced Euclidean Geometry*, Dover Publications, Inc., New York, 1929.

[13] P. MIRONESCU AND L. PANAITOPOL, *The existence of a triangle with prescribed angle bisector lengths*, Amer. Math. Monthly **101** (1994), 58–60.

[14] S. F. OSINKIN, *On the existence of a triangle with prescribed angle bisector lengths*, Forum Geom. **16** (2016), 399–405.

[15] S. SAVCHEV AND T. ANDREESCU, *Mathematical Miniatures*, Anneli Lax New Mathematical Library, No. 43, MAA, Washington, D. C., 2003.

(Received September 26, 2017)

Ghada Al-Afifi
Philadelphia University
Amman, Jordan
e-mail: ghalafifi@yahoo.com

Mowaffaq Hajja
Philadelphia University
Amman, Jordan
e-mail: mowhajja@yahoo.com

Ahmad Hamdan
Philadelphia University
Amman, Jordan
e-mail: ahamdan2011@yahoo.com

Panagiotis T. Krasopoulos
Unified Social Security Fund
28 Pireos St., 10437, Athens, Greece
e-mail: pan_kras@yahoo.gr; pankras@in.gr