

## SOME PROPERTIES OF ZIPF–MANDELBROT LAW AND HURWITZ $\zeta$ -FUNCTION

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*Abstract.* In this paper we deal with analytical properties of the Zipf-Mandelbrot law. If total mass of this law is spread all over positive integers we come to Hurwitz  $\zeta$ -function. As we show, it is very natural first to examine properties of Hurwitz  $\zeta$ -function to derive properties of Zipf-Mandelbrot law. Using some well-known inequalities such as Chebyshev's and Lyapunov's inequality we are able to deduce a whole variety of theoretical characterizations that include, among others, log-convexity, log-subadditivity, exponential convexity.

### 1. Introduction

In 1918 [5] Jean-Baptiste Estoup first notice the hyperbolic nature of word use. The first explanation of this regularity is done by George Kingsley Zipf in 1935, today's known as Zipf's law [5]. Zipf in [12] observed in the study of human languages that when words are ranked according to their occurrence frequency in the descending order, the frequency is inversely proportional to its rank  $k$  according to

$$f(k) = \frac{C}{k^s}, \quad (1)$$

where  $s$  is near 1 and  $C$  is the normalizing constant.

The more general model introduced Benoit Mandelbrot [6, pp. 503–512], by using arguments on the fractal structure of lexical trees:

$$f(k) = \frac{C}{(k+q)^s}. \quad (2)$$

When  $q = 0$ , we get indeed Zipf's law.

For  $N \in \mathbb{N}$ ,  $q \geq 0$ ,  $s > 0$ ,  $k \in \{1, 2, \dots, N\}$ , in a more clear form, Zipf-Mandelbrot law (probability mass function) is defined with

$$f(k, N, q, s) = \frac{1/(k+q)^s}{\zeta(N, s, q)}, \quad (3)$$

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where

$$\zeta(N, s, q) = \sum_{i=1}^N \frac{1}{(i+q)^s}, \quad (4)$$

$N \in \mathbb{N}$ ,  $q \geq 0$ ,  $s > 0$ ,  $k \in \{1, 2, \dots, N\}$ . If total number of words  $N$  tends to infinity we denote

$$f(k, q, s) = \frac{1/(k+q)^s}{\zeta(s, q)}, \quad (5)$$

where

$$\zeta(s, q) = \sum_{i=1}^{\infty} \frac{1}{(i+q)^s} \quad (6)$$

we recognize as Hurwitz  $\zeta$ -function. This infinite case, when total mass is spread over all set of positive integers, particularly, is studied in [7]. Note here, that we use more suitable version of Hurwitz  $\zeta$  function (see also [1]), since in the classical definition sum starts from zero and  $q > 0$ . However, this fact does not alter our conclusions about Hurwitz  $\zeta$ -function.

There are also quite different interpretations of Zipf-Mandelbrot law. As it is pointed out in [8] (see also [2], [11]), parameters in (3) can be interpreted in the following way:  $N$  is the number of species present and the parameters  $q$  and  $s$  have an ecological interpretation:  $q$  represents the diversity of the environment and  $s$  the predictability of the ecosystem, i.e. the average probability of the appearance of a species.

## 2. Monotonicity properties

As starting point, we use the next proposition on inequalities for sums of positive order ([9, pp. 36], [10, pp. 165]).

PROPOSITION 1. *If  $a_i \geq 0$ ,  $i \in \mathbb{N}$  then for  $0 < t < s$*

$$\left( \sum_{i=1}^{\infty} a_i^s \right)^{\frac{1}{s}} \leq \left( \sum_{i=1}^{\infty} a_i^t \right)^{\frac{1}{t}}. \quad (7)$$

THEOREM 1.

i) *The function  $s \mapsto [\zeta(N, s, q)]^{1/s}$  is decreasing i.e. for  $s > t > 0$*

$$[\zeta(N, s, q)]^{1/s} \leq [\zeta(N, t, q)]^{1/t}.$$

ii) *The function  $s \mapsto [f(k, N, q, s)]^{1/s}$  is increasing i.e. for  $s > t > 0$*

$$[f(k, N, q, s)]^{1/s} \geq [f(k, N, q, t)]^{1/t}.$$

iii) *The function  $s \mapsto [\zeta(s, q)]^{1/s}$  is decreasing i.e. for  $s > t > 0$*

$$[\zeta(s, q)]^{1/s} \leq [\zeta(t, q)]^{1/t}.$$

iv) The function  $s \mapsto [f(k, q, s)]^{1/s}$  is increasing i.e. for  $s > t > 0$

$$[f(k, q, s)]^{1/s} \geq [f(k, q, t)]^{1/t}.$$

*Proof.* i) We use the Proposition 1, for

$$a_i = \begin{cases} \frac{1}{i+q}, & i = 1, \dots, N; \\ 0, & i > N. \end{cases}$$

ii) Follows from i)-part and

$$\frac{1}{f(k, N, q, s)} = \sum_{i=1}^N \left( \frac{k+q}{i+q} \right)^s = (k+q)^s \zeta(N, s, q). \tag{8}$$

iii) Use Proposition 1 for  $a_i = \frac{1}{i+q}$ ,  $i \in \mathbb{N}$ .

iv) Follows from iii)-part and

$$\frac{1}{f(k, q, s)} = (k+q)^s \zeta(s, q). \quad \square \tag{9}$$

**THEOREM 2.** The function

$$s \mapsto (Nf(k, N, q, s))^{1/s} \tag{10}$$

is decreasing i.e. for  $s > t > 0$

$$(Nf(k, N, q, s))^{1/s} \leq (Nf(k, N, q, t))^{1/t}. \tag{11}$$

*Proof.* From (8) it follows

$$\frac{1}{Nf(k, N, q, s)} = \frac{1}{N} \sum_{i=1}^N \left( \frac{k+q}{i+q} \right)^s, \tag{12}$$

i.e.

$$(Nf(k, N, q, s))^{-1/s} = \left[ \frac{1}{N} \sum_{i=1}^N \left( \frac{k+q}{i+q} \right)^s \right]^{1/s}. \tag{13}$$

Denote  $x_i = \frac{k+q}{i+q}$ ,  $i = 1, \dots, N$ . Then the right-hand side of (13) is the power mean

$$M_N^{[s]}(x_{1,N}) := \left[ \frac{1}{N} \sum_{i=1}^N x_i^s \right]^{1/s}.$$

Using well-known fact, that  $s \mapsto M_N^{[s]}(x_{1,N})$  is increasing function (see for example [9, 10]) we conclude that the function

$$s \mapsto (Nf(k, N, q, s))^{1/s} \tag{14}$$

is decreasing.  $\square$

### 3. Log-convexity and exponential convexity

Let us recall well-known Lyapunov inequality, for sequences ([9, pp. 34], [10, pp. 117]).

PROPOSITION 2. *If  $a_i \geq 0$ ,  $i \in \mathbb{N}$ , then for  $0 < r < s < t$*

$$\left( \sum_{i=1}^{\infty} a_i^s \right)^{t-r} \leq \left( \sum_{i=1}^{\infty} a_i^r \right)^{t-s} \left( \sum_{i=1}^{\infty} a_i^t \right)^{s-r}. \quad (15)$$

If we set  $a_i = \frac{1}{i+q}$ ,  $i \in \mathbb{N}$  in (15) we get

COROLLARY 1. *For  $1 < r < s < t$*

$$\zeta^{t-r}(s, q) \leq \zeta^{t-s}(r, q) \zeta^{s-r}(t, q). \quad (16)$$

In the next theorem we prove, log-concavity of  $s \mapsto f(k, N, q, s)$  and log-convexity of  $s \mapsto \zeta(s, q)$ .

THEOREM 3. *Let  $\lambda \in (0, 1)$ .*

i) *For  $0 < r < t$ ,*

$$\zeta(N, \lambda r + (1 - \lambda)t, q) \leq \zeta^\lambda(N, r, q) \zeta^{1-\lambda}(N, t, q).$$

ii) *For  $0 < r < t$ ,*

$$(f(k, N, q, \lambda r + (1 - \lambda)t))^{-1} \leq (f(k, N, q, r))^{-\lambda} (f(k, N, q, t))^{-(1-\lambda)}.$$

iii) *For  $1 < r < t$ ,*

$$\zeta(\lambda r + (1 - \lambda)t, q) \leq \zeta^\lambda(r, q) \zeta^{1-\lambda}(t, q).$$

iv) *For  $1 < r < t$ ,*

$$(f(k, q, \lambda r + (1 - \lambda)t))^{-1} \leq (f(k, q, r))^{-\lambda} (f(k, q, t))^{-(1-\lambda)}.$$

*Proof.* i) For  $0 < r < t$  and  $\lambda \in (0, 1)$  we set

$$a_i = \begin{cases} \frac{1}{i+q}, & i = 1, \dots, N; \\ 0, & i > N. \end{cases}$$

and  $s = \lambda r + (1 - \lambda)t$  in (15):

$$\left( \sum_{i=1}^N \left( \frac{1}{i+q} \right)^{\lambda r + (1-\lambda)t} \right)^{t-r} \leq \left( \sum_{i=1}^N \left( \frac{1}{i+q} \right)^r \right)^{\lambda(t-r)} \left( \sum_{i=1}^N \left( \frac{1}{i+q} \right)^t \right)^{(1-\lambda)(t-r)}.$$

ii) Follows from (8) and i)-part.

iii) We set  $a_i = \frac{1}{i+q}$  and  $s = \lambda r + (1 - \lambda)t$  in (15).

iv) Follows from iii)-part and (9).  $\square$

We can conclude even more since this result can be extended to exponential convexity [4].

DEFINITION 1. A function  $h : I \rightarrow \mathbb{R}$  is exponentially convex on an interval  $I \subseteq \mathbb{R}$  if it is continuous and

$$\sum_{i,j=1}^n \xi_i \xi_j h\left(\frac{x_i + x_j}{2}\right) \geq 0$$

for all  $n \in \mathbb{N}$  and all choices  $\xi_i \in \mathbb{R}$ ,  $x_i \in I$ ,  $i = 1, \dots, n$ .

THEOREM 4. The function  $s \mapsto \zeta(s, q)$  is exponentially convex function on  $(1, \infty)$ .

Proof. For a given  $n \in \mathbb{N}$  let  $\xi_m \in \mathbb{R}$ ,  $s_m \in (1, \infty)$  ( $m = 1, \dots, n$ ) we have

$$\sum_{l,m=1}^n \xi_l \xi_m \zeta\left(\frac{s_l + s_m}{2}, q\right) = \sum_{l,m=1}^n \xi_l \xi_m \sum_{i=1}^{\infty} \frac{1}{(i+q)^{\frac{s_l + s_m}{2}}} \tag{17}$$

$$= \sum_{i=1}^{\infty} \sum_{l,m=1}^n \xi_l \xi_m \frac{1}{(i+q)^{\frac{s_l + s_m}{2}}} \tag{18}$$

$$= \sum_{i=1}^{\infty} \left( \sum_{m=1}^n \frac{1}{(i+q)^{\frac{s_m}{2}}} \right)^2 \geq 0. \tag{19}$$

Since the function  $s \mapsto \zeta(s, q)$  is continuous function on  $(1, \infty)$ , we conclude its exponential convexity on  $(1, \infty)$ .  $\square$

Using (9) we have also the next corollary.

COROLLARY 2. The function  $s \mapsto (f(k, q, s))^{-1}$  is exponentially convex function on  $(1, \infty)$ .

Proof. This is consequence of (9) and the fact that exponential convexity is closed under finite multiplication of exponentially convex functions.  $\square$

COROLLARY 3. The matrices  $[(\zeta(\frac{s_l + s_m}{2}, q))]_{l,m=1}^n$  and  $[(f(k, q, \frac{s_l + s_m}{2}))^{-1}]_{l,m=1}^n$  are positive semi definite for all  $n \in \mathbb{N}$ ,  $s_1, \dots, s_n$  in  $(1, \infty)$ .

We can also deduce exponential convexity from diversity point of view, notion mentioned in the introduction.

THEOREM 5. For any  $s > 0$ ,  $N \in \mathbb{N}$ , the function

$$q \mapsto \zeta(N, s, q)$$

is exponentially convex on  $(0, \infty)$ .

*Proof.* For  $k = 1, \dots, N$ , using the Laplace transform,

$$\frac{1}{(k+q)^s} = \int_0^{\infty} e^{-(k+q)t} \frac{t^{s-1}}{\Gamma(s)} dt$$

and the fact

$$\sum_{i,j=1}^n \xi_i \xi_j \exp \left[ - \left( k + \frac{q_i + q_j}{2} \right) t \right] = e^{-kt} \left( \sum_{i=1}^n \xi_i \exp \left( - \frac{q_i}{2} t \right) \right)^2 \geq 0,$$

we conclude exponential convexity of the function  $q \mapsto \frac{1}{(k+q)^s}$  on  $(0, \infty)$ . Now  $q \mapsto \zeta(N, s, q)$  is exponentially convex on  $(0, \infty)$  as a finite sum of exponentially convex functions.  $\square$

**THEOREM 6.** For any  $s > 1$ , the function

$$q \mapsto \zeta(s, q)$$

is exponentially convex on  $(0, \infty)$ .

*Proof.* Using Mellin transformation

$$\zeta(s, q) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1} e^{-(q+1)t}}{1 - e^{-t}} dt$$

and

$$\sum_{i,j=1}^n \xi_i \xi_j \exp \left( - \left( \frac{q_i + q_j}{2} + 1 \right) t \right) = \left( \sum_{i=1}^n \xi_i \exp \left( - \frac{q_i + 1}{2} t \right) \right)^2 \geq 0,$$

we conclude exponential convexity of  $q \mapsto \zeta(s, q)$  on  $(0, \infty)$ .  $\square$

**COROLLARY 4.** For  $s > 1$ , the matrix  $[\zeta(s, (\frac{q_l + q_m}{2}))]_{l,m=1}^n$  is positive semi definite for all  $n \in \mathbb{N}$ ,  $q_1, \dots, q_n \in (0, \infty)$ .

**COROLLARY 5.** For any  $s > 1$ , the function

$$q \mapsto \zeta(s, q)$$

is log-convex on  $(0, \infty)$ .

### 4. Log subadditivity

Let us recall Chebyshev’s inequality (see [9, pp. 27], [10, pp. 197]).

**THEOREM 7.** *Let  $(a_1, \dots, a_N)$  and  $(b_1, \dots, b_N)$  be two  $N$ -tuples of real numbers such that*

$$(a_i - a_j)(b_i - b_j) \geq 0, \text{ for } i, j = 1, \dots, N,$$

*and  $(w_1, \dots, w_N)$  be a positive  $n$ -tuple. Then*

$$\left( \sum_{i=1}^N w_i \right) \left( \sum_{i=1}^N w_i a_i b_i \right) \geq \left( \sum_{i=1}^n w_i a_i \right) \left( \sum_{i=1}^N w_i b_i \right). \tag{20}$$

**THEOREM 8.** *The function  $s \mapsto Nf(k, N, q, s)$  is log subadditive, i.e. for  $s, r > 0$*

$$Nf(k, N, q, s+r) \leq [Nf(k, N, q, s)] [Nf(k, N, q, r)]. \tag{21}$$

*Proof.* We apply Chebyshev’s inequality (20) for

$$a_i = \left( \frac{k+q}{i+q} \right)^s, \quad b_i = \left( \frac{k+q}{i+q} \right)^r, \quad w_i = \frac{1}{N}; \quad i = 1, \dots, N.$$

Hence we get

$$\begin{aligned} \frac{1}{N} \sum_{i=1}^N \left( \frac{k+q}{i+q} \right)^{s+r} &\geq \left( \frac{1}{N} \sum_{i=1}^N \left( \frac{k+q}{i+q} \right)^s \right) \left( \frac{1}{N} \sum_{i=1}^N \left( \frac{k+q}{i+q} \right)^r \right) \\ \Rightarrow \frac{1}{Nf(k, N, q, s+r)} &\geq \frac{1}{Nf(k, N, q, s)} \frac{1}{Nf(k, N, q, r)}, \end{aligned}$$

concluding (21).  $\square$

**THEOREM 9.** *The function  $u \mapsto [f(k, N, q, u^{-1})]^{-u}$  is log-convex.*

*Proof.* Using Lyapunov inequality in Proposition 2, for  $0 < r < s < t$

$$\left( \sum_{i=1}^N \left( \frac{k+q}{i+q} \right)^s \right)^{t-r} \leq \left( \sum_{i=1}^N \left( \frac{k+q}{i+q} \right)^r \right)^{t-s} \left( \sum_{i=1}^N \left( \frac{k+q}{i+q} \right)^t \right)^{s-r}. \tag{22}$$

Using (13) we rewrite this as

$$[f(k, N, q, s)]^{-\frac{1}{s}} \leq \left\{ [f(k, N, q, r)]^{-\frac{1}{r}} \right\}^{\frac{t-s}{s}} \left\{ [f(k, N, q, t)]^{-\frac{1}{t}} \right\}^{\frac{t}{s} \frac{s-r}{t-r}} \tag{23}$$

Now we substitute  $t = 1/x$ ,  $r = 1/y$ ,  $\lambda = \frac{t-s}{s} \frac{s-r}{t-r}$  in (23), and since  $1 - \lambda = \frac{t}{s} \frac{s-r}{t-r}$ ,  $s = [\lambda x + (1 - \lambda)y]^{-1}$ , we have

$$\begin{aligned} & \left[ f(k, N, q, [\lambda x + (1 - \lambda)y]^{-1}) \right]^{-[\lambda x + (1 - \lambda)y]} \\ & \leq \left\{ [f(k, N, q, x^{-1})]^{-x} \right\}^\lambda \left\{ [f(k, N, q, y^{-1})]^{-y} \right\}^{1-\lambda}, \end{aligned}$$

concluding log-convexity of the function  $u \mapsto [f(k, N, q, u^{-1})]^{-u}$ .  $\square$

### 5. Gini means and further monotonicity

For positive  $n$ -tuple  $(a_1, \dots, a_n)$ ,  $\alpha, \beta \in \mathbb{R}$ , Gini means are defined with

$$G(\alpha, \beta) = \begin{cases} \left( \frac{\sum_{i=1}^n a_i^\alpha}{\sum_{i=1}^n a_i^\beta} \right)^{\frac{1}{\alpha-\beta}}, & \alpha \neq \beta; \\ \exp \left( \frac{\sum_{i=1}^n a_i^\alpha \ln a_i}{\sum_{i=1}^n a_i^\alpha} \right), & \alpha = \beta. \end{cases} \quad (24)$$

It is known then see [10, pp. 119],

$$G(\alpha_1, \beta_1) \leq G(\alpha_2, \beta_2), \quad (25)$$

for  $\alpha_1 \leq \alpha_2$ ,  $\beta_1 \leq \beta_2$ ,  $\alpha_1 \neq \beta_1$ ,  $\alpha_2 \neq \beta_2$ .

If we choose  $a_i = \frac{k+q}{i+q}$  in (24) we will get Zip-Mandelbrot means:

$$Z(\alpha, \beta) = \begin{cases} \left( \frac{f(k, N, q, \beta)}{f(k, N, q, \alpha)} \right)^{\frac{1}{\alpha-\beta}}, & \alpha \neq \beta; \\ [(k+q)^\alpha \zeta(N, s, \alpha)]^{\frac{(k+q)^\alpha H_{N, q, \alpha}}{\alpha f(k, N, q, \alpha)}} \exp \left( -\frac{(k+q)^\alpha}{\alpha f(k, N, q, \alpha)} E(k, N, q, \alpha) \right), & \alpha = \beta. \end{cases} \quad (26)$$

where

$$E(k, N, q, \alpha) = -\sum_{k=1}^N f(k, N, q, \alpha) \ln f(k, N, q, \alpha)$$

denotes Shannon entropy of the law (3) (for related results see also [3]).

Using (25) we can now formulate the next theorem.

**THEOREM 10.** For  $0 < \alpha_1 \leq \alpha_2$ ,  $0 < \beta_1 \leq \beta_2$ ,  $\alpha_1 \neq \beta_1$ ,  $\alpha_2 \neq \beta_2$ ;

$$Z(\alpha_1, \beta_1) \leq Z(\alpha_2, \beta_2). \quad (27)$$

The expectation of the Zipf-Mandelbrot law is

$$\sum_{k=1}^N k f(k, N, q, s) = \frac{1}{\zeta(N, s, q)} \sum_{k=1}^N \frac{k+q-q}{(k+q)^s} = \frac{\zeta(N, s-1, q)}{\zeta(N, s, q)} - q.$$

This is a decreasing function over  $s$ , as the next theorem shows.

**THEOREM 11.** The function

$$s \mapsto \frac{\zeta(N, s-1, q)}{\zeta(N, s, q)}$$

is decreasing on  $\mathbb{R}_+$ .



*Proof.* We set  $a_i = \frac{1}{i+q}$ ,  $i = 1, \dots, N$  and  $\alpha = s-1$ ,  $\beta = s$  in (24). According (27), for  $0 < s < t$ , we have

$$\left( \frac{\zeta(N, s-1, q)}{\zeta(N, s, q)} \right)^{-1} \leq \left( \frac{\zeta(N, t-1, q)}{\zeta(N, t, q)} \right)^{-1}. \quad \square$$

Of course, result can be extended to Hurwitz  $\zeta$ -function.

COROLLARY 6. *The function*

$$s \mapsto \frac{\zeta(s-1, q)}{\zeta(s, q)}$$

*is decreasing on  $\mathbb{R}_+$ .*

REMARK 1. General remark in this section is that parameters  $\alpha$ ,  $\beta$  in (26) could be any real numbers, so Theorems 10 and 11 are also valid on  $\mathbb{R}^2$  and  $\mathbb{R}$ , respectively.

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